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NONLOCAL BOUNDARY VALUE PROBLEMS*

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The theory of the nonlocal linear boundary value problems is still on the level of examples. Any attempt to encompass them by a unified scheme sticks upon the lack of general methods. Here we are to outline an algebraic approach to linear nonlocal boundary value problems. It is based on the notion of convolution of linear operator and on operational calculus on it. Our operators are right inverses of the differentiation operator and its square. These right inverse operators are determined by the boundary value conditions of the problems we are to deal with. The idea of the algebraic approach consists in algebraization of the problem by reducing it to a single linear algebraic equation of the first degree in a corresponding commutative ring, containing as subrings: the function space, the multipliers ring of the convolution algebra and the number field. Thus we reduce all the consideration into a single algebraic system: the ring of the multiplier fractions. As applications it is possible to be considered the following nonlocal BVPs: 1) Nonlocal Cauchy problems; 2) Dezin BVP; 3) Samarskii – Ionkin problem; 4) Beilin problem; 5) Bitsadze – Samarskii problem. Following our algebraic approach we obtain for all of them explicit representations of the solutions. These representations may be considered as extensions of the classical Duhamel principle. They can be used for numerical calculation of the solutions using quadrature formulae.

1. Nonlocal Cauchy problems connected with the differentiation operator. Here we consider an elementary BVP for the differentiation operator $D_t = \dfrac{d}{dt}$ in the space $C([0,\infty))$:

$$y' = f(t),\ \chi\{y\} = 0 \tag{1.1}$$

with an arbitrary non-zero linear functional χ.

In order a solution to exist for arbitrary f, we are to assume $\chi\{1\} \neq 0$. We normalize χ by the restriction $\chi\{1\} = 1$.

The solution $y = lf\,(t)$ has the explicit form

$$lf\,(t) = \int_a^t f\,(\tau)\,d\tau - \chi_\tau\left\{\int_a^\tau f\,(\sigma)\,d\sigma\right\},\ a \in \Delta. \tag{1.2}$$

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In Dimovski [2] an operational calculus is developed for the operator l in $C(\Delta)$, the space of the continuous functions in an interval Δ. It follows the pattern of Mikusiński but using convolution (1.3) instead of Duhamel convolution (Dimovski [1], p. 52)

$$\left(f \overset{t}{*} g\right)(t) = \chi_\tau \left\{ \int_\tau^t f(t+\tau-\sigma)\, g(\sigma)\, d\sigma \right\} \tag{1.3}$$

such that $lf = \{1\} \overset{t}{*} f$.

This convolution as a rule, has a plenty of divisors of zero. Therefore C is only a commutative ring with the operations $+$ and $\overset{t}{*}$, but not an integrity domain. Nevertheless, the set N of the non-divisors of 0 in C is non-empty. This allows to use the process of "localization" from the general algebra (see S. Lang [4]). Then we consider the ring $\mathbf{M} = N^{-1}\mathrm{C}$ of the convolution fractions $\dfrac{f}{g}$ with $f \in C,\ g \in N$. $\mathbf{M}$ is the quotient ring $(C \times N)/\sim$ where the factorization is with respect to the equivalence relation

$$(f,\ g) \sim (f_1,\ g_1) \Leftrightarrow f \overset{t}{*} g_1 = f_1 \overset{t}{*} g.$$

To say it differently, the fraction $\dfrac{f}{g}$ is the class of all ordered pairs $(f_1,\ g_1)$, $f_1 \in C,\ g_1 \in N$, equivalent to (f, g), i.e.

$$\frac{f}{g} = \{(f_1,\ g_1) : (f_1, g_1) \sim (f,\ g)\}$$

The set $\mathbf{M}$=$N^{-1}C$ of the convolution fractions is a commutative ring, containing as a subring the space $C = C(\Delta)$ with the multiplication (1.3).

The operator l can be identified with the constant function $\{1\}$ in Δ, since $lf = \{1\} \overset{t}{*} f$, i.e. l =$\{1\}$.

As a nondivisor of 0 in $\mathbf{M}$, l has an inverse element

$$s = l^{-1}. \tag{1.4}$$

Let $\alpha \in$ C. Then the element $\dfrac{\alpha f}{f}$, $f \in N$ does not depend on f. Denoting it by $[\alpha]$ and following Mikusiński, we name it as *numerical operator.*

It is easy to show that $[\alpha + \beta] = [\alpha] + [\beta]$ and $[\alpha\beta] = [\alpha]\,[\beta]$ for arbitrary $\alpha,\ \beta \in$ C Hence, the ring of the numerical operators is isomorphic to (C, ., +) and we may consider C as subring of $\mathbf{M}$.

The element s is the algebraic analogue of the differentiation operator $\dfrac{d}{dt}$. The relation between f' and sf when $f \in C^1$ is given by

Theorem 1. *Let* $f \in C^1(\Delta)$. *Then*

$$f' = sf - \chi\{f\}, \tag{1.5}$$

where $\chi\{f\}$ *is considered as a numerical operator.*

Proof. It is easy to see that $lf'(t) = f(t) - \chi\{f\}$. This identity may be written in the form $lf'(t) = f(t) - \chi\{f\}\, l$. It remains to multiply it by s in order to obtain (1.5).

The identity (1.5) is the *basic formula* of our operational calculus.

By induction, from (1.5) it follows the formula

$$f^{(k)} = s^k f - \chi\{f\} s^{k-1} - \chi\{f'\} s^{k-2} - \cdots - \chi\left\{f^{(k-1)}\right\}. \tag{1.6}$$

Let $P(\lambda)$ be a non-zero polynomial and χ be a non-zero linear functional on a space $C(\Delta)$ of continuous functions in an interval Δ.

Consider the BVP

$$P\left(\frac{d}{dt}\right) y = f(t), \quad f \in C(\Delta), \tag{1.7}$$

$\chi\left\{y^{(k)}\right\} = \gamma_k$, $k = 0, 1, 2, \ldots \deg P - 1$ with given $\gamma_k \in \mathrm{C}$. It bears the name of *nonlocal Cauchy problem, determined by the functional* χ.

A nonlocal Cauchy problem is, in fact, any problem for solution of LODEs with constant coefficients in periodic functions. Indeed, if we are looking for a periodic solution of (1.7) with a period T, then this is equivalent to the special nonlocal Cauchy problem

$$P\left(\frac{d}{dt}\right) y = f(t), \; y^{(k)}(T) - y^{(k)}(0) = 0, \; k = 0, \; 1, \; 2, \; \ldots, \; \deg P - 1.$$

More generally, if we are looking for mean-periodic solutions of (1.7) with respect to a given linear functional χ in $C(\mathrm{R})$, i.e. for solutions $y(t)$ such that

$$\chi_\tau\{y(t+\tau)\} = 0, \; -\infty < t < \infty,$$

then this problem is equivalent to the nonlocal Cauchy problem

$$P\left(\frac{d}{dt}\right) y = f(t), \quad \chi_\tau\left\{y^{(k)}\right\} = 0, \; k = 0, \; 1, \; 2, \; \ldots, \; \deg P - 1. \tag{1.8}$$

with homogeneous boundary conditions.

The identities (1.6) allow to reduce each nonlocal Cauchy problem (1.7) to a single algebric equation

$$P(s)\, y = f + Q(s), \tag{1.9}$$

with a polynomial $Q(s)$ of degree $< n$.

The formal solution of (1.9) is $y = \dfrac{1}{P(s)} f + \dfrac{Q(s)}{P(s)}$. It is valid prodided $P(s)$ is non-divisor of zero in $\mathbf{M}$.

To decide if $P(s)$ is not a divisor of 0, we are to show that none of the zeros of $P(\lambda)$ is a zero of the *indicatrix* $E(\lambda) = \chi_\tau\{e^{\lambda\tau}\}$.

Further, as a routine operation, we use the Heaviside algorithm: 1) Factorize $P(s)$; 2) Develop $\dfrac{1}{P(s)}$ and $\dfrac{Q(s)}{P(s)}$ into partial fractions; 3) Interprete the partial fractions $\dfrac{1}{s-\lambda}$ and $\dfrac{1}{(s-\lambda)^k}$ as functions; 4) express $\dfrac{1}{P(s)} = \{G(t)\}$ and $\dfrac{Q(s)}{P(s)} = \{H(t)\}$ as functions; 5) Write down the solution as $y = G * f + H$.

In order to accomplish Step 3), we use the formulae:

$$\frac{1}{s-\lambda} = \left\{\frac{e^{\lambda t}}{E(\lambda)}\right\}, \quad \frac{1}{(s-\lambda)^k} = \left\{\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial\lambda^{k-1}}\left(\frac{e^{\lambda t}}{E(\lambda)}\right)\right\}, \; k = 2, \; 3, \; \ldots$$

If $P(s)$ is a divisor of 0, this is the so called *resonance case*. Obviously, then the solution, if it ever exists, is not unique. In order it to exist some additional conditions on the righthand side f should be imposed. The Heaviside algorithm in a modified form applies to the resonance case too. This application is not trivial (see Grozdev [6]).

At last, we state a generalization of the Duhamel principle for the homogeneous nonlocal Cauchy problem (1.8).

Theorem 2. *Let* $Y = \{Y(t)\}$ *be the solution of* (1.8) *for* $f(t) \equiv 1$. *Then*

$$y(t) = \frac{d}{dt}\{Y * f\} = \chi\{f\}\,Y(t) + \chi_\tau\left\{\int_\tau^t Y(t+\tau-\sigma)\,f'(\sigma)\,d\sigma\right\}$$

is the solution for arbitrary $f \in C^1(\Delta)$.

2. Nonlocal BVP for the square of the differentiation operator. Consider a space $C([0,\ a])$ of the continuous functions $f(x)$ on a segment $[0,\ a]$. Let Φ be a non-zero linear functional on $C^1([0,\ a])$. Further, for a technical simplification, we assume $\Phi_\xi\{\xi\} \neq 0$. Then, without any loss of generality, we may assume $\Phi_\xi\{\xi\} = 1$. Define the right inverse operator L of $\dfrac{d^2}{dx^2}$ as the solution $y = Lf(x)$ of the elementary nonlocal BVP:

$$y'' = f(x),\ y(0) = 0,\ \Phi\{y\} = 0.$$

Its solution has the explicit form:

$$Lf(x) = \int_0^x (x-\xi)f(\xi)\,d\xi - x\Phi_\xi\left\{\int_0^\xi (\xi-\eta)\,f(\eta)\,d\eta\right\}. \tag{2.1}$$

Theorem 3 (Dimovski [1], p.119)**.** *The operation*

$$\left(f \overset{x}{*} g\right)(x) = -\frac{1}{2}\Phi_\xi\left\{\int_0^\xi\left(\int_x^\eta f(\eta+x-\zeta)\,g(\zeta)\,d\zeta - \int_{-x}^\eta f(|\eta-x-\zeta|)\,g(|\zeta|)\operatorname{sgn}(\eta-x-\zeta)\,\zeta d\zeta\right)d\eta\right\} \tag{2.2}$$

is a bilinear, commutative and associative operation in $C([0,a])$ *such that* $Lf = \{x\} \overset{x}{*} f$.

Using convolution (2.2), it is possible to develop an operational calculus for the operator L, similar to the operational calculus for the operator l from Sect. 1

For details, see Dimovski [1], p. 136–137. It may be considered as a part of the operational calculus developed in the next section.

3. Multivariate operational calculi and multivariate nonlocal BVP.

Case 1. Operational calculi with one space variable and one time variable.

We consider the space $C = C([0,1]\times(0,\infty))$ of the functions $f(x,t)$ which are continuous for $0 \le x \le 1$, $0 \le t < \infty$ and let Φ be a non-zero linear functional on $C^1([0,1])$ and χ be a non-zero linear functional on $C([0,\infty))$. As in Sections 1 and 2, we assume $\chi\{1\} = 1$, $\Phi_\xi\{\xi\} = 1$. We will show that it is possible to define an explicit convolution $u*v$ in C such that the operators L and l to be multipliers of the convolution algebra $(C,*)$.

Theorem 4. *Let $u, vu, v \in C$. Then the operation*

$$(3.1)\qquad \begin{aligned}(u*v)(x,t) = -\frac{1}{2}\chi_\tau \Phi_\xi \Bigg\{ \int_0^\xi \Bigg(\int_\tau^t \int_x^\eta u(\eta + x - \zeta, t+\tau-\sigma)\, v(\zeta,\sigma)\, d\zeta\, d\sigma \\ - \int_\tau^t \int_{-x}^\eta u(|\eta - x - \zeta|, t+\tau-\sigma)\, v(|\zeta|,\sigma)\, \operatorname{sgn}(\eta - x - \zeta)\, \zeta d\zeta\, d\sigma \Bigg) d\eta \Bigg\}\end{aligned}$$

is a bilinear, commutative and associative operation in C such that $lL\, u = \{x\} * u$.

Lemma 1. *The operators l and L are multipliers of the convolution algebra $(C, *)$.*

The assertion is a special case of the more general fact that that the "partial" convolutional operators $\{\varphi(t)\} *^t$ and $\{f(x)\} *^x$ are multipliers in the convolution algebra$(C, *)$. Here we skip the routine proof. Next, we will introduce special notations for them.

Definition. If $f \in C([0,1])$ then by $[f]_t$ we denote the "partial" convolutional operator $f *^x$ and call it numerical operator with respect to the variable t, i.e. $[f]_t\, u = f \overset{x}{*} u$.

Similarly, if $\varphi \in C([0,\infty))$, then $[\varphi]_x\, u = \varphi \overset{t}{*} u$ is denoted by $[\varphi]_x$ and it is called numerical operator with respect to x , i.e. $[\varphi]_x\, u = \varphi \overset{t}{*} u$.

Case 2. Two space variables.

We consider the space $C = C([0,a] \times [0,b])$ of the continuous functions $u = u(x,y)$ on the rectangle $0 \le x \le a, 0 \le y \le b$. Let Φ and Ψ be non-zero linear functionals on the spaces $C^1([0,a])$ and $C^1([0,b])$, respectively. We assume additionally that $\Phi_\xi\{\xi\} \neq 0$ and $\Psi_\eta\{\eta\} \neq 0$. Then, without any loss of generality, we may assume that $\Phi_\xi\{\xi\} = 1$ and $\Psi_\eta\{\eta\} = 1$.

We define the right inverses L_x and L_y of the differential operators $(\partial/\partial x)^2$and $(\partial/\partial y)^2$ in $C = C([0,a] \times [0,b])$ by $(L_x u)(0,y) = 0$, $\Phi_\xi\{(L_x u)(\xi,y)\} = 0$ and $(L_y u)(x,0) = 0$, $\Psi_\eta\{(L_y u)(x,\eta)\} = 0$, correspondingly.

It is easy to find the following explicit expressions for these right inverse operators (Sect. 2):

$$(3.2)\qquad \begin{aligned} L_x u = \{x\} \overset{x}{*} u = \int_0^x (x-\xi) u(\xi,y)\, d\xi - x\Phi_\xi \left\{ \int_0^\xi (\xi - \alpha) u(\alpha, y)\, d\alpha \right\}, \\ L_y u = \{y\} \overset{y}{*} u = \int_0^y (y-\eta) u(x,\eta)\, d\eta - x\Psi_\eta \left\{ \int_0^\eta (\eta - \alpha) u(x, \alpha)\, d\alpha \right\}. \end{aligned}$$

Theorem 5. *The operation $u * v$ in C, defined by*

$$(3.3)\qquad (u*v)(x,y) = \frac{1}{4}\Phi_\xi \Psi_\eta \left\{ \int_0^\xi \int_0^\eta h(x,y,\tau,\sigma)\, d\tau\, d\sigma \right\}$$

where

$$h(x,y,\xi,\eta)=\int\limits_x^\xi\int\limits_y^\eta u(\xi+x-\sigma,\eta+y-\tau)v(\sigma,\tau)\,d\sigma\,d\tau$$

$$-\int\limits_{-x}^\xi\int\limits_y^\eta u(|\xi-x-\sigma|,\eta+y-\tau)v(|\sigma|,\tau)\operatorname{sgn}(\xi-x-\sigma)\sigma\,d\sigma\,d\tau$$

$$-\int\limits_x^\xi\int\limits_{-y}^\eta u(\xi+x-\sigma,|\eta-y-\tau|)v(\sigma,|\tau|)\operatorname{sgn}(\eta-y-\tau)\tau\,d\sigma\,d\tau$$

$$+\int\limits_{-x}^\xi\int\limits_{-y}^\eta u(|\xi-x-\sigma|,|\eta-y-\tau|)v(|\sigma|,|\tau|)\operatorname{sgn}(\xi-x-\sigma)(\eta-y-\tau)\sigma\tau\,d\sigma\,d\tau.$$

is a bilinear, commutative and associative operation, such that the product L_xL_y of the operators L_x and L_y has the representations $L_xL_yu=\{xy\}*u$. L_x and L_y are multipliers of the convolutional algebra $(C,*)$: $L_x=[x]_y$, $L_y=[y]_x$.

In both cases it is possible to develop a corresponding operational calculus. To this end one may follow an approach similar to Mikusiński's, using convolution fractions. But, as it concerns the multivariate case, such an approach is more involved than the alternative approach, using multiplier fractions. It is proposed in Dimovski [1], Sect. 1.4.

For the sake of uniformity, by C we denote one of the spaces $C([0,1]\times[0,\infty))$ or $C([0,a]\times[0,b])$. Next, we consider the ring M of the multipliers of the convolution algebra $(C,*)$. Let us remind that a linear operator $A:C\to C$ is a multiplier iff it holds $A(f*g)=(Af)*g$ identically for $f,g\in C$. The theory of the multipliers of Banach and other algebras is developed in Larsen [5]. An elementary, but basic for us result from this theory is the fact that M is a commutative ring . Elements of M are the operators l and L in Case 1, and the operators L_x and L_y in Case 2.

Let us denote by N the class of all non-zero non-divisors of zero in M. It is non-empty, since at least the identity operator I and the above mentioned operators are non-divisors of zero. Further, we apply the standard process of localization, known from the general algebra (see, e.g. Lang [4]). We used it yet in Sect. 1.

Definition. *The quotient ring $\mathbf{M}=N^{-1}M$ of $M\times N$ with respect to the equivalence relation*

$$(A,B)\sim(C,D)\Leftrightarrow AD=BC$$

i.e. $\mathbf{M}=(M\times N)/_\sim$ is said to be the ring of the multiplier fractions of the convolution algebra $(C,*)$.

We will denote the multiplier fractions as usual fractions $\dfrac{A}{B}$. As by usual fractions, we may consider the basic field ($\mathbb{R}$ or $\mathbb{C}$) as a part of $\mathbf{M}$ due to the embedding $\alpha\to[\alpha]/[1]$, where $[\alpha]$ and$[1]$ denote the corresponding numerical multipliers. The convolution algebras $\left(C([0,\ 1]),\overset{x}{*}\right)$ and $\left(C([0,\infty)),\overset{t}{*}\right)$ in Case 1, and the convolution algebras $\left(C([0,a]),\overset{x}{*}\right)$ and $\left(C([0,b]),\overset{y}{*}\right)$ in Case 2 also may be considered as parts of $\mathbf{M}$ due to the embeddings $f(x)\to[f(x)]_t/I$ and $\varphi(t)\to[\varphi(t)]_t/I$ and $f(x)\to[f(x)]_y/I$,

$g(y) \to [g(y)]_x/I$, correspondingly. Here I is the identity operator in C. As for the convolution algebra $(C, *)$ it also is considered as a part of $\mathbf{M}$ due to the embedding $u \to \{u\} * /I$.

For our aims, most important elements of $\mathbf{M}$ are: Case 1: $s = \dfrac{1}{l}$ and $S = \dfrac{1}{L}$; Case 2: $S_x = \dfrac{1}{L_x}$ and $S_y = \dfrac{1}{L_y}$. Here by 1 we denote the identity operator in C.

The relationship between the elements s and S and the corresponding partial differential operators $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial^2}{\partial x^2}$ in Case 1 is given by the following theorem:

Theorem 6. *Let $u \in C$ be such that the partial derivatives $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial^2 u}{\partial x^2}$ exist and are functions of C. The the following identities in $\mathbf{M}$ hold:*

$$\frac{\partial u}{\partial t} = su - [\chi_\tau\{u(x,\tau)\}]_t, \tag{3.4}$$

$$\frac{\partial^2 u}{\partial x^2} = Su - [\Phi_\xi\{u(\xi,t)\}]_x - S\{(1 - \Phi_\xi\{1\}x)u(0,t)\}, \tag{3.5}$$

where by [ ] partial numerical multipliers are denoted, and the brackets { } are use to denote functions of C.

The coresponding results for the Case 2 are stated in the next theorem:

Theorem 7. *Let $u \in C$ be a function with continuous partial derivatives $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial y^2}$. Then in $\mathbf{M}$ the following identities hold:*

$$\frac{\partial^2 u}{\partial x^2} = S_x u - [\Phi_\xi\{u(\xi,y)\}]_x - S_x\{(1 - \Phi_\xi\{1\}x)u(0,y)\} \tag{3.6}$$

and

$$\frac{\partial^2 u}{\partial y^2} = S_y u - [\Psi_\eta\{u(x,\eta)\}]_y - S_y\{(1 - \Psi_\eta\{1\}y)u(x,0)\}. \tag{3.7}$$

Let us prove (3.4). To this end, we calculate lu_t:

$$l\frac{\partial u}{\partial t} = \int_0^t u_t(x,\tau)d\tau - \chi_\tau\left\{\int_0^\tau u_t(x,\sigma)d\sigma\right\} =$$
$$= u(x,t) - u(x,0) - \chi_\tau\{u(x,\tau) - u(x,0)\} = u(x,t) - \chi_\tau\{u(x,\tau)\}.$$

Multiplying by s, we get (3.4). The proofs of (3.5)–(3.7) proceed in one and the same way. For definitness, let us prove (3.6). To this end we calculate $L_x u_{xx}$:

$$L_x\frac{\partial^2 u}{\partial x^2} = u(x,y) - u_x(0,y)x - u(0,y) - x(\Phi_\xi\{u(\xi,y)\} - u_x(0,y) - u(0,y)) =$$
$$= u(x,y) - x(\Phi_\xi\{u(\xi,y)\}) - (1 - \Phi_\xi\{1\}x)u(0,y).$$

Multiplying the identity obtained by S_x, we get (3.6).

By means of (3.4) it is possible to express the partial derivatives $\dfrac{\partial^k u}{\partial t^k}$ by means of

$s^k u$ using the boundary values $\chi_\tau \{u(x,\tau)\}, \chi_\tau \left\{\frac{\partial u}{\partial t}(x,\tau)\right\}, \ldots, \chi_\tau \left\{\frac{\partial^{k-1} u}{\partial t^{k-1}}(x,\tau)\right\}$ and $\frac{\partial^{2m} u}{\partial x^{2m}}$ by means of $S^m u$, using the boundary values: $u(0,t), \frac{\partial^2}{\partial x^2}u(x,0), \ldots, \frac{\partial^{2m-2}}{\partial x^{2m-2}}u(0,t)$ and $\Phi_\xi\{u(\xi,t)\}, \Phi_\xi \left\{\frac{\partial^2 u}{\partial x^2}(\xi,t)\right\}, \ldots, \Phi_\xi \left\{\frac{\partial^{2m-2} u}{\partial x^{2m-2}}(\xi,t)\right\}$. For example,

$$\frac{\partial^4 u}{\partial x^4}(x,t) = S\left\{\frac{\partial^2 u}{\partial x^2}\right\} - \Phi_\xi \left\{\frac{\partial^2 u}{\partial x^2}(\xi,t)\right\} - S\left\{(1-\Phi\{1\}x)\frac{\partial^2 u}{\partial x^2}(0,t)\right\} =$$

$$= S^2 u - S[\Phi_\xi\{u(\xi,t)\}]_x - S^2\{(1-\Phi\{1\}x)u(0,t)\} - \Phi_\xi\left\{\frac{\partial^2 u}{\partial x^2}(\xi,t)\right\} -$$

$$-S\left\{(1-\Phi\{1\}x)\frac{\partial^2 u}{\partial x^2}(0,t)\right\}.$$

Similar to the last formula are the corresponding formulae for the Case 2.

Now we are ready to outline the idea of algebraizing of BVPs for certain classes of PDEs in rectangular domains of the form $[0,1]\times[0,\infty)$ or $[0,a]\times[0,b]$.

Case 1. Evolution equations of the form

$$P\left(\frac{\partial}{\partial t}\right)u = Q\left(\frac{\partial^2}{\partial x^2}\right)u + F(x,t), 0 \le x \le 1, 0 \le t < \infty \tag{3.8}$$

with polynomials P and Q.

Case 2. Equations of the form

$$P\left(\frac{\partial^2}{\partial x^2}\right)u = Q\left(\frac{\partial^2}{\partial y^2}\right)u + F(x,y), \quad 0 \le x \le a, 0 \le y \le b \tag{3.9}$$

with polynomials P and Q.

Assume that $u = \{u(x,t)\}$ is a solution of (3.8). Using the formulae (3.4)–(3.5) and consider equation (3.8) in the ring $\mathbf{M}$ of the multiplier fractions, it takes the algebraic form

$$[P(s) - Q(S)]u = F + F_1, \tag{3.10}$$

where F_1 is a linear combination of boundary values of $u(x,t)$ of the form

$$\chi_\tau\left\{\frac{\partial^k}{\partial t^k}u(x,\tau)\right\}, 0 \le k \le \deg P - 1; \quad \frac{\partial^m}{\partial x^m}u(0,t), 0 \le m \le \deg Q - 1, \text{ and}$$

$$\Phi_\xi\left\{\frac{\partial^{2m}}{\partial x^{2m}}u(\xi,t)\right\}, 0 \le m \le \deg Q - 1.$$

If these boundary value functions are known, then the right hand side$F + F_1$ is also known as an element of **M.** Then the formal solution of (3.10) is

$$u = \frac{F + F_1}{P(s) - Q(S)} \tag{3.11}$$

It always exists, provided $P(s) - Q(S)$ is a non-divisor of zero in $\mathbf{M}$. This is true when there a uniqueness theorem holds. A similar approach applies for (3.9), but using formulae (3.6)–(3.7) instead of (3.4)–(3.5).

As illustrations, we consider two demonstrative examples.

Example 1. Solve the boundary value problem:

$$u_t = u_{xx},\quad 0 < x < 1,\quad 0 < t;\quad u\,(x,0) = 0,\quad u\,(1,t) = t,\quad u\,(x,0) = \frac{x^3}{6} - \frac{x}{6}.$$

Remark. It is taken the polynomial solution $u(x,t) = xt + \dfrac{x^3}{6} - \dfrac{x}{6}$ of the heat equation in order to test the method.

Solution. In our case $\chi\{\varphi\} = \varphi(0)$ and $\Phi\{f\} = f(1)$. It is easy to see that $L^2 = \left\{\dfrac{x^3}{6} - \dfrac{x}{6}\right\}$. Using formulae (3.4), we obtain $u_t = su - [u(x,0)]_t = su - L^2 = su - \dfrac{1}{S^2}$ and $u_{xx} = Su - [u(0,t)]_x = Su - [t]_x = Su - l^2 = Su - \dfrac{1}{s^2}$.

The problem reduces to a single algebraic equation: $su - \dfrac{1}{S^2} = Su - \dfrac{1}{s^2}$

Its solution in **M** is

$$u = \left(\frac{1}{S^2} - \frac{1}{s^2}\right)\frac{1}{s-S} = \frac{s+S}{S^2s^2} = \frac{1}{S^2s} + \frac{1}{Ss^2} = L^2l + Ll^2.$$

If we interprete this as a function $u = \{u\,(x,t)\}$, we obtain

$$u = \frac{x^3}{6} - \frac{x}{6} + xt.$$

The test happened to be successful.

Example 2. Obtain $u = xy$ as the solution of Bitsadze – Samarskii problem:

$$u_{xx} + u_{yy} = 0,\quad 0 \le x \le a,\quad 0 \le y \le b;$$
$$u\,(x,0) = u\,(0,y) = 0,\quad u\,(x,b) = bx,\quad u\,(a,y) - u\,(c,y) = (a-c)\,y,\quad 0 < c < a.$$

Solution. The boundary value functionals are $\Phi\{f\} = (f\,(a) - f\,(c))/(a-c)$ and $\Psi\{g\} = g\,(b)/b$.

By (3.11) we have $u_{xx} = S_x u - [y]_x = S_x u - L_y$ and $u_{yy} = S_y u - [x]_y = S_y u - L_x$. The BVP reduces to the single equation

$$(S_x + S_y)\,u = L_x + L_y,$$

Hence

$$u = (L_x + L_y)/(S_x + S_y) = \left(\frac{1}{S_x} + \frac{1}{S_y}\right)\Big/(S_x + S_y) = \frac{1}{S_xS_y} = L_xL_y = \{xy\}.$$

The test of the method for Case 2 happened to be successful too.

Some more representative examples are considered in [6].

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НЕЛОКАЛНИ ГРАНИЧНИ ЗАДАЧИ

Иван Димовски

Предложена е единна алгебрична схема за изследване на широк клас линейни нелокални гранични задачи за уравнения на математическата физика, включваща нелокалните гранични задачи на Дезин, Бицадзе-Самарски, Ионкин, Бейлин и др. Тя се основава на некласически конволюции, въведени от автора и използва нови директно изградени операционни смятания. Прави се пълно алгебризиране на граничната задача чрез свеждането ѝ до линейно алгебрично уравнение в комутативен пръстен от мултипликаторни частни. Полученото формално алгебрично решение се интерпретира като функция, която е класическо или обобщено решение на граничната задача.