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ON A GROUP OF ELLIPTIC LINEAR-FRACTIONAL TRANSFORMATIONS*

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Using a Sato's construction we define maps of the two-dimensional shape space into itself. The two-dimensional Möbius space and the unit disk are models of the two-dimensional shape space.We prove that the considered maps form a one-parameter subgroup of the Möbius group in the first model and the group of rotation in the second model.

1. Introduction. The equivalence classes of triangles with respect to the group $\mathrm{G} = \mathrm{Sim}^+(\mathbb{R}^2)$ of the direct similarities of the Euclidean plane $\mathbb{R}^2$ can be considered as a two-dimensional shape space Σ_2. This space and the first model of Σ_2, so called the shape sphere, were introduced by D. Kendall in [4]. There are several other models of the space Σ_2. One of these models is due to H. Sato (see [8]). For fixed non-degenerate triangle $\triangle abc$, he considered a point (x, y, z) in the Euclidean space $\mathbb{R}^3$, where $x = \sphericalangle(bac)$, $y = \sphericalangle(cba)$, $z = \sphericalangle(acb)$. Thus the points of the set

$$\Pi = \{(x, y, z) \mid x + y + z = \pi,\ x > 0,\ y > 0,\ z > 0\}$$

represent the equivalence classes of similar triangles in $\mathbb{R}^2$. Let $a(t)$, $b(t)$, $c(t)$ be points lying on the sides $\overline{ab}$, $\overline{bc}$, $\overline{ca}$ of $\triangle abc$ such that the corresponding affine ratios are $(aba(t)) = (bcb(t)) = (cac(t)) = t : (1 - t)$. H. Sato proves that the set of non-degenerate triangles $\triangle abc$

$$T(\triangle\, abc) = \{\triangle\, a(t)b(t)c(t) \mid t \in \mathbb{R}\}$$

is represented by a closed convex curve in Π.

Another representation of the classes of similar triangles is the Euclidean plane extended with a point at infinity. This interpretation is realized by J. Lester in [6]. For that purpose, the Euclidean plane is identified with the field of complex numbers $\mathbb{C}$ and it is adjoined a point at infinity, i.e. $\mathbb{C}_\infty = \mathbb{C} \bigcup \infty$.

Let us recall some basic facts from [6] and [2]. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three points in $\mathbb{C}$ and at most two of them are coinciding then it is defined a triangle $\triangle\mathbf{abc}$. Degenerated triangles with distinct collinear vertices or two coinciding vertices are allowed. There exists a complex number which determines the ordered triangle $\triangle\mathbf{abc}$ up to a direct plane similarity. In according to [6], this is the number

(1) $$\triangle_{\mathbf{abc}} = \frac{\mathbf{a} - \mathbf{c}}{\mathbf{a} - \mathbf{b}} \in \mathbb{C}_\infty,$$

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called a shape of the triangle $\triangle\mathbf{abc}$. In particular, $\triangle\mathbf{abc}$ is isosceles with apex at $\mathbf{a}$ whenever $|\triangle_{\mathbf{abc}}| = 1$, $\triangle\mathbf{abc}$ is equilateral when $\triangle_{\mathbf{abc}} = \omega = \frac{1}{2} + i.\frac{\sqrt{3}}{2}$ or $\triangle_{\mathbf{abc}} = \overline{\omega} = \frac{1}{2} - i.\frac{\sqrt{3}}{2}$ and $\triangle\mathbf{abc}$ is right-angled at $\mathbf{a}$ whenever $\triangle_{\mathbf{abc}}$ is imaginary. It is clear that $\triangle_{\mathbf{abc}} = \infty \iff \mathbf{a} = \mathbf{b} \neq \mathbf{c}$. For any degenerate triangle with $\mathbf{a} \neq \mathbf{b}$, $\triangle_{\mathbf{abc}} \in \mathbb{R}$.

Then, $\mathbb{C}_\infty$ is also a model of the two-dimensional shape space Σ_2. We call $\mathbb{C}_\infty$ the Lester's model of Σ_2.

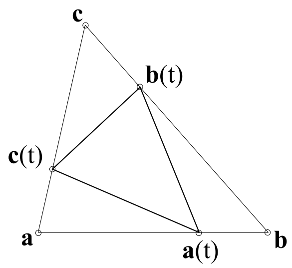

Fig. 1. Sato's construction

Now, we shall obtain a relation between the shapes of triangles $\triangle\mathbf{abc}$ and $\triangle_{\mathbf{a}(t)\mathbf{b}(t)\mathbf{c}(t)}$ for any $t \in \mathbb{R}$. Let $\mathbf{z} \in \mathbb{C}$ be the shape of the triangle $\triangle\mathbf{abc}$, i. e. $\triangle_{\mathbf{abc}} = \mathbf{z}$. Without lost of generality we may suppose $\mathbf{a} = 0$, $\mathbf{b} = 1$, $\mathbf{c} = \mathbf{z}$. If the points $\mathbf{a}(t) \in \overline{\mathbf{ab}}$, $\mathbf{b}(t) \in \overline{\mathbf{bc}}$ and $\mathbf{c}(t) \in \overline{\mathbf{ca}}$ (see Fig. 1) are such that $\mathbf{a}(t) = (1-t)\mathbf{a} + t\mathbf{b}$, $\mathbf{b}(t) = (1-t)\mathbf{b} + t\mathbf{c}$, $\mathbf{c}(t) = (1-t)\mathbf{c} + t\mathbf{a}$, where $t \in \mathbb{R}$, then $\mathbf{a}(t) - \mathbf{c}(t) = (1-t)(\mathbf{a} - \mathbf{c}) + t(\mathbf{b} - \mathbf{a}) = [(1-t)\mathbf{z} - t](\mathbf{a} - \mathbf{b})$ and $\mathbf{a}(t) - \mathbf{b}(t) = (1-t)(\mathbf{a} - \mathbf{b}) + t(\mathbf{b} - \mathbf{c}) = (1-2t)(\mathbf{a} - \mathbf{b}) + t(\mathbf{a} - \mathbf{c}) = (1 - 2t + t\mathbf{z})(\mathbf{a} - \mathbf{b})$. Using (1), we find that

(2) $$\mathbf{w} = \triangle_{\mathbf{a}(t)\mathbf{b}(t)\mathbf{c}(t)} = \frac{(1-t)\mathbf{z} - t}{t\mathbf{z} + 1 - 2t}, \quad t \in \mathbb{R}.$$

Three distinct points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{C}$ define in general six distinct ordered triangles. The triangles $\triangle\mathbf{abc}$, $\triangle\mathbf{bca}$ and $\triangle\mathbf{cab}$ have the same orientation and different shapes. We obtain the shapes of the triangles $\triangle_{\mathbf{abc}} = \mathbf{z}$, $\triangle_{\mathbf{bca}} = \frac{1}{1 - \mathbf{z}}$ and $\triangle_{\mathbf{cab}} = 1 - \frac{1}{\mathbf{z}}$ replacing t in (2) by 0, 1 and 1/2, respectively.

2. A special subgroup of the Möbius group. The linear-fractional transformations $\mathbf{z} \longmapsto \frac{\mathbf{az} + \mathbf{b}}{\mathbf{cz} + \mathbf{d}}$ of $\mathbb{C}_\infty = \mathbb{C} \cup \infty$, where $\mathbf{bc} - \mathbf{ad} \neq 0$, form a group which is the oriented two-dimensional Möbius group, denoted by $\text{Möb}^+(2)$ (see [1] and [5]). The generators of this group are the similarities $\mathbf{z} \longmapsto \mathbf{az} + \mathbf{b}$ and the complex inversion $\mathbf{z} \longmapsto \frac{1}{\mathbf{z}}$. Let $\varphi_t : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ be the transformation defined by

(3) $$\mathbb{C}_\infty \ni \mathbf{z} \longrightarrow \frac{(1-t)\mathbf{z} - t}{t\mathbf{z} + 1 - 2t} \in \mathbb{C}_\infty \quad \text{for} \quad t \in \mathbb{R} \cup \infty.$$

Theorem 1. *Any mapping φ_t is an elliptic transformation in the extended plane. Furthermore, φ_t is a product of a dilation with center at point 0, followed by a translation onto a real line and an inversion with a pole on the same line.*

Proof. Solving the equation $\mathbf{z} = \dfrac{(1-t)\mathbf{z}-t}{t\mathbf{z}+1-2t}$ with respect to $\mathbf{z}$ we find that ω and $\overline{\omega}$ are the unique fixed points of φ_t for any $t \in \mathbb{R}\cup\infty$. Hence, the normal form of the linear-fractional transformation φ_t is $\dfrac{\mathbf{w}-\omega}{\mathbf{w}-\overline{\omega}} = \dfrac{1-t-t\omega}{1-t-t\overline{\omega}}.\dfrac{\mathbf{z}-\omega}{\mathbf{z}-\overline{\omega}}$. Since $\left|\dfrac{1-t-t\omega}{1-t-t\overline{\omega}}\right| = 1$ it follows that φ_t is an elliptic transformation (see [3], p. 282). For the remaining assertion, let h be a similarity of the form $h(\mathbf{z}) = \lambda\mathbf{z}+\mu$, $\mathbf{z}\in\mathbb{C}_\infty$, and let ι be an inversion with a pole α and a power $\beta \in \mathbb{R}^+$, i.e. $\iota(\mathbf{z}) = \alpha + \dfrac{\beta}{\mathbf{z}-\alpha}$, $\mathbf{z}\in\mathbb{C}_\infty$. Then $\varphi_t = \iota\circ h$ whenever $\lambda = \dfrac{3t^2-3t+1}{(1-t)^2} \in \mathbb{R}^+$, $\mu = -\dfrac{t^2}{(1-t)^2}$, $\alpha = t \in \mathbb{R}$ and $\beta = (1-t)^2 \in \mathbb{R}^+$. □

In [8], H. Sato does not explore the case when the triangles $\triangle\mathbf{abc}$ are degenerate. Having in mind the considerations above there is no restriction to examine this case.

Corollary 1. *The mapping* $\varphi_t : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$, *defined by* (3), *preserves the real line for any* $t\in\mathbb{R}\cup\infty$.

Then the one-dimensional shape space $\{\triangle_{a(t)b(t)c(t)} \mid t\in\mathbb{R}\cup\infty\}$, corresponding to the degenerate triangles $\triangle\mathbf{abc}$ with a shape $\triangle_{\mathbf{abc}} \in \mathbb{R}\bigcup\infty$, is the real line supplied with the point at infinity ∞. Therefore the restriction $\varphi_t|_{\mathbb{R}\cup\infty} : \mathbb{R}\cup\infty \longrightarrow \mathbb{R}\cup\infty$ is defined by the equation

$$\varphi_t|_{\mathbb{R}\cup\infty}(x) = \frac{(1-t)x-t}{tx+1-2t},\ x\in\mathbb{R},\ t\in\mathbb{R}\cup\infty.$$

In other words, $\varphi_t|_{\mathbb{R}\cup\infty}$ is a homography of the real projective line.

Some special cases:

(a) If $\mathbf{a}=\mathbf{b}$, i. e. $x = \triangle_{\mathbf{abc}} = \infty$ then $\varphi_t|_{\mathbb{R}\cup\infty}(\infty) = \dfrac{1-t}{t}$;

(b) If $\mathbf{c}$ is the midpoint of the segment $[\mathbf{ab}]$ then $x = \triangle_{\mathbf{abc}} = 1/2$ and $\varphi_t|_{\mathbb{R}\cup\infty}(1/2) = \dfrac{1-3t}{t}$;

(c) If $\mathbf{b}=\mathbf{c}$ then $x = \triangle_{\mathbf{abc}} = 1$ and $\varphi_t|_{\mathbb{R}\cup\infty}(1) = \dfrac{1-2t}{t-1}$.

Now, let us denote by $\mathcal{E}$ the set $\{\varphi_t : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty \mid t\in\mathbb{R}\cup\infty\}$.

Theorem 2. *The transformations of* $\mathcal{E}$ *form a subgroup of Möb*$^+(2)$.

Proof. It is clear that $\mathcal{E} \subset$Möb$^+(2)$ since $-t^2-(1-t)(1-2t)\neq 0$ for $t\in\mathbb{R}$. When $t=\infty$ we obtain the transformation $\mathbf{z} \longmapsto \dfrac{-\mathbf{z}-1}{\mathbf{z}-2}$ which is also linear-fractional. The set $\mathcal{E}$ is a subgroup if the following conditions are satisfied:

(a) $\varphi_t^{-1}\in\mathcal{E}$ for any $\varphi_t\in\mathcal{E}$ and (b) $\varphi_{t_2}\circ\varphi_{t_1}\in\mathcal{E}$ for any $\varphi_{t_1}, \varphi_{t_2}\in\mathcal{E}$.

Starting with (a) we find that the reverse mapping φ_t^{-1} is determined by the equation $\mathbf{z} = \dfrac{(2t-1)\mathbf{w}-t}{t\mathbf{w}-1+t} = \dfrac{(1-s)\mathbf{w}-s}{s\mathbf{w}+1-2s}$, where $s = \dfrac{t}{3t-1}$. Consequently $\varphi_t^{-1}\in\mathcal{E}$. To verify (b) let $\varphi_{t_i}\in\mathcal{E}$, $\varphi_{t_i}(\mathbf{z}) = \dfrac{(1-t_i)\mathbf{z}-t_i}{t_i\mathbf{z}+1-2t_i}$, $t_i\in\mathbb{R}\cup\infty$, $i=1,2$. Hence,

$$\varphi_{t_2}\circ\varphi_{t_1}(\mathbf{z}) = \frac{(1-t_1-t_2)\mathbf{z}+3t_1t_2-t_1-t_2}{(t_1+t_2-3t_1t_2)\mathbf{z}+1-2t_1-2t_2+3t_1t_2} = \frac{(1-k)\mathbf{z}-k}{k\mathbf{z}+1-2k},$$

$k = \dfrac{t_1 + t_2 - 3t_1t_2}{1 - 3t_1t_2}$ and this completes the proof. $\square$

3. The unit disk as a model of the two dimensional shape space. The unit disk $\mathcal{B} := \{\mathbf{z} \in \mathbb{C} \mid |\mathbf{z}| \leq 1\}$ also can be interpreted as two-dimensional shape space, i. e. the points of the unit disk represent the equivalence classes of positive oriented similar triangles. H. Nakamura and K. Oguiso in [7] define the map $f : \mathcal{H} \longrightarrow \mathcal{B}^\circ$ by

$$f(\mathbf{z}) = \frac{\omega^2 - \omega\mathbf{z}}{\mathbf{z} + \omega^2}, \quad \mathbf{z} \in \mathcal{H}, \tag{4}$$

where $\mathcal{H} := \{\mathbf{z} \in \mathbb{C} \mid \operatorname{Im} \mathbf{z} > 0\}$ is the upper half plane and $\mathcal{B}^\circ$ is the open unit disk. The bijection f maps the similarity class of regular triangles with a shape ω in the origin $\mathbf{0}$. If $\triangle\mathbf{abc}$ is a triangle in $\mathbb{R}^2$ with a shape $\triangle_{\mathbf{abc}} = \mathbf{z}$ then $\triangle_{\mathbf{bca}} = \dfrac{1}{1-\mathbf{z}}$ and $\triangle_{\mathbf{cab}} = \dfrac{\mathbf{z}-1}{\mathbf{z}}$. Since $f\left(\dfrac{1}{1-\mathbf{z}}\right) = \omega^2 f(\mathbf{z})$ and $f\left(\dfrac{\mathbf{z}-1}{\mathbf{z}}\right) = \omega^4 f(\mathbf{z})$ it follows that the anticlockwise rotation $R_{\mathbf{0}}^{2\pi/3}$ of $\mathcal{B}$ around the origin through the angle $2\pi/3$ corresponds to the effect of "cycling" of the vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of the triangle $\triangle\mathbf{abc}$.

According to [7], the points of the open disk $\mathcal{B}^\circ$ represent similarity classes of non-degenerate triangles. We extend the map f on the real line $\mathbb{R} \subset \mathbb{C}$ supplied with a point at infinity ∞. Then the results in the previous section may be interpreted in $\mathcal{B}$.
Now, we shall describe some similarity classes of special triangles in $\overline{\mathcal{H}} := \mathcal{H} \cup \mathbb{R} \cup \infty$.

Proposition 1. *The points of the unit circle of $\mathcal{B}$ represent the similarity classes of degenerate triangles in $\mathbb{C}_\infty$.*

Proof. Let $\mathbf{w} = f(\mathbf{z})$ for $\mathbf{z} \in \overline{\mathcal{H}}$. The degenerate triangles in $\mathbb{C}_\infty$ have a shape $\mathbf{z} \in \mathbb{R} \cup \infty$. Since $f(\infty) = -\omega$ and $|\mathbf{w}| = 1 \Leftrightarrow (\omega^2 - \omega\mathbf{z})(\overline{\omega}^2 - \overline{\omega}\overline{\mathbf{z}}) = (\mathbf{z} + \omega^2)(\overline{\mathbf{z}} + \overline{\omega}^2) \Leftrightarrow \operatorname{Im} \mathbf{z} = 0$ the proof is completed.

Proposition 2. *Let c_1, c_2, c_3 be circles with radii $\sqrt{3}$ and centers at the points $2\overline{\omega}$, 2ω, -2 respectively. Then the points of the sections of the circles c_i, $i = 1, 2, 3$ with the open disk $\mathcal{B}^\circ$ represent similarity classes of positive oriented right-angled triangles in $\mathbb{R}^2$. Moreover, the points of the segments $(-\omega^2, \omega^2)$, $(\omega, -\omega)$, $(-1, 1)$ of $\mathcal{B}^\circ$ represent similarity classes of positive oriented isosceles triangles in $\mathbb{R}^2$ (see Fig. 2).*

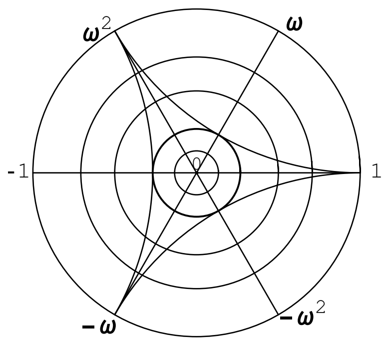

Fig. 2. The unit disk as a model of the 2-dimensional shape space

Proof. Let $\triangle\mathbf{abc}$ be a triangle in $\mathbb{R}^2 \cong \mathbb{C}$ with a shape $\mathbf{z} = \triangle_{\mathbf{abc}} \in \mathbb{C}$. The triangle $\triangle\mathbf{abc}$ is right-angled at $\mathbf{a}$ whenever $\operatorname{Re}\mathbf{z} = 0$. Solving the equation $\mathbf{w} = f(\mathbf{z})$ with respect to $\mathbf{z}$ we get $\mathbf{z} = \dfrac{\omega^2 - \omega^2\mathbf{w}}{\omega + \mathbf{w}}$, $\mathbf{w} \in \mathcal{B}^\circ$. Hence, $\mathbf{z} + \overline{\mathbf{z}} = 0 \Leftrightarrow \dfrac{\omega^2 - \omega^2\mathbf{w}}{\omega + \mathbf{w}} + \dfrac{\overline{\omega}^2 - \overline{\omega}^2\overline{\mathbf{w}}}{\overline{\omega} + \overline{\mathbf{w}}} = 0 \Leftrightarrow$ $4 - 4\operatorname{Re}(\omega\mathbf{w}) + |\mathbf{w}|^2 = 3 \Leftrightarrow |\mathbf{w} - 2\overline{\omega}|^2 = 3 \Leftrightarrow \mathbf{w} \in c_1 \cap \mathcal{B}^\circ$.

The triangle $\triangle\mathbf{abc}$ is isosceles with apex at $\mathbf{a}$ whenever $|\mathbf{z}| = 1$. We have that $|\mathbf{z}| = 1 \Leftrightarrow (\omega^2 - \omega^2\mathbf{w})(\overline{\omega}^2 - \overline{\omega}^2\overline{\mathbf{w}}) = (\omega + \mathbf{w})(\overline{\omega} + \overline{\mathbf{w}}) \Leftrightarrow \mathbf{w} = \omega^4\overline{\mathbf{w}} \Leftrightarrow \mathbf{w} = \pm|\mathbf{w}|\omega^2 \Leftrightarrow$ $\mathbf{w}$ lies on the segment $(-\omega^2, \omega^2)$ of $\mathcal{B}^\circ$. Finally, applying the rotation $R_{\mathbf{0}}^{2\pi/3}$ we obtain the representations in $\mathcal{B}^\circ$ for all similarity classes of positive oriented right-angled or isosceles triangles in $\mathbb{R}^2$. □

Now, let us define the map $\psi_t : \mathcal{B} \longrightarrow \mathcal{B}$ by the commutative diagram below

$$\begin{array}{ccc} \overline{\mathcal{H}} & \xrightarrow{\varphi_t} & \overline{\mathcal{H}} \\ f\big\downarrow & & \big\downarrow f \\ \mathcal{B} & \xrightarrow{\psi_t} & \mathcal{B} \end{array}$$

Theorem 3. *The map $\psi_t : \mathcal{B} \longrightarrow \mathcal{B}$ is a rotation.*

Proof. Let $\mathbf{z} \in \overline{\mathcal{H}}$. We have that $\overline{\mathcal{H}} \ni \mathbf{z} \overset{\varphi_t}{\longrightarrow} \varphi_t(\mathbf{z}) = \dfrac{(1-t)\mathbf{z} - t}{t\mathbf{z} + 1 - 2t} \in \overline{\mathcal{H}}$, where $t \in \mathbb{R} \cup \infty$. If $f(\mathbf{z}) = Z \in \mathcal{B}$ then replacing in (4) $\mathbf{z}$ by $\varphi_t(\mathbf{z})$ we get $f(\varphi_t(\mathbf{z})) = \dfrac{\omega^2 - \omega\varphi_t(\mathbf{z})}{\varphi_t(\mathbf{z}) + \omega^2} = \dfrac{\omega(2t-1) - t}{\omega(t-1) + t}.Z$. Therefore $\psi_t(Z) = \dfrac{\omega(2t-1) - t}{\omega(t-1) + t}.Z$. From $\left|\dfrac{\omega(2t-1) - t}{\omega(t-1) + t}\right| = 1$ it follows that ψ_t is a rotation of $\mathcal{B}$. □

So, if $\psi_t(Z) = e^{i\theta(t)}.Z$ for any $Z \in \mathcal{B}$ we get $\cos\theta(t) = \dfrac{(2-3t)^2 - 3t^2}{(2-3t)^2 + 3t^2}$ and $\sin\theta(t) = \dfrac{2\sqrt{3}t(3t-2)}{(2-3t)^2 + 3t^2}$.

Corollary 2. *The transformations of the set $\{\psi_t \mid t \in \mathbb{R} \cup \infty\}$ form the group of rotations of the unit disk $\mathcal{B}$.*

The proof is trivial.

REFERENCES

[1] M. Berger. Geometry I, II., Springer, Berlin, 1994.

[2] G. Georgiev, R. Encheva. One-dimensional shape spaces. *Math. and Education in Math.*, **34** (2005), 108–112.

[3] Kiyosi Itô (ed.) Encyclopedic dictionary of mathematics. The MIT Press, Cambridge, Massachusetts and London, England, 1993.

[4] D. Kendall. Shape manifolds, procrustean metric, and complex projective spaces. *Bull. London Math. Soc.* **16** (1984), 81–121.

[5] R. Langevin, P. Walczak. Holomorphic maps and pencils of circles. *American Mathematical Monthly*, **115**, No 8 (2008), 690–700.

[6] J. A. Lester. Triangles I: Shapes. *Aequationes Math.*, **52** (1996), 30–54.

[7] H. Nakamura, K. Oguiso. Elementary moduli space of triangles an iterative processes. *J. Math. Sci. Univ. Tokyo*, **10** (2003), 209–224.
[8] H. Sato. Orbits of triangles obtained by interior division of sides. *Proc. Japan Acad.*, **74** (1998), 4–9.

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ВЪРХУ ГРУПАТА ОТ ЕЛИПТИЧНИ ДРОБНО-ЛИНЕЙНИ ПРЕОБРАЗУВАНИЯ

Радостина Енчева, Георги Георгиев

Като използваме конструкция на Сато, дефинираме изображение в двумерното шейп пространство. Двумерното Мьобиусово пространство и единичният диск са модели на това шейп пространство. Доказваме, че разглежданите изображения образуват едно-параметрична подгрупа на Мьобиусовата група в първия модел и групата на ротациите във втория модел.