МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2009
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2009
Proceedings of the Thirty Eighth Spring Conference of
the Union of Bulgarian Mathematicians
Borovetz, April 1–5, 2009

MAXIMA OF MOVING AVERAGES WITH NOISE IN THE WEIBULL MAX-DOMAIN OF ATTRACTION*

Pavlina Jordanova

Here we obtain Invariance principle for maxima in two particular cases. The time-intersections of considered sequences of random processes are maxima of properly affine transformed stationary finite or infinite moving averages. The distribution function of the noise components belongs to the max-domain of attraction of Weibull distribution. The max-increments of such processes are dependent.
The limiting process prove to be max-stable. For finite moving averages case its time-intersections have Weibull distribution. In the case of infinite moving averages, they have Gumbel distribution.

1. Introduction. In 1964 J. Lamperty [4] proved Invariance principle for maxima (IPM) of independent identically distributed (iid) random variables (rv's).

The maxima of a linear process with subexponential noise is investigated mainly by R. Davis and S. Resnick in [3] and [1]. Another IPMs are given in Theorem 5.5.11 in [5] and Proposition 4.28 in [7].

In 1991 R. Davis and S. Resnick [2] studied the convergence of point processes which space coordinates have linear process presentation with noise in the Weibull max-domain of attraction (max-DA).

Here we prove IPM for strictly stationary sequence and precisely for moving averages (MA) sequence with noise in the max-DA of

$$(1)\qquad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^\alpha\}, & x \le 0, \\ 1, & x > 0, \end{cases} \qquad \alpha > 0 \quad \text{(Weibull distribution).}$$

In the paper $U \in RV_\alpha^\infty$ means that U is regularly varying at ∞ with exponent α, $U \in RV_\alpha^0$ means that U is regularly varying at 0 with exponent α, and x_F^R is the right end point of a distribution function (df) F, i.e.

$$x_F^R = \sup\{x : F(x) < 1\}.$$

For a nondecreasing function H, with the convention that $\inf\{\emptyset\} = +\infty$,

$$H^{\leftarrow}(x) = \inf\{y : H(y) \ge x\}$$

is the left-continuous inverse of H.

****2000 MSC:** 11H60, 60F17, 60G70, 60F05

Key words: moving average, invariance principles, extreme value, weak convergence

This research is partially supported by the NFSI, Grant VU-MI-105/2005 of the Ministry of Science and Education, Bulgaria.

Throughout the paper we suppose that $(\Omega, \mathcal{A}, P)$ is a given complete probability space with filtration $(\mathcal{A}_t)_{t\geq 0}$. We assume that all P-null sets of $\mathcal{A}$ are added to $\mathcal{A}_0$ and all discussed random elements are defined on $(\Omega, \mathcal{A}, P)$.

Let S be a complete, separable metric space and $\mathcal{S}$ be the Borel σ-algebra of subsets of S generated by the open sets. By $X_n \Longrightarrow X$ we denote the weak convergence of the sequence $\{X_n\}_{n\in N}$ of random elements in S to the random element X in S.

We denote by $\mathcal{M}([0,\infty))$ the space of non-decreasing, right-continuous functions $y(t) : [0,\infty) \to [0,\infty)$, with the finite left limits on $(0,\infty)$, endowed with the J_1-topology of Skorokhod.

If E is a locally compact space with countable base, as usually $M_p(E)$ is the set of point measures on E that are finite on compact subsets of E and $\mathcal{M}_p(E)$ is the σ-algebra generated by the vague open sets. We denote by $\xrightarrow{v}$ the vague convergence in $\mathcal{M}_p(E)$.

We represent a point measure $N \in M_p(E)$ by $N(A) = \sum_i \varepsilon_{x_i}(A),\ A \subset E$, where for $x \in E$ and $A \subset E$

$$\varepsilon_x(A) = \begin{cases} 0, & x \notin A, \\ 1, & x \in A. \end{cases}$$

2. The finite moving averages case. In this section, we suppose that $\{c_i\}_{i=1,2,\ldots,k}$ are positive real numbers, $\{\xi_i\}_{i\in\mathbf{Z}}$ is a sequence of iid rv's with df $F \in \max -DA(\Psi_\alpha)$, $\alpha > 0$ and that $\{X_n\}_{n\in\mathbf{N}}$ have representation as finite MA, i.e.

$$X_n = \sum_{j=1}^{k} c_j \xi_{n-j}, \quad n \in \mathbf{N}, \quad k \in \mathbf{N}, \quad k < \infty. \tag{2}$$

Hence the sequence $\{X_n\}_{n\in\mathbf{N}}$ is strictly stationary.

By Proposition 1.13 in [7] the centering and norming constants for maxima of $\{\xi_i\}_{i\in\mathbf{Z}}$ could be chosen correspondingly x_F^R and $x_F^R - \left(\frac{1}{1-F}\right)^{\leftarrow}(n)$.

The objective of this section is to find the centering and norming constants and non-degenerate weak limit of a sequence

$$Y_n(t) = \begin{cases} \dfrac{M_{[nt]} - a_n}{b_n}, & t \geq n^{-1}, \\[2ex] \dfrac{X_1 - a_n}{b_n}, & 0 < t < n^{-1}, \end{cases} \tag{3}$$

where $M_n = \max\{X_1, \ldots, X_n\}$ and $[s]$ stands for the biggest integer, less than or equal to s.

Theorem 2.1. *If $\{Y_n\}_{n\in\mathbf{N}}$ are defined in* (3) *with $\{X_n\}_{n\in\mathbf{N}}$ that have representation* (2)*) with $F \in \max -DA(\Psi_\alpha)$, $\alpha > 0$ and positive real numbers $\{c_i\}_{i=1,2,\ldots,k}$, then $Y_n \Longrightarrow Y$ in $\mathcal{M}([0,\infty))$ when $n \to \infty$, $a_n \sim x_F^R \sum_{j=1}^{k} c_j$, $b_n \sim x_F^R - F^{\leftarrow}(1-n^{-1/k})$ and the limiting process Y is $\Psi_{k\alpha}$-extremal process with df*

$$P(Y(t) \leq x) = \exp\{-t \frac{\Gamma^k(\alpha+1)}{\Gamma(\alpha k+1)} \prod_{i=1}^{k} c_i (-x)^{k\alpha}\}, \quad x < 0, \quad t > 0.$$

Proof. Because of $F \in \max -DA(\Psi_\alpha)$, by Proposition 1.13 in [7], $x_F^R < \infty$.

Let $c=\sum\limits_{j=1}^{k}c_j$,

$$N_n=\sum_{i=1}^{\infty}\varepsilon_{\left(\frac{i}{n},\frac{X_i-x_F^Rc}{b_n}\right)}\text{ and }N=\sum_{i=1}^{\infty}\varepsilon_{(t_i,-\eta_i)}.$$

By (2.11) in [2] we have $N_n\Longrightarrow N$ on $\mathcal{M}_p([0,\infty)\times(-\infty,0])$, when $n\to\infty$, $b_n\sim x_F^R-F^{\leftarrow}(1-n^{-1/k})$ and $\sum\limits_{j=1}^{\infty}\varepsilon_{(t_j,\eta_j)}$ is a Poisson random measure on $\mathcal{M}_p([0,\infty)^2)$ with mean measure (mm) $dt\times\nu_k(dx)$, such that for all $x\geq 0$,

$$\nu_k([0,x])=c(\alpha,k)x^{k\alpha},\quad c(\alpha,k)=\frac{\Gamma^k(\alpha+1)}{\Gamma(\alpha k+1)}\prod_{i=1}^{k}c_i^{\alpha}.$$

This can also be written down in the following way

$$\sum_{i=1}^{\infty}\varepsilon_{(t_j,\eta_j)}\sim PRM(\mathcal{M}_p([0,\infty)^2);dt\times\nu_k(dx)).$$

The above mentioned means that

$$N\sim PRM(\mathcal{M}_p([0,\infty)\times(-\infty,0]);dt\times\mu_k(dx)),$$

where $\mu_k([x,0])=c(\alpha,k)(-x)^{k\alpha},\quad x\leq 0.$

Define a functional $T:\mathcal{M}_p([0,\infty)\times(-\infty,0])\to\mathcal{M}([0,\infty))$ by relations:

$$(Tm)(t)=\bigvee_{k:\tau_k\leq t}\theta_k,\quad t:m([0,t]\times(-\infty,0])>0,\quad m=\sum_k\varepsilon_{(\tau_k,\theta_k)}\tag{4}$$

$$(Tm)(t)=\bigvee_{k:\tau_k=\sup\{s>0:m((0,s]\times(-\infty,0])=0\}}\theta_k,\quad t:m([0,t]\times(-\infty,0])=0.$$

Analogously to Proposition 4.20 [7] we obtain that T is almost surely (a.s.) continuous.

By Continuous mapping theorem (CMTh) we have $TN_n\Longrightarrow TN$ on $\mathcal{M}([0,\infty))$, when $n\to\infty$. By definition of T, if we chose $a_n\sim x_F^Rc$ and $b_n\sim x_F^R-F^{\leftarrow}(1-n^{-1/k})$ we obtain that for all $t>0$, $Y_n(t)=(TN_n)(t)$.

We denote (TN) by Y. Because of

$$N\sim PRM(\mathcal{M}_p([0,\infty)\times(-\infty,0]);dt\times\mu_k),$$

Y is an extremal process with homogeneous max-increments and

$$P(Y(1)\leq x)=P(N([0,1]\times(x,0])=0)=\exp\{-c(\alpha,k)(-x)^{k\alpha}\},\ x<0,$$

i.e. it is $\Psi_{k\alpha}$-extremal process.

3. The infinite moving averages case. Let $\{\xi_i\}_{i\in\mathbf{Z}}$ be a sequence of iid rv's with df $F\in\max-DA(\Psi_\alpha)$, $\alpha>0$ and $\{c_i\}_{i\in\mathbf{N}}$ be positive real numbers. Suppose $\{X_n\}_{n\in\mathbf{N}}$ have representation as infinite MA, i.e.

$$X_n=\sum_{j=1}^{\infty}c_j\xi_{n-j},\quad n\in\mathbf{N}.\tag{5}$$

Again the sequence $\{X_n\}_{n\in\mathbf{N}}$ is strictly stationary.

We are interested in the centering and norming constants and the weak limit of a sequence $\{Y_n\}_{n\in\mathbf{N}}$, defined in (3).

Theorem 3.1. *Assume that* $\{Y_n\}_{n\in\mathbf{N}}$ *are defined in* (3) *with* $\{X_n\}_{n\in\mathbf{N}}$ *that have representation* (5) *with* $F \in max - DA(\Psi_\alpha)$, $\alpha > 0$ *and real numbers* $c_i > 0$. *Denote by* $c_{(1)} \geq c_{(2)} \geq \dots$ *the same sequence* $c_1, c_2, \dots$ *but ordered.*

If for some $q > 2$, $c_{(j)} \sim O(j^{-q})$, $j \to \infty$ *and for all* $s \in (0,1)$

$$\lim_{n\to\infty} s^n \sum_{j=s^{-n}}^{\infty} \frac{c_{(j)}^2}{c_{(n)}^2} = 0,$$

then the sequence $\{Y_n\}_{n\in\mathbf{N}}$ *is weakly convergent, i.e.* $Y_n \Longrightarrow Y$ *in* $\mathcal{M}([0,\infty))$ *when* $n \to \infty$, *the limiting process* Y *is* Λ*-extremal process with* $\Lambda(x) = \exp\{-\exp\{-x\}\}$, $x \in \mathbf{R}$, $a_n \sim F_\nu^{\leftarrow}\left(1 - \dfrac{1}{n}\right)$, *where* $\nu = \sum_{j=1}^{\infty} c_j \xi_j$ *and* $b_n \sim (r^{\leftarrow}(F_\nu^{\leftarrow}(1-n^{-1}))^{-1}$, *where*

$$r(\lambda) = \sum_{j=1}^{\infty} \frac{c_j E(\xi_1 \exp\{\lambda c_j \xi_1\})}{E \exp\{\lambda c_j \xi_1\}},\ \lambda > 0.$$

Proof. Because of $F \in \max - DA(\Psi_\alpha)$, by Proposition 1.13 in [7], $x_F^R < \infty$ and $\overline{U}(y) = 1 - F(x_F^R - \frac{1}{y}) \in RV_{-\alpha}^{\infty}$.

Then for the df, say G, of $x_F^R - \xi_1$ we obtain, that for all $t > 0$,

$$\lim_{x\to 0} \frac{G(xt)}{G(x)} = \lim_{x\to\infty} \frac{G(tx^{-1})}{G(x^{-1})} = \lim_{x\to\infty} \frac{P((x_F^R - \xi_1)^{-1} \geq xt^{-1})}{P((x_F^R - \xi_1)^{-1} \geq x)} = \lim_{x\to\infty} \frac{\overline{U}(xt^{-1})}{\overline{U}(x)} = t^\alpha,$$

i.e. $G \in RV_\alpha^0$. Conditions of Theorem 4.1 of [2] are satisfied for G (that is why we use upper index G) and

$$\sum_{i=1}^{\infty} \varepsilon_{\left(\frac{i}{n}, \frac{X_i^G - a_n^G c}{b_n^G}\right)} \Longrightarrow \sum_{i=1}^{\infty} \varepsilon_{(t_j,\eta_j)} \text{ on } \mathcal{M}_p([0,\infty) \times [-\infty,\infty)),\ n \to \infty,$$

where $\sum_{i=1}^{\infty} c_i = c < \infty$ and $\sum_{i=1}^{\infty} \varepsilon_{(t_j,\eta_j)} \sim PRM(\mathcal{M}_p([0,\infty) \times [-\infty,\infty)); dt \times \mu(dx))$,

$$\mu([-\infty, x]) = e^x,\ x \in \mathbf{R},$$

(6) $$a_n^G \sim (F_\nu^G)^{\leftarrow}(n^{-1}),\quad b_n^G \sim \frac{1}{(m^G)^{\leftarrow}(a_n^G)} \quad \textit{and}$$

(7) $$m^G(\lambda) = \sum_{j=1}^{\infty} \frac{c_j E((x_F^R - \xi_1)\exp\{-\lambda c_j (x_F^R - \xi_1)\})}{E \exp\{-\lambda c_j (x_F^R - \xi_1)\}}, \lambda > 0.$$

These mean that

(8) $$\sum_{i=1}^{\infty} \varepsilon_{\left(\frac{i}{n}, \frac{a_n^G c - X_i^G}{b_n^G}\right)} \Longrightarrow N \text{ on } \mathcal{M}_p([0,\infty) \times (-\infty,\infty]),\ n \to \infty,$$

where $N = \sum_{i=1}^{\infty} \varepsilon_{(t_j,-\eta_j)} \sim PRM(\mathcal{M}_p([0,\infty) \times (-\infty,\infty]); dt \times \mu_1(dx))$,

$$\mu_1([x,\infty]) = e^{-x},\quad x \in R.$$

By definition of X_n^G we have $X_n^G = \sum_{j=1}^{\infty} c_j (x_F^R - \xi_{n-j}) = x_F^R c - X_n,\quad n \in \mathbf{N}$.

Because of

$$F_\nu^G(x) = P\left(\sum_{j=0}^{\infty} c_j(x_F^R - \xi_j) < x\right)$$
$$= P(x_F^R c - \nu < x) = P(\nu > x_F^R c - x) = 1 - F_\nu(x_F^R c - x),$$

we have $(F_\nu^G)^{\leftarrow}(y) = x_F^R c - (F_\nu)^{\leftarrow}(1-y)$ and

$$(9)\qquad a_n^G \sim (F_\nu^G)^{\leftarrow}(n^{-1}) = x_F^R c - F_\nu^{\leftarrow}\left(1 - \frac{1}{n}\right) = x_F^R c - \left(\frac{1}{\overline{F}_\nu}\right)^{\leftarrow}(n).$$

By (8), when $n \to \infty$,

$$(10)\qquad \sum_{i=1}^{\infty} \varepsilon_{\left(\frac{i}{n}, \frac{X_i - F_\nu^{\leftarrow}(1-n^{-1})}{b_n^G}\right)} \Longrightarrow \sum_{i=1}^{\infty} \varepsilon_{(t_j, -\eta_j)} \text{ on } \mathcal{M}_p([0,\infty) \times (-\infty,\infty]).$$

By (7), for $\lambda > 0$,

$$m^G(\lambda) = c x_F^R - \sum_{j=1}^{\infty} \frac{c_j E(\xi_1 \exp\{-\lambda c_j(x_F^R - \xi_1)\})}{E\exp\{-\lambda c_j(x_F^R - \xi_1)\}} = c x_F^R - \sum_{j=1}^{\infty} \frac{c_j E(\xi_1 \exp\{\lambda c_j \xi_1\})}{E\exp\{\lambda c_j \xi_1\}}.$$

If we denote the last sum by $r(\lambda)$ we obtain $m^G(\lambda) = c x_F^R - r(\lambda)$.

Then $(m^G)^{\leftarrow}(y) = r^{\leftarrow}(c x_F^R - y),\ y > 0$ and by (6) and (9) we have,

$$b_n^G \sim \frac{1}{(m^G)^{\leftarrow}(a_n^G)} \sim \frac{1}{(m^G)^{\leftarrow}\left(c x_F^R - F_\nu^{\leftarrow}\left(1 - \frac{1}{n}\right)\right)} = \frac{1}{r^{\leftarrow}(F_\nu^{\leftarrow}(1 - \frac{1}{n}))}.$$

When we substitute the last in (10) we get

$$(11)\qquad \sum_{i=1}^{\infty} \varepsilon_{\left(\frac{i}{n}, \frac{X_i - F_\nu^{\leftarrow}(1-n^{-1})}{b_n^G}\right)} = \sum_{i=1}^{\infty} \varepsilon_{\left(\frac{i}{n}, \frac{X_i - F_\nu^{\leftarrow}(1-n^{-1})}{(r^{\leftarrow}(F_\nu^{\leftarrow}(1-n^{-1})))^{-1}}\right)}.$$

We denote the above point process by N_n.

Define a functional $T_1 : \mathcal{M}_p([0,\infty) \times (-\infty,\infty]) \to \mathcal{M}([0,\infty))$ by relations:

$$(12)\qquad (T_1 m)(t) = \bigvee_{k:\tau_k \le t} \theta_k, \quad t : m([0,t] \times (-\infty,\infty]) > 0, \quad m = \sum_k \varepsilon_{(\tau_k, \theta_k)} \quad \text{and}$$

$$(T_1 m)(t) = \bigvee_{k:\tau_k = sup\{s>0:m((0,s]\times(-\infty,\infty])=0\}} \theta_k, \quad t : m([0,t] \times (-\infty,\infty]) = 0.$$

In Proposition 4.20 [7] is obtained that T_1 is a.s. continuous in $\mathcal{M}([0,\infty))$ at $\mathcal{M}_p([0,\infty) \times (-\infty,\infty])$.

So, by CMTh we have $T_1 N_n \Longrightarrow T_1 N$ on $\mathcal{M}([0,\infty))$ when $n \to \infty$.

By (12) for $a_n \sim F_\nu^{\leftarrow}(1 - n^{-1})$ and $b_n \sim (r^{\leftarrow}(F_\nu^{\leftarrow}(1 - n^{-1})))^{-1}$, $Y_n(\cdot) = (T_1 N_n)(\cdot)$.

We denote $T_1 N$ by Y. Because of $N \sim PRM(\mathcal{M}_p([0,\infty) \times (-\infty,\infty]); dt \times \mu_1)$, Y is an extremal process with homogeneous max-increments and

$$P(Y(1) \le x) = P(N([0,1] \times (x,\infty]) = 0) = \exp\{-e^{-x}\},\ x \in \mathbf{R},$$

i.e. it is Λ-extremal process, where Λ is the Gumbel df.

REFERENCES

[1] R. A. Davis, S. I. Resnick. Extremes of moving averages of random variables from the domain of attraction of the double exponential distribution, *Stochastic Processes Appl.*, **30**, (1988), 41–68.
[2] R. A. Davis, S. I. Resnick. Extremes of moving averages of random variables with finite endpoint, Ann. Probability, **19**, (1991), 312–328.
[3] R. A. Davis, S. I. Resnick. Limit theory for moving averages of random variables with regularly varying tail probabilities, *Annals of Probability*, **13**, (1985), 179–197.
[4] J. Lampery. On Extreme Order Statistics, *Ann. Math. Statist.*, **35**, (1964), 1726–1737.
[5] P. Embrechts, Cl. Klueppelberg, T. Mikosch. Modelling Extremal Events for Insurance and Finance, Berlin, Springer Verlag, 1997.
[6] M. R. Leadbetter, G. Lindgren, H. Rootzen. Extremes and Related Properties of Random Sequences and Processes, Berlin, Springer, 1983.
[7] S. I. Resnick. Extreme Values, Regular Variation, and Point Processes, New York, Springer, 1987.
[8] W. Whitt. Stochastic-processes Limits, London, Springer, 2002.

Pavlina K. Jordanova
Faculty of Mathematics and Informatics
Shumen University
115, Universitetska Str.
9712 Shumen, Bulgaria
e-mail: pavlina_kj@abv.bg

МАКСИМУМИ НА ПЛЪЗГАЩИ СЕ СРЕДНИ СЪС СМУЩЕНИЯ В МАКС-ОБЛАСТТА НА ПРИВЛИЧАНЕ НА УЕЙБУЛ

Павлина Йорданова

Получен е принцип за инвариантност на максимума в два частни случая. Моментните сечения на разглежданите редици от случайни процеси са максимуми на подходящо афинно трансформирани стационарни крайни или безкрайни плъзгащи се средни. Функцията на разпределение на шумовите компоненти принадлежи на макс-областта на привличане на разпределението на Уейбул. Макснарастванията на тези процеси са зависими.
Граничният процес се оказва макс-устойчив. За случая с крайните плъзгащи се средни неговите моментни сечения имат разпределение на Уейбул. В случая на безкрайни плъзгащи се средни те имат разпределение на Гумбел.