МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2009
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2009
Proceedings of the Thirty Eighth Spring Conference of
the Union of Bulgarian Mathematicians
Borovetz, April 1–5, 2009

A COMPATIBLE METRIC FOR COMPUTING THE DIMENSION DIAMETERS OF SUBSETS OF ESSENTIAL SYSTEMS

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Let X be a normal space. In this note we propose a construction of a pseudometric r in X which allows an easy calculation of the n-dimensional diameters of some subsets of X. Some consequences of this result are discussed.

1. Basic concepts and definitions. 1.1. Let X be a normal topological space and consider a system

$$\mathcal{E} = \{(A_1, B_1); (A_2, B_2); \dots ; (A_n, B_n)\}$$

consisting of n disjoint pairs of closed subsets of X. X is a normal space an hence for every $i = 1, 2, \dots , n$ one can construct a continuous function $f_i : X \to [0, 1]$ such that $f_i(A_i) = 0$ and $f_i(B_i) = 1$. The system $\mathcal{E}$ generates a pseudometric p by natural way:

$$p(x, y) = \max_{1\le i\le n} |f_i(x) - f_i(y)|.$$

Obviously the topology in X is compatible with the pseudometric p in a sense that every "open ball" respective to p is an open subset of X. Note furthermore, that if $(X, \widetilde{\varrho})$ is metrizable with a metric $\widetilde{\varrho}$ in it then

$$\varrho^*(x, y) = \widetilde{\varrho}(x, y) + \max_{1\le i\le n} |f_i(x) - f_i(y)|$$

is an equivalent metric in X.

Indeed, let us denote by $O^{\varrho^*}_\varepsilon(x)$ and by $O^{\widetilde{\varrho}}_\varepsilon(x)$ the open ε-balls respective the metrics ϱ^* and $\widetilde{\varrho}$ respectively. Clearly, $O^{\widetilde{\varrho}}_\varepsilon(x) \supset O^{\varrho^*}_\varepsilon(x)$ for every x because $\varrho^* \ge \widetilde{\varrho}$. Conversely let $\varepsilon > 0$ be a given positive number and consider the open ball $O^{\widetilde{\varrho}}_\varepsilon(x)$. The function f_i is continuous at x and hence for every i there exists $\delta_i > 0$ such that $|f(y) - f(x)| < \dfrac{\varepsilon}{2}$ whenever $\widetilde{\varrho}(x, y) < \delta_i$. Now put $\delta = \min\left\{\dfrac{\varepsilon}{2};\ \min\limits_{1\le i\le n} \delta_i\right\} > 0$. It is easy to see that if $\widetilde{\varrho}(x, y) < \delta$ then one has $\varrho^*(x, y) < \varepsilon$. In the sequel in case when X is metrizable we shall suppose that it is equipped with the metric ϱ^* and that, in addition $\varrho^*(x, y) \le 1$ – otherwise one may replace ϱ^* with the equivalent metric $\varrho(x, y) = \min\{1, \varrho^*(x, y)\}$.

1.2. Further, we recall some general facts in dimension theory. For an arbitrary subset Y of X by *diameter* $d^p(Y)$ we mean as usual the number $d^p(Y) = \sup\{p(x, y)|x \in Y, y \in Y\}$. Now, let $\mathcal{U}$ be an open covering of a pseudometric space X. By $\mathrm{mesh}^p\mathcal{U}$ we mean the number $\sup\{d^p(U)|\, U \in \mathcal{U}\}$.

If $Y \subset X$ then the n-dimensional diameter $d_n^p(Y)$ of the subset Y is the number $\inf\{\mathrm{mesh}^p(\mathcal{U}_Y)\}$, where $\mathcal{U}$ runs the set of all open coverings of X with $\mathrm{ord}(\mathcal{U}_Y) \leq n+1$, where $\mathcal{U}_Y = \{U \cap Y | U \in \mathcal{U}\}$.

Note that some authors refer to d_n as d_{n+1} ([2], [1]). In these papers d_n is called an *n-dimensional degree*. Here we follow the terminology which is adopted in [3], so we shall call d_n an *n-dimensional diameter*.

1.3. Definition [4]**.** By a *metric dimension* $\mu^p - \dim(Y)$ of a subspace Y of X we mean the least of the integers n such that for every $\varepsilon > 0$ there is an open locally finite covering $\mathcal{U}$ of Y for which $\mathrm{ord}(\mathcal{U}) \leq n+1$ and $\mathrm{mesh}(\mathcal{U}) < \varepsilon$.

For a metric space (X, ϱ), the inequality $d_n(X) > 0$ means that the metric dimension $\mu - \dim X$ of X is not less than $n+1$ [6]. Clearly, for compact metric spaces, $d_n(X) = 0$ if and only if $\dim X \leq n$.

1.4. Definition [3]**.** Let n be an integer and let ε be a positive number. The space X is referred to as $(n, \varepsilon)-$connected between the closed sets P and Q if for an arbitrary partition C between P and Q in X we have $d_{n-2}^p(C) > \varepsilon$.

Consider again the system $\mathcal{E} = \{(A_1, B_1); (A_2, B_2); \ldots; (A_n, B_n)\}$ consisting of n disjoint pairs of closed subsets of X.

1.5. Definition. The system $\mathcal{E}$ is *essential* (or *$n-$defining* [6]), if for any closed sets P_i, $i = 1, \ldots, n$, separating A_i and B_i, the intersection $\bigcap\limits_{i=1}^{n} P_i$ is nonempty.

Obviously the essential system is an analogue of the system of opposite faces of the n-dimensional cube I^n; $I = [0, 1]$. Because of that we shall call A_i and B_i *faces* of $\mathcal{E}$.

1.6. Definition. The subset $M \subset X$ *does not cut* X between the sets $P \subset X$ and $Q \subset X$ if one can find a connected closed set $K \subset X$ for which $K \subset X \setminus M$ and $K \cap P \neq \emptyset \neq K \cap Q$.

2. n-defining systems. Everywhere below we consider a pseudometric which is generated by an n-defining system.

In the sequel we use the following Lemma [5].

2.1. Lemma. Let $\mathcal{E}$ be an essential system in the normal space X and $\mathcal{U}$ be a locally finite open covering of X with $\mathrm{ord}\,\mathcal{U} \leq n$. Then some element of $\mathcal{U}$ intersects two opposite faces of $\mathcal{E}$.

Lemma 2.2. Suppose that L cuts the normal space X between the disjoint closed nonempty sets P and Q. Then every open neighborhood O of L contains a (closed in X) partition C between P and Q.

Proof. First, suppose that O contains P or Q (or both of them). Let, for example, $O \supset P$. Then $O^* = O \setminus Q$ is an open set and $O^* \supset P$ because $P \cap Q = \emptyset$. Keeping in mind that X is a normal space, we obtain an open set $U \subset O^*$, whose closure $\overline{U}$ is contained in O^* and $P \subset U$. Evidently, the boundary $C = \partial U$ of U is a partition between P and Q, and $C \subset O$.

Therefore, the essential part of the proof appears in the case when $P^* = P \setminus O$ and $Q^* = Q \setminus O$ are nonempty (and obviously closed) subsets of $Y = X \setminus O$. Note that Y is a closed subset of X, and $Y \cap L = \emptyset$. By the condition of the Lemma, it follows that Y is not connected between P^* and Q^*. That is to say, there exist two disjoint and open in Y sets U^* and V^* for which $U^* \supset P^*$, $V^* \supset Q^*$ and $Y = U^* \cup V^*$.

The last equality implies that U^* and V^* are also closed in Y, and hence, in X. Further, let us put $A = P \cup U^*$ and $B = Q \cup V^*$. Apparently, A and B are closed subsets of X and $A \supset P$, $B \supset Q$.

To finish the proof, we shall show that $A \cap B = \emptyset$:

$$A \cap B = (P \cup U^*) \cap (Q \cup V^*) = (P \cap Q) \cup (P \cap V^*) \cup (U^* \cap Q) \cup (U^* \cap V^*) = \emptyset.$$

To prove this, note that clearly $P \cap Q = \emptyset$, $U^* \cap V^* = \emptyset$ and for the remaining parts of the union, we have

$$P = (P \setminus O) \cup (P \cap O) = P^* \cup (P \cap O) \subset U^* \cup O$$

since $P^* \subset U^*$ and $P \cap O \subset O$. Hence,

$$P \cap V^* \subset (U^* \cup O) \cap V^* = (U^* \cap V^*) \cup (O \cap V^*) = \emptyset.$$

Proceeding in the same way, we obtain that $U^* \cap Q = \emptyset$. Thus, we have constructed closed sets $A \supset P$ and $B \supset Q$ with $A \cap B = \emptyset$.

By condition X was normal, so we may choose disjoint open in X neighborhoods $U \supset A$ of A and $V \supset B$ of B. To finish the proof, let us put $C = X \setminus (U \cup V)$. Now, from $U \supset A \supset U^*$ and $V \supset B \supset V^*$ it follows that

$$C = X \setminus (U \cup V) \subset X \setminus (U^* \cup V^*) = X \setminus Y = X \setminus (X \setminus O) = O.$$

As an immediate consequence of the above lemmas one can obtain the following result for normal spaces:

2.3. Theorem. Suppose that X is a normal space and let $\dim X \geq n$. Then for every normally placed subset $M \subset X$ with $d^p_{n-2}(M) < 1$ we have $d^p_1(X \setminus M) > 0$. That means that in some sense M "does not cut" X between every pair (A, B) of opposite faces of $\mathcal{E}$.

Proof. Note first that if A is a face of the system $\mathcal{E}$ then $d^p_{n-2}A = 1$. Now, let us suppose that M cuts X between A and B, and consider an open covering $\mathcal{U} = \{U_1, \ldots, U_p\}$ of M with $\operatorname{ord}(\mathcal{U}) \leq n-1$ and $\operatorname{mesh}(\mathcal{U}) < 1$. Next, let $O = \cup U_i$ be the body $|\mathcal{U}|$ of $\mathcal{U}$. Then $A \setminus O \neq \emptyset \neq B \setminus O$ because $d_{n-2}(A) = d_{n-2}(B) = 1$ and by construction $d_{n-2}(O) < 1$. Next, we apply Lemma 2.2 to obtain that O contains a partition C between A and B. It is easy to see that if C is a partition between two faces of $\mathcal{E}$ then $d^p_{n-1}C = 1$, so we have obtained a contradiction.

The above theorem shows that the complement $X \setminus M$ of a low dimensional normally placed subset M of a normal space X with $\dim X \geq n$ is "connected" in the sense that the dimensional diameter of $X \setminus M$ is greater than zero. Note that X can be even totally disconnected (even one can say that $\mu^p - \dim(X \setminus M) \geq 1$).

3. The space X is compact. Let X be a compact T_2 space with n-defining system $\mathcal{E}$ in it. Then the following theorem holds:

Theorem 3.1. Let X be a compact space with n-defining system $\mathcal{E}$ and let as usual p be the pseudometric, generated by $\mathcal{E}$. Next suppose that $\{F_i\}_{i=1}^{\infty}$ is a countable system of closed subsets in X such that the inequality

$$d^p_{n-2}\left(\bigcup_{i \neq j}(F_i \cap F_j)\right) < 1$$

holds. Then $X \neq \sum\limits_{i=1}^{\infty} F_i$.

Proof. Suppose the contrary and denote by M the sum $M = \bigcup\limits_{i \neq j}(F_i \cap F_j)$. Note that M is a F_σ subset of X and hence it is normally placed in X. Now we can apply Theorem 2.3 to obtain an open set O containing M an such that $d^p_{n-2}(O) < 1$. Then O

does not contain a partition between every pair of opposite faces of $\mathcal{E}$. In other words, $Y = X \setminus O$ does not separate for example the faces A_1 and B_1. Therefore the space Y is a compactum which is connected between A_1 and B_2 and $Y = \bigcup_{i=1}^{\infty} Y_i$ where $Y_i = F_i \cap Y$. Obviously $Y_i \cap Y_j = \emptyset$ if $i \neq j$ which contradicts the well known Sierpinski theorem [7].

Various results from [9], [8] and [6] can be directly obtained as corollories of the above theorem.

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ПОЛЕЗНА МЕТРИКА ЗА ПРЕСМЯТАНЕТО НА РАЗМЕРНОСТНИ ДИАМЕТРИ НА ПОДМНОЖЕСТВА НА СЪЩЕСТВЕНИ СИСТЕМИ

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Нека X е нормално пространство. В тези бележки предлагаме конструкция на псевдометрика r в X, която позволява лесно пресмятане на размерностните диаметри на някои подмножества на X. Дискутират се също така някои следствия от предложената техника.