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AN ELEMENTARY PROOF OF THE LOCALIZED SUBELLIPTIC ESTIMATES FOR NON-DEGENERATE PSEUDODIFFERENTIAL OPERATORS OF PRINCIPAL TYPE*

Petar Popivanov

It is given a short elementary proof of the localized subelliptic estimates for non-degenerate pseudodifferential operators of principal type.

1. Introduction. There is a long history of the subelliptic estimates stated for the first time by Hörmander in [2] and proved by Hörmander, Egorov and Trèves (see [6], [7], [5]). The aim of this short note is to propose an elementary proof of the subelliptic estimates in the case of non-degenerate pseudodifferential operators of principal type. We remind of the reader that a classical pseudodifferential operator $P \in L^m_{ce}\Omega$ with principal symbol $p^0_m(x,\ \xi)$ is called operator of principal type if $p^0_m(x,\ \xi) = 0$, $\xi \neq 0 \Rightarrow \bigtriangledown_\xi p^0_m(x,\ \xi) \neq 0$. Definitions. properties and detailed study of pseudodifferential operators can be found in [6], [7]. The operator $P(x,\ D)$ is nondegenerate and of principal type if $p^0_m(x,\ \xi) = 0$, $\xi \neq 0 \Rightarrow \bigtriangledown_{x,\xi}\Re p^0_m(x,\ \xi) \not\parallel \bigtriangledown_{x,\xi}\Im p^0_m(x,\ \xi)$. We shall say that the pseudodifferential operator $P(x,D) \in L^m_{ce}(\Omega)$ is subelliptic in Ω if for each compact $K \subset \Omega$ there exist constants $C(K)$, s and δ such that in the Sobolev spaces H_s the following inequality holds:

$$\|u\|_s \leq C_s(K)\left(\|Pu\|_{s-m+\delta} + \|u\|_{s-1}\right), \quad \forall C^\infty_0(K), \quad 0 \leq \delta < 1. \tag{1}$$

Due to Hörmander, Egorov and Trèves there exists a fill characterization of the scalar subelliptic operators (1) via the algebraic properties of their principal symbol $p^0_m(x,\ \xi)$. The necessary and sufficient condition for the fulfilment of (1) is proved in several steps and by simplifying microlocally the symbol p^0_m to the normal form $\xi_1 + i\xi_2 + iq(x,\ \xi)$, $q(x,\ t\xi) = tq(x,\ \xi)$, $\forall t \geq 1$ near to the point $\rho^0 = (x_0,\ \xi^0)$, $\xi^0 = (\xi^0_1, \ldots, \xi^0_n) \neq 0$, $\xi^0_1 = \xi^0_2 = q(x_0, \xi^0) = \bigtriangledown_{x,\xi} q(x_0, \xi^0) = 0$. Certainly, in this case $m = 1$, $s = 1 - \delta$.

The details of the above mentioned reduction could be found in Chapter VIII of [6]. The crucial result in proving of (1) for nondegenerate pseudodifferential operators of principal type can be formulated as follows:

Theorem 1 (Egorov, Th. 4.3 from Chapter VIII of [6])**.** *Let* $Q = \{(t,x) \in \mathbb{R}^2,\ |t| \leq 1, |x| \leq 1\}$. *Then the estimate*

$$\lambda\|u\|_0 \leq C\left\|u'_t(t,x) + D_x + \lambda^{k+1}B(t,x)\right\|_0, \tag{2}$$

$\forall u \in C^\infty_0(Q)$, $\forall\lambda$ *sufficiently large,* $C = \text{const} > 0$ *and with* $B(t,x)$ *– polynomial of order* k *having real coefficients is valid if an only if*

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(i) $\partial_t B(t,x) \le 0$

(ii) *there exists a constant* $c_0 > 0$ *and such that*

$$\sum_{i+j\le k-1} \left|\partial_t^{i+1}\partial_x^j B(t,x)\right| \ge c_0 \ in\ Q.$$

Certainly, $D_x = \frac{1}{i}\partial_x$ *in* (2).

The necessary part of the proof of Theorem 1 is more or less simpler. Because of this fact we shall concentrate on the proof of the sufficiency part of Theorem 1.

2. Proof of the sufficiency of Theorem 1. 1. We shall divide the proof into several propositions.

Proposition 1. *Let* $P = \frac{\partial}{\partial t} + D_x + \lambda^{k+1}B$ *satisfy the condition* **(i)**. *Then*

$$\|P^*u\| \le \|Pu\|, \quad \forall u \in C_0^\infty(Q), \tag{3}$$

where $P^* = -\partial_t + D_x + \lambda^{k+1}B$ *is the* $L_2(Q)$ *adjoint operator of* $P = \partial_t + D_x + \lambda^{k+1}B$.

Proof. We start from the identity

$$\|Pu\|_0^2 = \|P^*u\|_0^2 + \Re\left([C_1u,u],u\right), \quad \forall u \in C_0^\infty(Q)$$

and $C_1 = [P^*,P] = -2\lambda^{k+1}\partial_t B \ge 0$. Therefore, (3) holds.

Put $L_1 = \partial_t$, $L_2 = -\big(\partial_x + i\lambda^{k+1}B\big)$. Evidently, $P = L_1 + iL_2$. Consequently, the commutator $[L_1,L_2] = -i\lambda^{k+1}\frac{\partial B}{\partial t}$, $[L_1,[L_1,L_2]] = -i\lambda^{k+1}\frac{\partial^2 B}{\partial t^2}$, $[L_2,[L_1,L_2]] = +i\lambda^{k+1}\frac{\partial^2 B}{\partial t\partial x}$. Then we shall use induction. Let L_{i_Γ} stand for L_1 or L_2. Then the repeated commutator $[L_{i_1},\dots,L_{i_\Gamma-1},[L_1,L_2]] = \pm i\lambda^{k+1}\partial_t^{s'}\partial_x^{s''}B$, where s' is the number of such i'_j for which $i'_j = 1$ and s'' is the number of such i''_j, $i''_j = 2$; $s' \ge 1$, $s'+s'' = \Gamma \ge 1$. Having in mind that ord $Q = k$ we conclude that the vector fields L_1, L_2 form a nilpotent Lie algebra of rank k.

Proposition 2. *The estimate*

$$\lambda\|u\|_0 \le \|Pu\|_0, \quad \forall u \in C_0^\infty(Q) \tag{4}$$

and for each λ *– sufficiently large is valid under the conditions* **(i)**, **(ii)**.

Proposition 2 coincides with Egorov's Theorem 1.

Before proving (4) we shall make the following remark.

Remark 1. The estimate (4) can be localized in Q.

In fact, let V_j, $j = 1,\dots,s$, be a finite open covering of the closed rectangle Q and $\varphi_j \in C_0^\infty(V_j)$ form a partition of unity, i. e. $\sum_{j=1}^{s}\varphi_j^2 = 1$ near Q, $0 \le \varphi_j \le 1$. Assume that $\lambda\|\psi\|_0 \le \|P\psi\|_0$, $\forall\psi \in C_0^\infty(V_j)$, $\lambda \ge \lambda_0 \gg 1$, $1 \le j \le s$. Thus, $\forall u \in C_0^\infty(Q)$ we have that

$$\lambda\|\varphi_j u\|_0 \le \|P(\varphi_j u)\|_0 = \|uT_j\varphi_j + \varphi_j Pu\| \le \|\varphi_j Pu\| + \|u\|_0\|T_j\varphi_j\|_0,$$

where $T_j = \partial_t\varphi_j + D_x\varphi_j \in C_0^\infty(V_j)$.

The elementary inequality $(a+b)^2 \le 2(a^2+b^2)$, $a \ge 0$, $b \ge 0$, enables us to conclude that

$$(5) \qquad \lambda^2 \sum_{j=1}^{s} \|\varphi_j u\|_0^2 \le 2 \sum_{j=1}^{s} \|\varphi_j P u\|_0^2 + 2sc^2 \|u\|_0^2, \quad c = \max_j \|T_j \varphi_j\|_0,$$

i. e. $\lambda^2 \|u\|_0^2 \le 2\|Pu\|_0^2 + \mathrm{const}\, \|u\|_0^2$, $\forall u \in C_0^\infty(Q)$ and for λ sufficiently large.

2. Suppose that $(t_0, x_0) \in Q$. According to (ii) one can find an open neighbourhood $V_{(t_0,x_0)}$ of (t_0, x_0) and integers $i(t_0, x_0)$, $j(t_0, x_0)$ such that $\left|\partial_t^{i+1} \partial_x^j B\right| \ge c_0' > 0$ in $V_{(t_0,x_0)}$. As $V_{(t_0,x_0)}$ form an open covering of Q there exists a finite open covering $\{V_\tau\}_{\tau=1}^s$ of Q. Put $\varphi_\tau \in C_0^\infty(V_\tau)$ for the functions forming the corresponding partition of unity in Q. Having in mind that (4) is localizable in Q and $\left|\partial_t^{i(\tau)+1} \partial_x^{j(\tau)} B\right| \ge c_0' > 0$ in V_τ we can assume without loss of generality that $\left|\partial_t^{i+1} \partial_x^j B\right| \ge c_0' > 0$ everywhere in Q with some integers i, j, $0 \le i + j \le k - 1$. Put $M = \max\limits_{\substack{i,j \\ i+j \le k-1}} \sup\limits_Q \left|\partial_t^{i+1} \partial_x^j B\right|^{\frac{1}{i+j+1}} \ge \mathrm{const} > 0$.

Consider now the following Cauchy problem in $L_2(Q)$

$$(6) \qquad \left| \begin{array}{l} \partial_h z_j = L_j z_j, \quad j = 1, 2, \quad h \in \mathbb{R}^1 \\ z_j|_{h=0} = v(t, x) \in C_0^\infty(Q). \end{array} \right.$$

In the Appendix of the paper we shall show that (6) possesses a unique solution $z_j(h, t, x)$ written in the form $z_j = e^{hL_j} v$, i. e. $\partial_h \left(e^{hL_j} v\right) = L_j e^{hL_j} v$.

Moreover, $\|e^{hL_j}\| = 1$, $j = 1, 2$. We shall put everywhere hL_1, hL_2 instead of L_1, L_2 and we will obtain the following formula for the repeated commutators:

$$(7) \qquad [hL_{i_1}, \ldots, hL_{i_\Gamma - 1}, [hL_1, hL_2]] = \pm i \lambda^{k+1} h^{\Gamma+1} \partial_t^{s'} \partial_x^{s''} B(t, x),$$

$s' + s'' = \Gamma \ge 1$, $\Gamma \le k$. The famous Campbell-Hausdorff formula on nilpotent Lie groups of rank k (see [1], [3], [4]) gives us that

$$(8) \qquad e^{\left[L_{i_1}, \ldots, L_{i_\Gamma - 1}, [L_1, L_2]\right] h^{\Gamma+1}} = \prod e^{\pm h L_j}, \quad \Gamma \le k$$

and in the right-hand side of (8) are participating finitely many factors $e^{\pm hL_j}$, $j = 1, 2$ (may be huge amount of factors). Thus, (7) implies:

$$(9) \qquad e^{\pm i \lambda^{k+1} h^{\Gamma+1} \partial_t^{i+1} \partial_x^j B(t,x)} = \prod e^{\pm h L_j}, \quad i + j = \Gamma \ge 1.$$

Put now $h = \dfrac{1}{M} \lambda^{-\frac{k+1}{\Gamma+1}} \Rightarrow \dfrac{1}{h} = \lambda^{\frac{k+1}{\Gamma+1}} M$, $|h| \le \mathrm{const}$, as $\lambda \ge 1$, $\lambda^{k+1} h^{\Gamma+1} \partial_x^j \partial_t^{i+1} B = \dfrac{\partial_t^{i+1} \partial_x^j B}{M^{i+j+1}}$, i. e. $0 < c_0'' \le \dfrac{|\partial_t^{i+1} \partial_x^j B|}{M^{i+j+1}} \le 1$; $1 \le \Gamma \le k \Rightarrow \dfrac{k+1}{\Gamma+1} \ge 1$. As either $\partial_t^{i+1} \partial_x^j B > 0$ in Q or $\partial_t^{i+1} \partial_x^j B < 0$ in Q it is evident from geometrical reasons that there exists a constant $d_0 > 0$ and such that

$$(10) \qquad \left| e^{\pm i \lambda^{k+1} h^{\Gamma+1} \partial_t^{i+1} \partial_x^j B} - 1 \right| \ge d_0 > 0.$$

Remark 2. Let A_p, $p = 1, \ldots, s$, be bounded operators in L_2. Then

$$(11) \qquad \|A_1 \ldots A_s \psi - \psi\| \le C \sum_{p=1}^{s} \|A_p \psi - \psi\|, \quad \forall \psi \in L_2(Q).$$

The proof is obvious but we shall verify it only for $p = 3$ (induction could be used in the general case). In fact, $A_1 A_2 A_3 \psi - \psi = A_1 A_2 (A_3 \psi - \psi) + A_1 A_2 \psi - \psi = A_1 A_2 (A_3 \psi -$

$\psi) + A_1(A_2\psi - \psi) + A_1\psi - \psi$, etc.

Combining (8), (9), (10), (11) we get

$$\begin{aligned} d_0\|u\| &\leq \left\| \left(e^{[L_{i_1},\dots,L_{i_{\Gamma-1}},[L_1,L_2]]h^{\Gamma+1}} - 1 \right) u \right\| \leq \\ &\leq C \sum \left\| \left(e^{\pm hL_j} - 1 \right) u \right\|_0, \quad \forall u \in C_0^\infty(Q) \end{aligned} \tag{12}$$

and the sum in the right-hand side of (12) is finite.

Proposition 3([3], [4])**.** *One can find a constant $C > 0$ and such that*

$$\left\| \left(e^{hL_j} - 1 \right) \psi \right\|_0 \leq C|h| \|L_j \psi\|_0, \quad \forall \psi \in C_0^\infty(Q), \quad j = 1, 2. \tag{13}$$

We shall prove in the Appendix that e^{hL_j} is unitarian operator in L_2, i. e. $\|e^{hL_j}\| = 1$. Consider now $0 \leq \Gamma_j(h) = \left\| \left(e^{hL_j} - 1 \right) \psi \right\|_0^2$, $\Gamma_j(0) = 0$. Then

$$\begin{aligned} \Gamma_j'(h) &= \left(L_j e^{hL_j}\psi,\ e^{hL_j}\psi - \psi \right) + \left(e^{hL_j}\psi - \psi,\ L_j e^{hL_j}\psi \right) \\ &\Rightarrow |\Gamma_j'(h)| \leq 2\| \left(e^{hL_j} - 1 \right) \psi\|_0 \|L_j\psi\|_0 \\ &= 2\sqrt{\Gamma_j(h)} \|L_j\psi\|_0 \Rightarrow \sqrt{\Gamma_j(h)} \leq |h| \|L_j\psi\|_0, \text{ etc.} \end{aligned}$$

According to the definition of h, (12), (13) and with some constant $C > 0$, $\|u\| \leq C|h|(\|L_1 u\| + \|L_2 u\|)$, $\forall u \in C_0^\infty(Q)$, i. e.

$$\|u\| \leq \text{const}\, \lambda^{-\frac{k+1}{\Gamma+1}} \left(\|L_1 u\|_0 + \|L_2 u\|_0 \right), \quad \forall u \in C_0^\infty(Q). \tag{15}$$

The simple observation $P + P^* = 2iL_2$, $P - P^* = 2\partial_t$, Proposition 1 and (15) complete the proof of Proposition 2. Thus Egorov's Theorem 1 is proved.

Appendix.

1. Consider in $L_2(Q)$ the Cauchy problem

$$\left| \begin{array}{l} \partial_h v = L_1 v = \partial_t v \\ v|_{h=0} = \psi, \quad \psi = \psi(t,x) \in C_0^\infty(Q). \end{array} \right. \tag{16}$$

Evidently, $v(h,t,x) = \psi(t+h,x) = e^{hL_1}\psi(t,x) \Rightarrow$ for each $h \in \mathbb{R}^1$ fixed, $\|e^{hL_1}\psi\|_0 = \|\psi\|_0$, i. e. $\|e^{hL_1}\| = 1$.

2. Consider now in $L_2(Q)$

$$\left| \begin{array}{l} \partial_h v = L_2 v = \partial_x v + i\lambda^{k+1} B v \\ v|_{h=0} = \psi(t,x) \in C_0^\infty(Q). \end{array} \right. \tag{17}$$

The standard change $\left| \begin{array}{l} p = h + x \\ q = x \end{array} \right.$ in (17) transforms (17) into

$$\left| \begin{array}{l} \dfrac{\partial v}{\partial q} + i\lambda^{k+1} B(t,q) v = 0 \\ v|_{q=p} = \psi(t,p), \end{array} \right. \tag{18}$$

i. e. $v = e^{-i\lambda^{k+1}\int_p^q B(t,r)dr} \cdot A(t,p) \Rightarrow A(t,p) = \psi(t,p)$ and therefore $v(h,t,x) = e^{hL_2}\psi(t,x) = \psi(t,x+h) e^{-i\lambda^{k+1}\int_{h+x}^x B(t,r)dr} \Rightarrow \|e^{hL_2}\psi\|_0 = \|\psi\|_0$ as B is real valued polynomial.

REFERENCES

[1] J. P. Serre. Lie Algebras and Lie groups, Lectures given at Harvard Univ., New York – Amsterdam, Benjamin, 1965.

[2] L. Hörmander. Pseudodifferential operators and non-elliptic boundary value problems. *Ann. of Math.*, **83** (1966), 129–209.

[3] L. Hörmander. Hypoelliptic second order differential operators. *Acta Math.*, **119**, (1967), 147–171.

[4] Yu. V. Egorov. Nondegenerate pseudodifferential operators of principal type. *Math. Sbornik*, **82**, (1970), 323–342 (in Russian).

[5] F. Trèves. A new method of proof of the subelliptic estimates. *Comm. Pure Appl. Math.*, **24**, (1971), 71–115.

[6] L. Hörmander. The Analysis of Linear Partial Differential Operators, vol. I–IV. Springer Verlag, 1983–1985.

[7] Yu. V. Egorov. Linear differential operators of principal type. Nauka, Moscow, 1984 (in Russian).

Petar Popivanov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
1113 Sofia, Bulgaria
e-mail: popivano@math.bas.bg

ЕЛЕМЕНТАРНО ДОКАЗАТЕЛСТВО НА ЛОКАЛИЗИРАНИТЕ СУБЕЛИПТИЧНИ ОЦЕНКИ ЗА НЕИЗРОДЕНИ ПСЕВДОДИФЕРЕНЦИАЛНИ ОПЕРАТОРИ ОТ ГЛАВЕН ТИП

Петър Попиванов

Предложено е кратко елементарно доказателство на локализирана субелиптична оценка за неизродени оператори от главен тип.