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RIEMANNIAN MANIFOLDS WITH IDEMPOTENT JACOBI OPERATORS*

Vesselin Videv, Zhivko Zhelev

We survey and analyze the Riemannian manifolds with idempotent Jacobi operators in addition to the research in [1], [2], [3], [9] and [10], and we also supply the necessary algebraic motivation for such a survey. Similar problems in the Pseudo-Riemannian case remain open and we believe in the existence of lots of fecund results there in contrast to the sterility in the Riemannian case.

First, we shall present some elementary algebraic observations. Let $\mathcal{A}$ be a ring. Then the following is a well-known definition:

Definition 1. *An element $a \in \mathcal{A}$ is called an idempotent if $a^2 = a$.*

It is obvious then $a^n = a$, $n \in \mathbb{Z}$, $n \geq 2$ as well. $0_{\mathcal{A}}$ and $\mathrm{Id}_{\mathcal{A}}$ are trivial idempotents. But in general there exist idempotent elements which are not trivial and in this case $\mathcal{A}$ is a ring with zero divisors. Indeed, if $a \neq 0_{\mathcal{A}}, \mathrm{Id}_{\mathcal{A}}$, it follows that $\mathrm{Id}_{\mathcal{A}} - a \neq 0_{\mathcal{A}}$ and $a(\mathrm{Id}_{\mathcal{A}} - a) = a - a^2 = a - a = 0_{\mathcal{A}}$. Now let $\mathcal{A}$ be a ring, X be an arbitrary set and $\mathfrak{R} = \mathfrak{F}(X, \mathcal{A})$ be the set of all maps from X to $\mathcal{A}$. We define $(\mathfrak{f} + \mathfrak{g})(x) = \mathfrak{f}(x) + \mathfrak{g}(x)$ and $\mathfrak{f}\mathfrak{g}(x) = \mathfrak{f}(x) \cdot \mathfrak{g}(x)$. Then $(\mathfrak{R}, +, \cdot)$ is a ring. If $X = \emptyset$ or $\mathcal{A}$ is trivial, $\mathfrak{R}$ is also trivial (in both cases there exists only one map $X \longrightarrow \mathcal{A}$). The following facts follow immediately:

Proposition 1. *Let $X \neq \emptyset$. Then $\mathfrak{R}$ is commutative iff $\mathcal{A}$ is commutative.*

Proposition 2. *Let $\mathcal{A}$ be a non-trivial ring and X contains at least two elements. Then $\mathfrak{R}$ contains non-trivial idempotents.*

Proof. Indeed, let $S \subset X$, $S \neq X$, $S \neq \emptyset$ and $\chi\colon S \longrightarrow \mathcal{A}$ such that
$\chi\colon = \begin{cases} \chi(x) = 1_{\mathcal{A}}, & \text{if} \quad x \in S \\ \chi(x) = 0_{\mathcal{A}}, & \text{if} \quad x \notin S. \end{cases}$ Then $\chi \neq 1_{\mathcal{A}}$, $0_{\mathcal{A}}$ and $\chi^2 = \chi$. □

Now let (M, g) be an n-dimensional Riemannian manifold with a metric tensor g. Let also $\mathcal{F}(M)$ be the associative algebra of all smooth functions on M and $\mathcal{X}(M)$ be the $\mathcal{F}(M)$-module of all smooth vector fields over M. In particular, $\mathcal{F}(M)$ is a ring and we can consider the case $\mathcal{A} = X = \mathcal{F}(M)$ and $\mathfrak{R} = \mathcal{X}(M)$ since if $X \in \mathcal{X}(M)$ is a smooth vector field, there is a map $X\colon \mathcal{F}(M) \longrightarrow \mathcal{F}M$ $(f(p) \mapsto Xf(p),\ p \in M)$ such that $X(\alpha f + \beta g)(p) = \alpha X f(p) + \beta X f(p)$, $\alpha, \beta \in \mathbb{R}$; $f, g \in \mathcal{F}(M)$ and $X(fg)(p) = Xf(g) \cdot g(p) + f(p) \cdot Xg(p)$. Now in this algebraic setting by Prop.2 we can deduce that $\mathcal{X}(M)$ contains non-trivial idempotents since the ring of all differentiable functions is a non-trivial ring.

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Let ∇ be the Levi-Chivita connection and let $\mathcal{R}(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ be the associated *skew-symmetric* curvature operator. The *Jacobi operator* is a self-adjoint endomorphism defined by:

$$\mathcal{J}(X)\colon U \longrightarrow \mathcal{R}(U,X)X, \tag{1}$$

where $U \in X^{\perp}$ [5], [7], [8], [14]. We assume that (1) is such a non-trivial idempotent, i.e.

$$\mathcal{J}(X) \circ \mathcal{J}(X) = \mathcal{J}(X). \tag{2}$$

Definition 2. *An algebraic curvature tensor $\mathcal{R}$ is called Osserman if the eigenvalues of the Jacobi operator do not depend on the choice of the unit tangent vector $X \in M_p$. A Riemannian manifold (M,g) is called a pointwise Osserman if its curvature tensor is Osserman at any point $p \in M$. If in addition, the eigenvalues of $\mathcal{J}(X)$ are constant on M, the manifold is called a globally Osserman manifold* [5], [7], [11], [14].

Let (M,g) be locally isotropic, i.e. for each point $p \in M$ and $X,Y \in M_p$ with $g(X,X) = g(Y,Y)$, there is a local isometry of (M,g) of a neighborhood of p which fixes p and exchanges X and Y. Then it is a two-point homogenous space, i. e. the group of local isometries acts transitively on the unit sphere bundle of (M,g) and hence (M,g) is Osserman. The lack of other examples led Osserman to conjecture that the converse might also be true [14] which was proved by Chi when $n \neq 4m$, $m \geq 1$ [5] and by Nikolayevsky when $n \neq 16$ [11], [12], [13]. Following these results one can formulate

Theorem 1. *Let V be a n-dimensional vector space endowed with an algebraic curvature tensor $\mathcal{R}^0\colon V \times V \times V \longrightarrow V$ by $\mathcal{R}^0(X,Y)Z = g(Y,Z)X - g(X,Z)Y$* [7, p. 6]*. Then:*

†) *If n is odd, we have a curvature tensor $\mathcal{R} = \kappa\mathcal{R}^0$, where κ is a constant;*

‡) *If n is even and $n \neq 16$ there exists an almost complex structure J on V such that $J^2 = -\operatorname{Id}_V$ and $g(X,Y) = g(JX,JY)$ or equivalently $g(JX,Y) + g(X,JY) = 0$ for any $X,Y \in V$. In this case we have that $\mathcal{R} = \lambda\mathcal{R}^0 + \dfrac{1}{3}(\lambda-\mu)\mathcal{R}^J$, $\lambda,\mu \in \mathbb{R}$, where $\mathcal{R}^J(X,Y,Z) := g(Y,JZ)JX - g(X,JZ)JY - 2g(X,JY)JZ$.*

From this result immediately follows

Theorem 2. *Riemannian manifolds (M,g) of signature $(0,n)$, $n \neq 16$, are global Osserman manifolds iff (M,g) are two-point homogenous spaces with a curvature tensor of one of the forms (†) or (‡).*

Let (M,g) be a n-dimensional Riemannian manifold with a curvature tensor of the form (†). Then (M,g) is a Riemannian manifold of a constant sectional curvature. Let (M,g,J) be a Riemannian manifold of signature $(0,2m)$, $m \geq 1$, endowed with an almost complex structure J, i. e. there exists an endomorphism $J\colon M_p \longrightarrow M_p$ such that $J^2 = -\operatorname{Id}(M_p)$ and $g(X,Y) = g(JX,JY)$, $X,Y \in M_p$, $p \in M$. The triple (M,g,J) is called *an almost Hermitian* or AH manifold. If additionally $\mathcal{R}(X,Y,Z,U) = \mathcal{R}(JX,JY,JZ,JU)$, $X,Y,Z \in M_p$ and $p \in M$, (M,g,J) is called AH_3-manifold. If (M,g,J) is an AH_3-manifold with a curvature tensor of the form (‡), then (M,g,J) is called an AH_3-manifold of a pointwise sectional curvature $\mu = \mu(p)$ and of a pointwise skew-holomorphic sectional curvature $\lambda = \lambda(p)$ at any point $p \in M$ [16].

Our main goal is to investigate these classes of Riemannian manifolds which satisfy (2) for any unit tangent vector $X \in M_p$ at any point $p \in M$.

Example 1. Let (M,g) be a Riemannian manifold of signature $(0,n)$ and of a constant sectional curvature of the form (†). Then $\mathcal{J}(X)$ has the representation

$$\mathcal{J}(X) = \kappa(U - g(U,X)X). \tag{3}$$

Using the properties of $\mathcal{R}$ and the definition of $\mathcal{J}(X)$ we get that

$$\begin{aligned}\mathcal{J}(X)\circ\mathcal{J}(X) &= \mathcal{R}(\mathcal{J}(X)(U),X,X) = \mathcal{R}(\kappa((U-g((U,X)X,X,X))) = \\ &\kappa\mathcal{R}(U,X,X) - g(U,X)\mathcal{R}(X,X,X) = \kappa\mathcal{R}(U,X,X) = \kappa\mathcal{J}(X)(U).\end{aligned}$$

If $\kappa = 1$ in (3) the relation above is equivalent to (2). Hence, if (M,g) is an n-dimensional Riemannian manifold of constant sectional curvature $\kappa = 1$, any Jacobi operator is an idempotent operator for any unit tangent vector $X \in M_p$ at any point $p \in M$. In this case we have that

$$\mathcal{R}(X,Y,Z) = g(Y,Z)X - g(X,Z)Y = \mathcal{R}^0(X,Y,Z) \text{ or shortly } \mathcal{R} = \mathcal{R}^0. \tag{4}$$

Example 2. Let (M,g) be an AH_3-manifold of a pointwise holomorphic sectional curvature $\mu = \mu(p)$ and of a pointwise skew-holomorphic sectional curvature $\lambda = \lambda(p)$, and with a curvature tensor of the form (‡). Then for the Jacobi operator we have the representation

$$\mathcal{J}(X)(U) = \lambda(U - g(U,X)X) + (\mu-\lambda)g(U,JX)JX. \tag{5}$$

From here we get that

$$\mathcal{J}(JX)\circ\mathcal{J}(X) = \lambda(JX - g(JX,X)X) + (\mu-\lambda)g(JX,JX)JX = \lambda JX + (\mu-\lambda)JX = \mu JX,$$

$$\text{or } \ \mathcal{J}(JX)\circ\mathcal{J}(X) = \mu JX, \tag{6}$$

using the properties of the almost complex structure J, more precisely, the fact that $g(JX,X) = 0$, $g(JX,JX) = 1$, $X \in M_p$, $p \in M$. Now we have

$$\begin{aligned}&\mathcal{J}(X)\circ\mathcal{J}(X)(U) = \mathcal{R}(\mathcal{J}(X)(U),X,X) = \\ &\mathcal{R}(\lambda((U-g(U,X)X)) + (\mu-\lambda)g((U,JX)JX,X,X) = \\ &\lambda\mathcal{R}(U,X,X) + (\mu-\lambda)g(U,JX)\mathcal{R}(JX,X,X) = \\ &\lambda\mathcal{J}(X)(U) + (\mu-\lambda)g(U,JX)\cdot\mathcal{J}(X)(JX).\end{aligned} \tag{7}$$

From (6) and (7) we derive

$$\mathcal{J}(X)\circ\mathcal{J}(X)(U) = \lambda\mathcal{J}(X)(U) + (\mu-\lambda)\mu g(U,JX)JX. \tag{8}$$

Assuming that (2) holds we get from (8) that

$$(*) \quad (1-\lambda)\mathcal{J}(X)(U) + (\lambda-\mu)\mu g(U,JX)JX = 0.$$

Let us suppose $\mathcal{J}(X)(U)$ and JX be linearly independent for at least one U (X is fixed). Then from $(*)$ we arrive at the possibilities $\lambda = 1$ and $\mu = 1$ or $\lambda = 1$ and $\mu = 0$. In the first case the curvature tensor coincides with (4), and (M,g,J) is a space of constant sectional curvature $\kappa = 1$. In the second case we have that

$$\begin{aligned}&\mathcal{R}(X,Y,Z) = \\ &g(Y,Z)X - g(X,Z)Y + \frac{1}{3}(g(Y,JZ)JX - g(X,JZ)JY - 2g(X,JY)JZ) = \\ &\left(\mathcal{R}^0 + \frac{1}{3}\mathcal{R}^{\mathcal{J}}\right)(X,Y,Z) \text{ or shortly } \mathcal{R} = \mathcal{R}^0 + \frac{1}{3}\mathcal{R}^{\mathcal{J}}.\end{aligned} \tag{9}$$

It is easy to compute using (5) that in the case (9) any Jacobi operator $\mathcal{J}(X)$ has eigenvalues $0,0,1,\dots,1$ which correspond to the eigenvectors $X, JX, Y_1, Y_2, \dots, Y_{n-2}$, $(Y_1, Y_2, \dots, Y_{n-2} \perp X, JX)$, $X \in M_p$, $p \in M$.

Let now $\mathcal{J}(X)(U)$ depend on JX for any tangent vector U. Then $\mathcal{J}(X)(U) = cJX$, $c \in \mathbb{R}$, $c \neq 0$. Now if $Y \perp X, JX$ we get that $g(\mathcal{J}(X)(U), Y) = g(\mathcal{R}(U, X, X), Y) = 0$ and from the properties of the curvature tensor we have $g(\mathcal{R}(U, X, X), Y) = g(\mathcal{R}(Y, X, X), U) = 0$, $U \in M_p$. And since g is a Riemannian metric [15] it follows that

$$\mathcal{R}(Y, X, X) = \mathcal{J}(X)(Y) = 0. \tag{10}$$

That means that for any unit tangent vector X, any unit tangent vector $Y \perp X, JX$ is an eigenvector of the Jacobi operator with a corresponding eigenvalue 0. Hence, $\mathcal{J}(X)$ has eigenvectors $JX, X, Y_1, Y_2, \ldots, Y_{n-2}$, $(Y_1, Y_2, \ldots, Y_{n-2} \perp X, JX)$, with corresponding eigenvalues $\mu, 0, 0, \ldots, 0$. But from (5) it follows that

$$\mathcal{J}(X)(Y) = \lambda Y, \quad Y \perp X, JX;\ Y \in M_p,\ p \in M \tag{11}$$

for every unit tangent vector $Y \perp X, JX$, $Y \in M_p$ and $p \in M$. Now from (10) and (11) we get that $\lambda = 0$ and in this case by (‡), we obtain that

$$\begin{aligned} &\mathcal{R}(X, Y, Z) = \frac{1}{3}\mu(2g(X, JY)JZ + g(X, JZ)JY - g(Y, JZ)JX) = \frac{\mu}{3}\mathcal{R}^{\mathcal{J}}(X, Y, Z) \\ &\text{or shortly } \ \mathcal{R} = \frac{\mu}{3}\mathcal{R}^{\mathcal{J}}. \end{aligned} \tag{12}$$

From here we get that

$$\mathcal{J}(X)(U) = \mu g(U, JX)JX, \tag{13}$$

and consequently

$$\begin{gathered} \mathcal{J}(X) \circ \mathcal{J}(X)(U) = \mathcal{R}(\mathcal{J}(X)(U), X, X) = \\ \mathcal{R}(\mu g(U, JX)JX, X, X) = \mu g(U, JX)\mathcal{R}(JX, X, X) = \mu g(U, JX)\mathcal{J}(X)(JX). \end{gathered} \tag{14}$$

Now using (6) and (14) we get that $\mathcal{J}(X) \circ \mathcal{J}(X)(U) = \mu^2 g(U, JX)JX$ and from our assumption (2) and (13) we arrive at $\mu g(U, JX)JX = \mu^2 g(U, JX)JX$. From here we have either the case $\mu = 0$ (but since $\lambda = 0$ it follows that $R \equiv 0$), or $g(U, JX) = \mu g(U, JX)$ and putting $U = JX$ we get that $\mu = 1$. Thus, $\mu = 1$ and $\lambda = 0$ and we have that

$$\begin{aligned} &\mathcal{R}(X, Y, Z) = \frac{1}{3}(2g(X, JY)JZ + g(X, JZ)JY - g(Y, JZ)JX) = \frac{1}{3}\mathcal{R}^{\mathcal{J}}(X, Y, Z) \\ &\text{or } \mathcal{R} = \frac{1}{3}\mathcal{R}^{\mathcal{J}}. \end{aligned} \tag{15}$$

and $\mathcal{J}(X)(U) = g(U, JX)JX$ respectively. In the latter case $\mathcal{J}(X)$ has eigenvectors JX, X, Y_1, Y_2, $\ldots$, Y_{n-2}, $(Y_1, Y_2, \ldots, Y_{n-2} \perp X, JX)$ with corresponding eigenvalues $1, 0, 0, \ldots, 0$.

Now we shall briefly show that only Riemannian manifolds with curvature tensors of the forms (4), (9) and (15) have idempotent Jacobi operators $\mathcal{J}(X)$ for any unit tangent vector $X \in M_p$, $p \in M$. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of M_p. We put $X := e_1$ and let the Jacobi operators $\mathcal{J}_{e_1} \circ \mathcal{J}_{e_1}$ and $\mathcal{J}_{e_1}$ correspond to the matrix A and B respectively. For the diagonal elements we get:

$$\begin{aligned} a_{11} &= K_{12}^2 + R_{2112}^2 + \cdots + R_{211n}^2 \\ a_{22} &= K_{13}^2 + R_{3112}^2 + \cdots + R_{311n}^2 \\ a_{33} &= K_{14}^2 + R_{4112}^2 + \cdots + R_{411n}^2, \\ &\vdots \\ a_{nn} &= K_{1n}^2 + R_{n112}^2 + \cdots + R_{(n-1)11n}^2 \end{aligned} \tag{16}$$

and

$$b_{11} = K_{12},\ b_{22} = K_{13}, \ldots, b_{nn} = K_{1n}, \tag{17}$$

where R_{ijks} and K_{pr} are the curvature tensor and the sectional curvature tensor components respectively [15]. Let us suppose now that $\{e_1, e_2, \ldots, e_n\}$ be the eigenvectors of the Jacobi operator $\mathcal{J}_{e_1}$ with corresponding eigenvalues $\{0, c_2, \ldots, c_n\}$. Then the following relations hold:

$$\begin{gathered} R_{i11j} = 0, \quad i \neq j,\ i, j = 2, 3, \ldots, n; \\ c_k = K_{1k}, \quad k = 2, 3, \ldots, n. \end{gathered} \tag{18}$$

Now because of (2) we have that $\mathcal{J}_{e_1} \circ \mathcal{J}_{e_1} = \mathcal{J}_{e_1}$. That means that $a_{ii} = b_{ii}$, $i = 1, 2, \ldots, n$ and from (16)–(18) we get that $c_k(c_k - 1) = 0, \quad k = 2, 3, \ldots, n$. Since $X = e_1$ can be arbitrary chosen, the last is true for any Jacobi operator $\mathcal{J}(X)$ for any unit tangent vector $X \in M_p$ at any point $p \in M$. Thus, from $\mathcal{J}(X)$ being an idempotent follows that all eigenvalues of the Jacobi operator are equal to 0 or 1 and, hence, they are global constants on (M, g). Therefore (M, g) is a global Osserman manifold.

Finally we can formulate the main result:

Theorem 3. *Let (M, g) be a non-flat Riemannian manifold of signature $(0, n)$, $n \neq 16$. Then any Jacobi operator $\mathcal{J}(X)$ is idempotent for any unit tangent vector $X \in M_p$ at any point $p \in M$ iff one of the following is true:*

- (i) *(M, g) is an n-dimensional Riemannian manifold of constant sectional curvature 1 and curvature tensor $\mathcal{R} = \mathcal{R}^0$ or;*
- (ii) *There exists an almost complex structure J on M such that (M, g, J) is a $2n$-dimensional AH_3-manifold of a pointwise holomorphic sectional curvature $\mu = 0$ and of a pointwise skew-holomorphic sectional curvature $\lambda = 1$, and curvature tensor $\mathcal{R} = \mathcal{R}^0 + \dfrac{1}{3}\mathcal{R}^{\mathcal{J}}$ or;*
- (iii) *There exists an almost complex structure J on M such that (M, g, J) i an $2n$-dimensional AH_3-manifold of a pointwise holomorphic sectional curvature $\mu = 1$ and of a pointwise skew-holomorphic sectional curvature $\lambda = 0$, and curvature tensor $\mathcal{R} = \dfrac{1}{3}\mathcal{R}^{\mathcal{J}}$.*

Corollary. *Let (M, g, J) be an AH-manifold, $\dim M \neq 16$ and let also $\mathcal{R}^0 \cdot \mathcal{R}^{\mathcal{J}} = -\mathcal{R}^{\mathcal{J}} \circ \mathcal{R}^0$ at any point $p \in M$. Then $\mathcal{J}(X)$ is an idempotent operator at any point $p \in M$ for any tangent vector $X \in M_p$ if and only if (i) or (iii) of Theorem 3 is true.*

All geometric results above allow a subtle algebraic interpretation. Let V be a finite dimensional vector space, $V_1, V_2 \subset V$ and $f \colon V \longrightarrow V$ be a linear operator. Then it is a well-known algebraic fact that $f^2 = f$ if and only if $f(V) = V_1$ and $V = V_1 \oplus V_2 = \operatorname{Im} f \oplus \ker f$. Then f is called a *projector* of the vector space V onto the linear space V_1 over V_2. Now we can formulate the following algebraic

Theorem 4. *Let $\mathfrak{M} := (V, \langle\cdot,\cdot\rangle, A)$ be a 0-model, i. e. $\langle\cdot,\cdot\rangle$ is a non-degenerate inner product of signature (p, q) on a finite dimensional vector space V of dimension $m = p + q$ and $A \in \otimes^4 V^*$ is an algebraic curvature tensor, i.e. A is a 4-tensor which satisfies the symmetries of the Riemann curvature tensor:*

$$A(v_1, v_2, v_3, v_4) = A(v_3, v_4, v_1, v_2) = -A(v_2, v_1, v_3, v_4),$$
$$A(v_1, v_2, v_3, v_4) + A(v_2, v_3, v_1, v_4) + A(v_3, v_1, v_2, v_4) = 0\,.$$

The associated curvature operator $\mathcal{A}$ and Jacobi operator $\mathcal{J}$ are then characterized, respectively, by the identities [9]:

$$\langle \mathcal{A}(v_1, v_2)v_3, v_4\rangle = A(v_1, v_2, v_3, v_4) \quad \textit{and} \quad \langle \mathcal{J}(v_1)v_2, v_3\rangle = A(v_2, v_1, v_1, v_3).$$

Then Jacobi operator $\mathcal{J}$ is a projector over V iff one of the following is true:

(*i*) $V = \operatorname{Im}\mathcal{J} \oplus \ker\mathcal{J}$ *and* $\dim(\operatorname{Im}\mathcal{J}) = m - 1$, $\dim(\ker\mathcal{J}) = 1$ *or;*

(*ii*) $V = \operatorname{Im}\mathcal{J} \oplus \ker\mathcal{J}$ *and* $\dim(\operatorname{Im}\mathcal{J}) = m - 2$, $\dim(\ker\mathcal{J}) = 2$ *or;*

(*iii*) $V = \operatorname{Im}\mathcal{J} \oplus \ker\mathcal{J}$ *and* $\dim(\operatorname{Im}\mathcal{J}) = 1$, $\dim(\ker\mathcal{J}) = m - 1$.

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Vesselin Videv
Department of Mathematics and Informatics
Trakia University, Rektorat, FE, 463
6003 St. Zagora, Bulgaria
e-mail: videv@uni-sz.bg

Zhivko Zhelev
Department of Mathematics and Informatics
Trakia University, Rektorat, FE, 367A
6003 St. Zagora, Bulgaria
e-mail: zhelev@uni-sz.bg

РИМАНОВИ МНОГООБРАЗИЯ С ИДЕМПОТЕНТНИ ОПЕРАТОРИ НА ЯКОБИ

Веселин Видев, Живко Желев

В настоящата статия ние изследваме риманови многообразия с идемпотентни оператори на Якоби като представяме и необходимата алгебрична мотивация за едно такова изследване. Същите проблеми в псевдоримановата геометрия остават все още неизследвани, но ние вярваме в наличието на доста богати резултати в този конкретен случай.