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NILPOTENCY IN INVOLUTION MATRIX ALGEBRAS OVER ALGEBRAS WITH INVOLUTION*

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Given a ring with involution $(A,*)$, we may consider two involutions $\sharp$ and $\flat$ on the 2×2 matrix ring $M_2(A,*)$. In the paper we find necessary and sufficient conditions for the index of nilpotency of the symmetric and skew-symmetric due to the two involutions elements of $(M_2(A,*),\sharp)$ and $(M_2(A,*),\flat)$ in the cases when A is a finite dimensional Grassmann algebra (without 1) of dimension 3 and of dimension 7. For A being the concrete four-dimensional algebra with almost polynomial growth of the $*$-codimensions described by Mishchenko and Valenti we find necessary and sufficient conditions for the nilpotency of index 2 of the symmetric and skew-symmetric variables due to the two considered involutions.

1. Preliminaries. We consider the 2×2 matrix algebra $M_2(G_k)$ over the finite dimensional Grassmann algebra G_k for $k=2$ and $k=3$. The notations are the following:

Let G denote the infinite dimensional Grassmann algebra, namely

$$G = G(V) = K\langle v_1, v_2, \dots \mid v_iv_j + v_jv_i = 0 \; i,j=1,2,\dots\rangle.$$

The field K is of characteristic zero. The algebra G' (without 1) has a basis $v_{i_1}v_{i_2}\dots v_{i_k}$, where $1\le i_1<i_2<\dots<i_k$. The elements v_i are called generators of G' while the elements $v_{i_1}v_{i_2}\cdots v_{i_k}$ for $1\le i_1<i_2<\dots<i_k$ are called basic monomials of G'. For $G=K+G'$ a generator is 1 as well. The algebras G and G' are PI-equivalent (they satisfy the same polynomial identities). It is easy to be seen that $G'=J(G)$, where $J(G)$ is the Jacobson radical of the algebra.

The algebra G is in the mainstream of resent research in PI-theory. Its importance is connected with the structure theory for the T-ideals of identities of associative algebras developed by Kemer. In [3, Theorem 1.2] he proved that any T-prime T-ideal can be obtained as the T-ideal of identities of one of the following algebras: $M_n(K)$, $M_n(G)$ and $M_{n,u}(G)$, the latter being the algebra of $n\times n$ supermatrices over $G=G_0\oplus G_1$ with G_0-blocks (with entries of even degree) of sizes $u\times u$ and $(n-u)\times(n-u)$ and with G_1-blocks (with entries of odd degree) of sizes $u\times(n-u)$ and $(n-u)\times u$.

The Grassmann algebra is one of the fundamental structures in PI-theory since it also generates a minimal variety of exponential growth [4].

Well known facts concerning the algebra G are the following:

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Proposition 1 [4, Corollary, p. 437]. *The T-ideal $T(G)$ is generated by the identity* $[x_1, x_2, x_3] = 0$.

Some identities for $M_2(G)$ were found in [8, 6].

Proposition 2 [2, Exercise 5.3]. *For $G_k = G(V_k)$ over a k-dimensional vector space V_k all identities follow from the identity $[x_1, x_2, x_3] = 0$ and the standard identity*

$$S_{2p}(x_1, \dots, x_{2p}) = \sum_{\sigma \in \mathrm{Sym}(2p)} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(2p)} = 0,$$

where p is the minimal integer such that $2p > k$.

As the algebra G'_n is nilpotent of index $n+1$, the same is true for the matrix algebra $M_k(G'_n)$. Thus its elements are nilpotent of index $s \le n+1$.

We are interested in those elements having index of nilpotency $s < n+1$. It means that $s = 2$ for the elements of $M_2(G'_2)$ and s is either 2 or 3 for the elements of $M_2(G'_3)$.

Equipping the Grassmann algebra G_k for $k = 2, 3$ with involution $*$ we define two involutions on the matrix algebras $M_2(G'_k)$ and define the index s for the symmetric and skew-symmetric due to the considered involutions elements of the corresponding matrix algebras.

2. Nilpotency of the $\sharp$- and $\flat$-symmetric and skew-symmetric elements of $(M_2(G'_2, *), \sharp)$ and $(M_2(G'_2, *), \flat)$, respectively. In [1] two involutions $*$ are considered on the infinite dimensional Grassmann algebra G: the identical on the generators of G one which we shall denote $* = \mathrm{id}$ and the involution $* = \phi_{12}$ acting on the generators by $\phi_{12}(e_{2k-1}) = e_{2k}$ and $\phi_{12}(e_{2k}) = e_{2k-1}$ for $k = 1, 2, \dots$. When we work on G_2 over the vector space V_2 the involution ϕ_{12} means that $\phi_{12}(e_1) = e_2$ and $\phi_{12}(e_2) = e_1$. Then $\phi_{12}(e_1e_2) = \phi_{12}(e_2)\phi_{12}(e_1) = e_1e_2$.

For $a_i \in (G_2, * = \phi_{12})$, $i = 1, \dots, 4$, we define [7]

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^\sharp = \begin{pmatrix} a_1^* & a_3^* \\ a_2^* & a_4^* \end{pmatrix}. \tag{1}$$

Let $a_i = \alpha_i e_1 + \beta_i e_2 + \gamma_i e_1 e_2$ for $i = 1, \dots, 4$. Thus a matrix $M_s \in (M_2(G'_2, \phi_{12}), \sharp)$ is $\sharp$-symmetric for

$$\alpha_1 = \beta_1, \quad \alpha_2 = \beta_3, \quad \alpha_3 = \beta_2, \quad \alpha_4 = \beta_4, \quad \text{i.e.}$$

$$M_s = \begin{pmatrix} \alpha_1(e_1 + e_2) + \gamma_1 e_1 e_2 & \alpha_2 e_1 + \alpha_3 e_2 + \gamma_2 e_1 e_2 \\ \alpha_3 e_1 + \alpha_2 e_2 + \gamma_2 e_1 e_2 & \alpha_4(e_1 + e_2) + \gamma_4 e_1 e_2 \end{pmatrix}.$$

Nilpotency of index $s = 2$ of M_s leads to two cases for the coefficients, namely:

$$\begin{aligned} &1.\ \alpha_2 = \alpha_3 \\ &2.\ \alpha_2 = -\alpha_3, \quad \alpha_1 = \alpha_4. \end{aligned} \tag{2}$$

Thus we get:

Theorem 1. *There are two types of $\sharp$-symmetric matrices from $(M_2(G'_2, \phi_{12}), \sharp)$, which are nilpotent of index $s = 2$, namely*

$$M_{s1} = \begin{pmatrix} \alpha_1(e_1 + e_2) + \gamma_1 e_1 e_2 & \alpha_2(e_1 + e_2) + \gamma_2 e_1 e_2 \\ \alpha_2(e_1 + e_2) + \gamma_2 e_1 e_2 & \alpha_4(e_1 + e_2) + \gamma_4 e_1 e_2 \end{pmatrix}$$

and

$$M_{s2} = \begin{pmatrix} \alpha_1(e_1 + e_2) + \gamma_1 e_1 e_2 & \alpha_2(e_1 - e_2) + \gamma_2 e_1 e_2 \\ -\alpha_2(e_1 - e_2) + \gamma_2 e_1 e_2 & \alpha_1(e_1 + e_2) + \gamma_4 e_1 e_2 \end{pmatrix}.$$

Corrolary 1. *The subsets* $W_1 = \{M_{s1}\}$, $W_2 = \{M_{s2}\}$ *of* $(M_2(G'_2, \phi_{12}), \sharp)$ *are vector spaces.*

Proof. For any two matrices $A_1, B_1 \in W_1$ and $A_2, B_2 \in W_2$ we have $A_1B_1 = 0$ and $A_2B_2 + B_2A_2 = 0$. Thus $(\alpha A_i + \beta B_i)^2 = 0$ for $i = 1, 2$ and $\alpha, \beta \in K$.

Now we consider the skew-symmetric case. A skew-symmetric matrix has the form

$$M_{ss} = \begin{pmatrix} \alpha_1(e_1 - e_2) & \alpha_2 e_1 - \alpha_3 e_2 + \gamma_2 e_1 e_2 \\ \alpha_3 e_1 - \alpha_2 e_2 - \gamma_2 e_1 e_2 & \alpha_4(e_1 - e_2) \end{pmatrix}.$$

Nilpotency of index $s = 2$ gives the same conditions (2) on the coefficients of the entries of M_{ss}. Thus we get:

Theorem 2. *There are two types of* $\sharp$*-skew-symmetric matrices from* $(M_2(G'_2, \phi_{12}), \sharp)$, *which are nilpotent of index* $s = 2$, *namely*

$$M_{ss1} = \begin{pmatrix} \alpha_1(e_1 - e_2) & \alpha_2(e_1 - e_2) + \gamma_2 e_1 e_2 \\ \alpha_2(e_1 - e_2) - \gamma_2 e_1 e_2 & \alpha_4(e_1 - e_2) \end{pmatrix}$$

and

$$M_{ss2} = \begin{pmatrix} \alpha_1(e_1 - e_2) & \alpha_2(e_1 + e_2) + \gamma_2 e_1 e_2 \\ -\alpha_2(e_1 + e_2) - \gamma_2 e_1 e_2 & \alpha_1(e_1 - e_2) \end{pmatrix}.$$

Corrolary 2. *The subsets* $W_3 = \{M_{ss1}\}$, $W_4 = \{M_{ss2}\}$ *of* $(M_2(G'_2, \phi_{12}), \sharp)$ *are vector spaces.*

Now we consider the $\flat$-involution [7] defined by

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^{\flat} = \begin{pmatrix} a_4^* & a_2^* \\ a_3^* & a_1^* \end{pmatrix}. \tag{3}$$

Analogously as above we get:

Theorem 3. *There are two classes of* $\flat$*-symmetric matrices from* $(M_2(G'_2, \phi_{12}), \flat)$, *which are nilpotent of index* $s = 2$, *namely*

$$M_{s1} = \begin{pmatrix} \alpha_1(e_1 + e_2) + \gamma_1 e_1 e_2 & \alpha_2(e_1 + e_2) + \gamma_2 e_1 e_2 \\ \alpha_3(e_1 + e_2) + \gamma_3 e_1 e_2 & \alpha_1(e_1 + e_2) + \gamma_1 e_1 e_2 \end{pmatrix}$$

and

$$M_{s2} = \begin{pmatrix} \alpha_1 e_1 + \beta_1 e_2 + \gamma_1 e_1 e_2 & \gamma_2 e_1 e_2 \\ \gamma_3 e_1 e_2 & \beta_1 e_1 + \alpha_1 e_2 + \gamma_1 e_1 e_2 \end{pmatrix}.$$

Corrolary 3. *The subsets* $W_5 = \{M_{s1}\}$, $W_6 = \{M_{s2}\}$ *of* $(M_2(G'_2, \phi_{12}), \flat)$ *are vector spaces.*

Proof. For any two matrices A and B from W_5 (or from W_6) we have $AB + BA = 0$.

Theorem 4. *There are two classes of* $\flat$*-skew-symmetric matrices from* $(M_2(G'_2, \phi_{12}), \flat)$, *which are nilpotent of index* $s = 2$, *namely*

$$M_{ss1} = \begin{pmatrix} \alpha_1(e_1 - e_2) + \gamma_1 e_1 e_2 & \alpha_2(e_1 - e_2) \\ \alpha_3(e_1 - e_2) & \alpha_1(e_1 - e_2) - \gamma_1 e_1 e_2 \end{pmatrix}$$

and

$$M_{ss2} = \begin{pmatrix} \alpha_1 e_1 + \beta_1 e_2 + \gamma_1 e_1 e_2 & 0 \\ 0 & -\beta_1 e_1 - \alpha_1 e_2 - \gamma_1 e_1 e_2 \end{pmatrix}.$$

Corrolary 4. *The subsets* $W_7 = \{M_{ss1}\}$ *and* $W_8 = \{M_{ss2}\}$ *of* $(M_2(G_2', \phi_{12}), \flat)$ *are vector spaces.*

Remark 1. Analogues of Theorems 1 – 4 can be formulated in the case of $* = \mathrm{id}$.

3. Nilpotency of the $\sharp$- and $\flat$-symmetric and skew-symmetric elements of $(M_2(G_3', *), \sharp)$ and $(M_2(G_3', *), \flat)$. Now the involution on G_3 could be only $* = \mathrm{id}$. Thus the $\sharp$-involution, defined by (1), leads to the following conditions:

Let a matrix from $M_2(G_3', \mathrm{id})$ have the form $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, where

$$\begin{aligned} a_1 &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_1 e_2 + \alpha_4 e_3 + \alpha_5 e_1 e_3 + \alpha_6 e_2 e_3 + \alpha_7 e_1 e_2 e_3, \\ a_2 &= \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_1 e_2 + \beta_4 e_3 + \beta_5 e_1 e_3 + \beta_6 e_2 e_3 + \beta_7 e_1 e_2 e_3, \\ a_3 &= \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_1 e_2 + \gamma_4 e_3 + \gamma_5 e_1 e_3 + \gamma_6 e_2 e_3 + \gamma_7 e_1 e_2 e_3, \\ a_4 &= \delta_1 e_1 + \delta_2 e_2 + \delta_3 e_1 e_2 + \delta_4 e_3 + \delta_5 e_1 e_3 + \delta_6 e_2 e_3 + \delta_7 e_1 e_2 e_3. \end{aligned}$$

Then

$$\begin{aligned} a_1 * &= \alpha_1 e_1 + \alpha_2 e_2 - \alpha_3 e_1 e_2 + \alpha_4 e_3 - \alpha_5 e_1 e_3 - \alpha_6 e_2 e_3 - \alpha_7 e_1 e_2 e_3, \\ a_2 * &= \beta_1 e_1 + \beta_2 e_2 - \beta_3 e_1 e_2 + \beta_4 e_3 - \beta_5 e_1 e_3 - \beta_6 e_2 e_3 - \beta_7 e_1 e_2 e_3, \\ a_3 * &= \gamma_1 e_1 + \gamma_2 e_2 - \gamma_3 e_1 e_2 + \gamma_4 e_3 - \gamma_5 e_1 e_3 - \gamma_6 e_2 e_3 - \gamma_7 e_1 e_2 e_3, \\ a_4 * &= \delta_1 e_1 + \delta_2 e_2 - \delta_3 e_1 e_2 + \delta_4 e_3 - \delta_5 e_1 e_3 - \delta_6 e_2 e_3 - \delta_7 e_1 e_2 e_3. \end{aligned}$$

For a symmetric matrix M_s from $(M_2(G_3', \mathrm{id}), \sharp)$ we have

$$\alpha_3 = \alpha_5 = \alpha_6 = \alpha_7 = 0,$$

$$\gamma_1 = \beta_1, \quad \gamma_2 = \beta_2, \quad \gamma_3 = -\beta_3, \quad \gamma_4 = \beta_4, \quad \gamma_5 = -\beta_5, \quad \gamma_6 = -\beta_6, \quad \gamma_7 = -\beta_7,$$

$$\delta_3 = \delta_5 = \delta_6 = \delta_7 = 0, \quad \text{i.e.}$$

$$\begin{aligned} a_1 &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_4 e_3, \\ a_2 &= \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_1 e_2 + \beta_4 e_3 + \beta_5 e_1 e_3 + \beta_6 e_2 e_3 + \beta_7 e_1 e_2 e_3, \\ a_3 &= \beta_1 e_1 + \beta_2 e_2 - \beta_3 e_1 e_2 + \beta_4 e_3 - \beta_5 e_1 e_3 - \beta_6 e_2 e_3 - \beta_7 e_1 e_2 e_3, \\ a_4 &= \delta_1 e_1 + \delta_2 e_2 + \delta_4 e_3. \end{aligned}$$

Due to the remarks at the end of Section 1 we have to consider two cases:

Nilpotency of index $s = 2$: For $M_s^2 = (a_{ij})$ we have

$$\begin{aligned} a_{11} &= a_{22} = 0, \\ a_{12} &= (\alpha_1\beta_2 - \alpha_2\beta_1 + \beta_1\delta_2 - \beta_2\delta_1) e_1 e_2 \\ &\quad + (\alpha_1\beta_4 - \alpha_4\beta_1 + \beta_1\delta_4 - \beta_4\delta_1) e_1 e_3 \\ &\quad + (\alpha_2\beta_4 - \alpha_4\beta_2 + \beta_2\delta_4 - \beta_4\delta_2) e_2 e_3 \\ &\quad + (\alpha_1\beta_6 - \alpha_2\beta_5 + \alpha_4\beta_3 + \beta_3\delta_4 - \beta_5\delta_2 + \beta_6\delta_1) e_1 e_2 e_3, \\ a_{21} &= -a_{12}, \quad \text{i.e.} \end{aligned}$$

Theorem 5. *A symmetric matrix* $M_s = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ *from* $(M_2(G_3', \mathrm{id}), \sharp)$, *where*

$$\begin{array}{lrcl} & a_1 & = & \alpha_1 e_1 + \alpha_2 e_2 + \alpha_4 e_3, \\ (4) & a_2 & = & \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_1 e_2 + \beta_4 e_3 + \beta_5 e_1 e_3 + \beta_6 e_2 e_3 + \beta_7 e_1 e_2 e_3, \\ & a_3 & = & \beta_1 e_1 + \beta_2 e_2 - \beta_3 e_1 e_2 + \beta_4 e_3 - \beta_5 e_1 e_3 - \beta_6 e_2 e_3 - \beta_7 e_1 e_2 e_3, \\ & a_4 & = & \delta_1 e_1 + \delta_2 e_2 + \delta_4 e_3 \end{array}$$

is nilpotent of index $s = 2$ *iff*

$$\begin{aligned} (\alpha_1 - \delta_1)\beta_2 - (\alpha_2 - \delta_2)\beta_1 &= 0, \\ (\alpha_1 - \delta_1)\beta_4 - (\alpha_4 - \delta_4)\beta_1 &= 0, \\ (\alpha_2 - \delta_2)\beta_4 - (\alpha_4 - \delta_4)\beta_2 &= 0, \\ \alpha_1\beta_6 - \alpha_2\beta_5 + \alpha_4\beta_3 + \beta_3\delta_4 - \beta_5\delta_2 + \beta_6\delta_1 &= 0. \end{aligned}$$

By analogous considerations we get:

Theorem 6. *A skew-symmetric matrix* $M_s = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ *from* $(M_2(G_3', \mathrm{id}), \sharp)$, *where*

$$\begin{array}{lrcl} & a_1 & = & \alpha_3 e_1 e_2 + \alpha_5 e_1 e_3 + \alpha_6 e_2 e_3 + \alpha_7 e_1 e_2 e_3, \\ (5) & a_2 & = & \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_1 e_2 + \beta_4 e_3 + \beta_5 e_1 e_3 + \beta_6 e_2 e_3 + \beta_7 e_1 e_2 e_3, \\ & a_3 & = & -\beta_1 e_1 - \beta_2 e_2 + \beta_3 e_1 e_2 - \beta_4 e_3 + \beta_5 e_1 e_3 + \beta_6 e_2 e_3 + \beta_7 e_1 e_2 e_3, \\ & a_4 & = & \delta_3 e_1 e_2 + \delta_5 e_1 e_3 + \delta_6 e_2 e_3 + \delta_7 e_1 e_2 e_3 \end{array}$$

is nilpotent of index $s = 2$ *iff*

$$(\alpha_3 + \delta_3)\beta_4 - (\alpha_5 + \delta_5)\beta_2 + (\alpha_6 + \delta_6)\beta_1 = 0.$$

Nilpotency of index $s = 3$**:**

Theorem 7. *A* $\sharp$*-symmetric matrix with entries* (4) *is nilpotent of index* $s = 3$ *iff at least one of the elements* $(\alpha_1 - \delta_1)\beta_2 - (\alpha_2 - \delta_2)\beta_1$, $(\alpha_1 - \delta_1)\beta_4 - (\alpha_4 - \delta_4)\beta_1$ *and* $(\alpha_2 - \delta_2)\beta_4 - (\alpha_4 - \delta_4)\beta_2$ *of the field* K *is different from* 0 *and*

$$\begin{aligned} (\alpha_1\beta_2 - \alpha_2\beta_1 + \beta_1\delta_2 - \beta_2\delta_1)\beta_4 &- (\alpha_1\beta_4 - \alpha_4\beta_1 + \beta_1\delta_4 - \beta_4\delta_1)\beta_2 \\ &+ (\alpha_2\beta_4 - \alpha_4\beta_2 + \beta_2\delta_4 - \beta_4\delta_2)\beta_1 = 0, \\ (\alpha_1\beta_2 - \alpha_2\beta_1 + \beta_1\delta_2 - \beta_2\delta_1)\delta_4 &- (\alpha_1\beta_4 - \alpha_4\beta_1 + \beta_1\delta_4 - \beta_4\delta_1)\delta_2 \\ &+ (\alpha_2\beta_4 - \alpha_4\beta_2 + \beta_2\delta_4 - \beta_4\delta_2)\delta_1 = 0, \\ (\alpha_1\beta_2 - \alpha_2\beta_1 + \beta_1\delta_2 - \beta_2\delta_1)\alpha_4 &- (\alpha_1\beta_4 - \alpha_4\beta_1 + \beta_1\delta_4 - \beta_4\delta_1)\alpha_2 \\ &+ (\alpha_2\beta_4 - \alpha_4\beta_2 + \beta_2\delta_4 - \beta_4\delta_2)\alpha_1 = 0. \end{aligned}$$

Theorem 8. *A* $\sharp$*-skew-symmetric matrix with entries* (5) *is nilpotent of index* 3 *iff*

$$\alpha_3\beta_4 - \alpha_5\beta_2 + \alpha_6\beta_1 + \beta_1\delta_6 - \beta_2\delta_5 + \beta_4\delta_3 \neq 0.$$

Remark 2. The case of the $\flat$-involution is treated analogously.

4. Nilpotency of index 2 of the ♯- and ♭-symmetric and skew-symmetric variables of a matrix algebra over a special finite dimensional algebra M. Mishchenko and Valenti [5] construct a finite dimensional K-algebra M with involution $*$ and study its $*$-polynomial identities. They show that the $*$-variety generated by M, denoted by $\mathrm{var}(M,*)$, has almost polynomial growth. It means that the sequence of $*$-codimensions of M (i.e. the dimension of the space of multilinear polynomials in $1,2,\dots$ $*$-variables in the corresponding relatively free algebra with involution) cannot be bounded by any polynomial function but any proper $*$-subvariety of $\mathrm{var}(M,*)$ has polynomial growth.

We recall the construction of M, given in [5]:

Let e_{ij} denote the usual matrix units and define the subalgebra

$$A = K(e_{11}+e_{33}) \oplus Ke_{12} \oplus Ke_{13} \oplus Ke_{22} \oplus Ke_{23}$$

of the 3×3 upper triangular matrix algebra over K. The algebra A has an involution $*$ obtained by reflecting the matrices along the second diagonal, i.e.

$$\begin{pmatrix} u & r & t \\ 0 & v & s \\ 0 & 0 & u \end{pmatrix}^* = \begin{pmatrix} u & s & t \\ 0 & v & r \\ 0 & 0 & u \end{pmatrix}.$$

Notice that $I = Ke_{13}$ is a two-sided $*$-invariant ideal of A. Then $M = A/I$ is a four-dimensional algebra with induced involution that we shall denote $*$. If

$$e_{11}+e_{33}+I=a, \quad e_{22}+I=b,$$
$$e_{12}+I=c, \quad e_{23}+I=c^*,$$

then $M = \mathrm{Span}_K\{a,b,c,c^*\}$ with multiplication table

$$\begin{array}{llll} a\cdot a=a, & a\cdot b=0, & a\cdot c=c, & a\cdot c^*=0, \\ b\cdot a=0, & b\cdot b=b, & b\cdot c=0, & b\cdot c^*=c^*, \\ c\cdot a=0, & c\cdot b=c, & c\cdot c=0, & c\cdot c^*=0, \\ c^*\cdot a=c^*, & c^*\cdot b=0, & c^*\cdot c=0, & c^*\cdot c^*=0. \end{array} \tag{6}$$

Now we consider the 2×2 matrix algebra over M with the ♯- and the ♭-involution, defined in (1) and (3), respectively. Let

$$A = \begin{pmatrix} \alpha_1 a+\beta_1 b+\gamma_1 c+\delta_1 c^* & \alpha_2 a+\beta_2 b+\gamma_2 c+\delta_2 c^* \\ \alpha_3 a+\beta_3 b+\gamma_3 c+\delta_3 c^* & \alpha_4 a+\beta_4 b+\gamma_4 c+\delta_4 c^* \end{pmatrix}$$

be a matrix with entries from M.

A. We start with the $*=\sharp$-involution. Due to (1) and (6) we have

$$A^\sharp = \begin{pmatrix} \alpha_1 a+\beta_1 b+\delta_1 c+\gamma_1 c^* & \alpha_3 a+\beta_3 b+\delta_3 c+\gamma_3 c^* \\ \alpha_2 a+\beta_2 b+\delta_2 c+\gamma_2 c^* & \alpha_4 a+\beta_4 b+\delta_4 c+\gamma_4 c^* \end{pmatrix}.$$

The symmetry of the matrix A leads to the conditions

$$\alpha_2=\alpha_3, \quad \beta_2=\beta_3, \quad \gamma_1=\delta_1, \quad \gamma_2=\delta_3, \quad \gamma_3=\delta_2, \quad \gamma_4=\delta_4.$$

The index of nilpotency $s=2$ for the matrix

$$A_s = \begin{pmatrix} \alpha_1 a+\beta_1 b+\gamma_1(c+c^*) & \alpha_2 a+\beta_2 b+\gamma_2 c+\delta_2 c^* \\ \alpha_2 a+\beta_2 b+\delta_2 c+\gamma_2 c^* & \alpha_4 a+\beta_4 b+\gamma_4(c+c^*) \end{pmatrix} \tag{7}$$

gives

$$
\begin{aligned}
&(\alpha_1^2+\alpha_2^2)a+(\beta_1^2+\beta_2^2)b+(\alpha_1\gamma_1+\alpha_2\delta_2+\beta_1\gamma_1+\beta_2\gamma_2)(c+c^*)=0,\\
&\alpha_2(\alpha_1+\alpha_4)a+\beta_2(\beta_1+\beta_4)b+(\alpha_1\gamma_2+\alpha_2\gamma_4+\beta_2\gamma_1+\beta_4\gamma_2)c\\
&+(\alpha_2\gamma_1+\alpha_4\delta_2+\beta_1\delta_2+\beta_2\gamma_4)c^*=0,\\
&\alpha_2(\alpha_1+\alpha_4)a+\beta_2(\beta_1+\beta_4)b+(\alpha_2\gamma_1+\alpha_4\delta_2+\beta_1\delta_2+\beta_2\gamma_4)c\\
&+(\alpha_1\gamma_2+\alpha_2\gamma_4+\beta_2\gamma_1+\beta_4\gamma_2)c^*=0,\\
&(\alpha_2^2+\alpha_4^2)a+(\beta_2^2+\beta_4^2)b+(\alpha_2\gamma_2+\alpha_4\gamma_4+\beta_2\delta_2+\beta_4\gamma_4)(c+c^*)=0, \quad \text{i.e.}
\end{aligned}
$$

Theorem 9. *A $\sharp$-symmetric matrix* (7) *is nilpotent of index $s=2$ iff*

$$
\begin{aligned}
\alpha_1^2+\alpha_2^2=0, \quad \beta_1^2+\beta_2^2 &= 0,\\
\alpha_1\gamma_1+\alpha_2\delta_2+\beta_1\gamma_1+\beta_2\gamma_2 &= 0,\\
\alpha_2(\alpha_1+\alpha_4)=0, \quad \beta_2(\beta_1+\beta_4) &= 0,\\
\alpha_1\gamma_2+\alpha_2\gamma_4+\beta_2\gamma_1+\beta_4\gamma_2 &= 0,\\
\alpha_2\gamma_1+\alpha_4\delta_2+\beta_1\delta_2+\beta_2\gamma_4 &= 0,\\
\alpha_2^2+\alpha_4^2=0, \quad \beta_2^2+\beta_4^2 &= 0,\\
\alpha_2\gamma_2+\alpha_4\gamma_4+\beta_2\delta_2+\beta_4\gamma_4 &= 0.
\end{aligned}
$$

The skew-symmetry case gives:

Theorem 10. *A $\sharp$-skew-symmetric matrix*

$$
A_{ss}=\begin{pmatrix} \gamma_1(c-c^*) & \alpha_2 a+\beta_2 b+\gamma_2 c+\delta_2 c^* \\ -\alpha_2 a-\beta_2 b-\delta_2 c-\gamma_2 c^* & \gamma_4(c-c^*) \end{pmatrix}
$$

is nilpotent of index $s=2$ iff $\alpha_2=\beta_2=0$.

B. Now we consider the $\flat$-involution.

Theorem 11. *A $\flat$-symmetric matrix*

$$
B_s=\begin{pmatrix} \alpha_1 a+\beta_1 b+\gamma_1 c+\delta_1 c^* & \alpha_2 a+\beta_2 b+\gamma_2(c+c^*) \\ \alpha_3 a+\beta_3 b+\gamma_3(c+c^*) & \alpha_1 a+\beta_1 b+\delta_1 c+\gamma_1 c^* \end{pmatrix}
$$

is nilpotent of index $s=2$ iff

$$
\begin{aligned}
\alpha_1^2+\alpha_2\alpha_3=0, \quad \beta_1^2+\beta_2\beta_3 &= 0,\\
\alpha_1\gamma_1+\alpha_2\gamma_3+\beta_1\gamma_1+\beta_3\gamma_2 &= 0,\\
\alpha_1\delta_1+\alpha_3\gamma_2+\beta_1\delta_1+\beta_2\gamma_3 &= 0,\\
\alpha_1\alpha_2=0, \quad \beta_1\beta_2 &= 0,\\
\alpha_1\gamma_2+\alpha_2\delta_1+\beta_1\gamma_2+\beta_2\gamma_1 &= 0,\\
\alpha_1\alpha_3=0, \quad \beta_1\beta_3 &= 0,\\
\alpha_1\gamma_3+\alpha_3\gamma_1+\beta_1\gamma_3+\beta_3\delta_1 &= 0.
\end{aligned}
$$

Theorem 12. *A $\flat$-skew-symmetric matrix*

$$
B_{ss}=\begin{pmatrix} \alpha_1 a+\beta_1 b+\gamma_1 c+\delta_1 c^* & \gamma_2(c-c^*) \\ \gamma_3(c-c^*) & -\alpha_1 a-\beta_1 b-\delta_1 c-\gamma_1 c^* \end{pmatrix}
$$

is nilpotent of index $s = 2$ *iff* $\alpha_1 = \beta_1 = 0$.

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НИЛПОТЕНТНОСТ В МАТРИЧНИ АЛГЕБРИ С ИНВОЛЮЦИЯ НАД АЛГЕБРИ С ИНВОЛЮЦИЯ

Цецка Рашкова

За пръстен с инволюция $(A, *)$ се разглеждат две инволюции $\sharp$ и $\flat$ в матричния пръстен на квадратните матрици от втори ред $M_2(A, *)$. В статията са намерени необходими и достатъчни условия за класа на нилпотентност на симетричните и анти-симетричните относно всяка от двете инволюции елементи в $(M_2(A, *), \sharp)$ и $(M_2(A, *), \flat)$ в случая, когато A е крайномерна Грасманова алгебра (без 1) от размерност 3 и от размерност 7. Когато A е конкретната четиримерна алгебра с инволюция с почти полиномен ръст на $*$-коразмерностите, описана от Мищенко и Валенти, са намерени необходими и достатъчни условия за нилпотентност от клас 2 на симетричните и анти-симетричните елементи относно двете разглеждани инволюции.