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ON EVOLUTES OF NODARY AND UNDULARY DELAUNAY CURVES*

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Following the genuine geometrical construction invented by Delaunay we have found the explicit parameterizations of the evolutes of the nodary and undulary curves associated with the profile curves behind Delaunay nodoids and unduloids. The parameterizations of the latter curves are found also in explicit form.

1. Introduction. The interesting subclass of the surfaces of revolution with constant mean curvature have been introduced many years ago by the French geometer Delaunay, who gave also their complete classification [2]. The list of these surfaces which nowadays bears his name includes planes, cylinders, catenoids, spheres, nodoids and unduloids. It is an amazing fact discovered again by Delaunay that the last two types of surfaces arise via two purely geometrical constructions.

According to the second of them the profile curves of these surfaces can be identified with *roulettes*, i.e., the traces of the foci of conics obtained when they roll along some line without slipping. This construction which is well documented (see e.g. [3, 5] and [14]) has been generalized recently in a way to produce the so called anisotropic Delaunay surfaces [10].

In an Appendix to the abovementioned Delaunay paper Sturm presents a third – namely, variational characterization of these surfaces as those surfaces of revolution having a minimal lateral area at a fixed volume. That in turn revealed why these surfaces make their appearance as soap bubbles and liquid drops [7, 13] or cells under compression [17]. Modern expositions of the variational viewpoint on the subject can be found in [13] and [14].

Using entirely different methods the constant mean curvature surfaces of revolution have been treated by Kenmotsu [9] and Konopelchenko & Taimanov [11]. E.g., Kenmotsu had found and solved a complex nonlinear differential equation which describes the surfaces of revolution in $\mathbb{R}^3$ with a given mean curvature. The respective generating curve of the surface is presented in terms of the generalized Fresnel's integrals built on the specified mean curvature. Strange enough, one has to notice that the original

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Delaunay approach was based on finding first the corresponding evolute and then the generating curve itself. As far as we are aware the evolutes of the profile curves were not discussed in the literature and that is why one of the purposes of the present work is to present them in explicit form. The other one is to provide new parameterizations of the curves generating the Delaunay surfaces.

2. The Genuine Delaunay Construction. It was already mentioned that Delaunay starts with finding the evolute $\tilde{C}$ of the sought profile curve C. In what follow we will keep his notation as far as possible. If ρ is the radius of the curvature of C and $\tilde{s}$ is the natural parameter on its evolute $\tilde{C}$ by its very definition one has

$$\rho = \mathring{c} - \tilde{s}, \tag{1}$$

where $\mathring{c}$ is an arbitrary real parameter. Let $\mathfrak{n}$ denote the part of the tangent $\tilde{\mathbf{T}}$ of $\tilde{C}$ between M and its intersection with the symmetry axis X (see Fig. 1). The condition

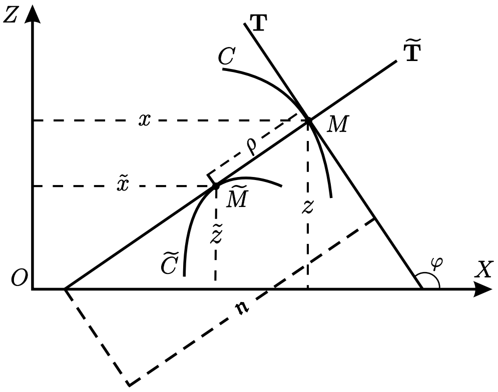

Fig. 1

that the surface $\mathcal{S}$ obtained by revolving C about X has a constant mean curvature $\frac{1}{2a}$ in which a is another real parameter can be written as

$$\frac{1}{\rho} + \frac{1}{\mathfrak{n}} = \frac{1}{a} \cdot \tag{2}$$

By inspecting Fig. 1 one easily finds also that

$$\mathfrak{n} = \tilde{z}\frac{\mathrm{d}\tilde{s}}{\mathrm{d}\tilde{z}} + \mathring{c} - \tilde{s} \tag{3}$$

and therefore

$$\frac{1}{\mathring{c} - \tilde{s}} + \frac{1}{\tilde{z}\frac{\mathrm{d}\tilde{s}}{\mathrm{d}\tilde{z}} + \mathring{c} - \tilde{s}} = \frac{1}{a} \cdot \tag{4}$$

Integrating the last equation one gets

$$\tilde{z}^2 = \alpha(\mathring{c} - \tilde{s})(2a - \mathring{c} + \tilde{s}) \tag{5}$$

where α is the integration constant. This equation can be solved for $\tilde{s}$ and after that the result differentiated with respect to $\tilde{z}$ in order to obtain

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{s}} = -\frac{\alpha\sqrt{a^2 - \tilde{z}^2/\alpha}}{\tilde{z}} \tag{6}$$

along with

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{s}} = \sqrt{1 - \left(\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{s}}\right)^2} = \frac{\sqrt{(1+\alpha)\tilde{z}^2 - a^2\alpha^2}}{\tilde{z}} \cdot \tag{7}$$

An inspection of (6) and (7) leads to the conclusion that in the above expressions the constant α can take all positive values and that in this case $\tilde{z}$ will vary in the interval $\left[\frac{a\alpha}{\sqrt{1+\alpha}}, a\sqrt{\alpha}\right]$. If α is negative it can take values between -1 and 0 while $|\tilde{z}|$ can take any value greater than $-\frac{a\alpha}{\sqrt{1+\alpha}}\cdot$ This means that in the last case the evolute will have infinite branches. The two alternatives just described will be considered below separately.

3. Nodary. Assuming that $0 < \alpha \leq \infty$ and introducing

$$m^2 = \frac{a^2\alpha^2}{1+\alpha}, \qquad n^2 = a^2\alpha, \qquad n^2 > m^2 \tag{8}$$

equations (6) and (7) can be combined into the form

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{z}} = -\frac{\sqrt{(1+\alpha)\tilde{z}^2 - a^2\alpha^2}}{\alpha\sqrt{a^2 - \tilde{z}^2/\alpha}} = -\sqrt{\frac{1+\alpha}{\alpha}}\frac{\sqrt{\tilde{z}^2 - m^2}}{\sqrt{n^2 - \tilde{z}^2}} \cdot \tag{9}$$

The last expression suggests that it can be uniformized via

$$\tilde{z} = \frac{m}{\mathrm{dn}(u,k)}, \qquad m = \frac{a\alpha}{\sqrt{1+\alpha}} = a\alpha k, \qquad k = \frac{1}{\sqrt{1+\alpha}}, \tag{10}$$

where $\mathrm{dn}(u,k)$ is one of the three *Jacobian elliptic functions*, u is its argument and the parameter k is known as an elliptic modulus. Details about elliptic functions and integrals can be found in [8]. Making use of (10) one can immediately find from (9) that

$$\tilde{x} = m(u - E(\mathrm{am}(u,k),k)), \tag{11}$$

where $\mathrm{am}(u,k)$ is Jacobi's *amplitude function*, $E(\psi,k)$ denotes the so called incomplete elliptic integral of the second kind and the integration constant is omitted. Taken together (10) and (11) provide the explicit parameterization of the evolute $\tilde{C}$. Its involute, i.e., the profile curve C of the Delaunay surface of constant mean curvature $\frac{1}{2a}$ can be found relying on direct geometrical relations (or consulting some of the textbooks [1, 6, 13–16]

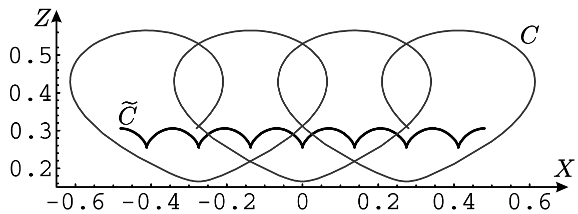

Fig. 2. The evolute $\tilde{C}$ of the *nodary* C generated with $\alpha = 2.333$ and $a = 0.2$ by (10), (11) and (14).

on classical differential geometry)

(12) $$x=\tilde{x}+\rho\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{s}},\qquad z=\tilde{z}+\rho\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{s}}\cdot$$

By (1), (5–7) and (10) one easily find

(13) $$\rho=a-\sqrt{a^2-\frac{\tilde{z}^2}{\alpha}}=a\left(1-k\frac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}\right),\quad \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{s}}=\mathrm{sn}(u,k),\quad \frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{s}}=-\mathrm{cn}(u,k)$$

which taken together give the parameterization of the *nodary*

(14) $$\begin{aligned} x[u] &= m(u-E(\mathrm{am}(u,k),k))+a(1-k\frac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)})\mathrm{sn}(u,k)\\ z[u] &= \frac{m}{\mathrm{dn}(u,k)}-a(1-k\frac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)})\mathrm{cn}(u,k). \end{aligned}$$

Both, the *nodary* C and its evolute $\tilde{C}$ are depicted in Fig. 2 for a concrete values of the parameters α and a.

4. Undulary. Following the plan announced above we will consider in this section the case when $-1<\alpha<0$. Because the treatment will be quite parallel to that of nodary the details will be just outlined but in order to distinguish the cases we use in respective formulas the bars which will reminiscent that we are dealing with negative α. In this setting we will have

(15) $$\frac{\mathrm{d}\bar{\tilde{x}}}{\mathrm{d}\tilde{s}}=\frac{\sqrt{(1+\alpha)\bar{\tilde{z}}^2-a^2\alpha^2}}{\bar{\tilde{z}}},\qquad \frac{\mathrm{d}\bar{\tilde{z}}}{\mathrm{d}\tilde{s}}=\frac{\sqrt{-\alpha}\sqrt{\bar{\tilde{z}}^2-a^2\alpha}}{\bar{\tilde{z}}}$$

and respectively

(16) $$\frac{\mathrm{d}\bar{\tilde{x}}}{\mathrm{d}\bar{\tilde{z}}}=\frac{1}{\sqrt{-\alpha}}\sqrt{\frac{(1+\alpha)\bar{\tilde{z}}^2-a^2\alpha^2}{\bar{\tilde{z}}^2-a^2\alpha}}=\sqrt{\frac{1+\alpha}{-\alpha}}\sqrt{\frac{\bar{\tilde{z}}^2-\bar{m}^2}{\bar{\tilde{z}}^2+\bar{n}^2}},$$

where

(17) $$\bar{m}^2=\frac{a^2\alpha^2}{1+\alpha},\qquad \bar{n}^2=-a^2\alpha,\qquad \bar{m}^2+\bar{n}^2=-\frac{a^2\alpha}{1+\alpha}\cdot$$

This time the uniformization can be accomplished by

(18) $$\bar{\tilde{z}}=\sqrt{\bar{m}^2+\bar{n}^2}\,\frac{\mathrm{dn}(u,\bar{k})}{\mathrm{cn}(u,\bar{k})}=\sqrt{\frac{-\alpha}{1+\alpha}}\,\frac{\mathrm{dn}(u,\bar{k})}{\mathrm{cn}(u,\bar{k})},\quad \bar{k}^2=\frac{\bar{n}^2}{\bar{m}^2+\bar{n}^2}=1+\alpha.$$

Doing this, we obtain

(19) $$\mathrm{d}\bar{\tilde{x}}=-a\frac{\mathrm{cn}^2(u,\bar{k})}{\mathrm{sn}^2(u,\bar{k})}\,\mathrm{d}u$$

and consequently

(20) $$\bar{\tilde{x}}=a(E(\mathrm{am}(u,\bar{k}),\bar{k})+\frac{\mathrm{cn}(u,\bar{k})\mathrm{dn}(u,\bar{k})}{\mathrm{sn}(u,\bar{k})})$$

in which as before, the integration constant is omitted. Further on, it is easy to find also that

(21) $$\bar{\rho}=a(1-\frac{1}{\bar{k}}\frac{1}{\mathrm{sn}(u,\bar{k})}),\qquad \frac{\mathrm{d}\bar{\tilde{x}}}{\mathrm{d}\tilde{s}}=\bar{k}\,\frac{\mathrm{cn}(u,\bar{k})}{\mathrm{dn}(u,\bar{k})},\qquad \frac{\mathrm{d}\bar{\tilde{z}}}{\mathrm{d}\tilde{s}}=\frac{\sqrt{-\alpha}}{\mathrm{dn}(u,\bar{k})}$$

which immediately gives

$$\bar{x}[u] = \tilde{\tilde{x}} + \bar{\rho}\frac{\mathrm{d}\tilde{\tilde{x}}}{\mathrm{d}\tilde{s}} = a(\bar{k} - \frac{1}{\mathrm{sn}(u,\bar{k})})\frac{\mathrm{cn}(u,\bar{k})}{\mathrm{dn}(u,\bar{k})}$$

(22)

$$\bar{z}[u] = \tilde{\tilde{z}} + \bar{\rho}\frac{\mathrm{d}\tilde{\tilde{z}}}{\mathrm{d}\tilde{s}} = \frac{a\sqrt{-\alpha}}{\bar{k}}\left(\frac{\mathrm{dn}(u,\bar{k})}{\mathrm{sn}(u,\bar{k})} + a(\bar{k} - \frac{1}{\mathrm{sn}(u,\bar{k})})\frac{1}{\mathrm{dn}(u,\bar{k})}\right).$$

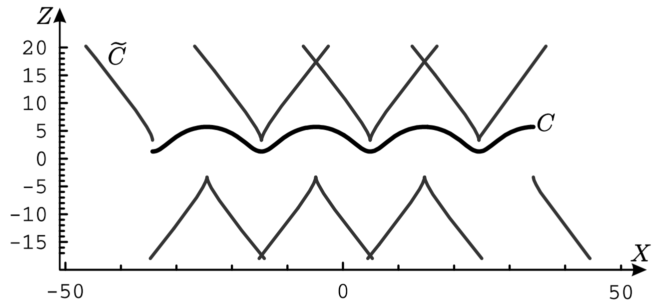

Fig. 3. The evolute $\tilde{C}$ of the *undulary* C generated with $\alpha = 3.5$ and $a = -0.6$ by (18), (20) and (22)

5. Concluding Remarks. The generating curves of the nodoids and the unduloids called by Eells [3] *nodary*, respectively *undulary* and their evolutes, have been found following the original Delaunay construction. These curves are periodic along the symmetry axis and have one local minimum and one local maximum in each period and do not depend on the chosen point on the evolutes as the constant $\mathring{c}$ desappears from all formulae. The parameterizations (up to integration) found by Delaunay himself [2] are given below for a comparison with those derived here and elsewhere [4, 12], i.e.,

$$\tilde{x} = -\frac{a\alpha\tan\varphi}{\sqrt{1+\alpha-\sin^2\varphi}} + \int_0^{\varphi}\frac{a\alpha\mathrm{d}\varphi}{\cos^2\varphi\sqrt{1+\alpha-\sin^2\varphi}}$$

(23)

$$\tilde{z} = \frac{a\alpha}{\sqrt{1+\alpha-\sin^2\varphi}}, \qquad \varphi\in\mathbb{R}$$

parametrize the evolutes, and

$$x[\varphi] = a\sin\varphi - a\tan\varphi\sqrt{1+\alpha-\sin^2\varphi} + \int_0^{\varphi}\frac{a\alpha\mathrm{d}\varphi}{\cos^2\varphi\sqrt{1+\alpha-\sin^2\varphi}}$$

(24)

$$z[\varphi] = -a\cos\varphi + a\sqrt{1+\alpha-\sin^2\varphi}$$

do the same for the nodary and undulary. The parameters α and a have the same meaning as specified before.

Finally, it can be easily realized that the integrals which appear in Delaunay formulas exist only on restricted intervals on which the evolute can be found.

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ВЪРХУ ЕВОЛЮТИТЕ НА КРИВИТЕ ГЕНЕРИРАЩИ НОДОИДИТЕ И УНДУЛОИДИТЕ НА DELAUNAY

Мариана Хаджилазова, Ивайло М. Младенов

Следвайки оригиналната геометрична конструкция на Delaunay е намерена явна параметризация на еволютите на профилните криви, генериращи въведените от него повърхнини известни като нодоиди и ундулоиди. В аналитичен вид са намерени и параметризациите на генериращите криви на споменатите повърхнини.