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RIEMANN'S HYPOTHESIS

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In 1859, i.e. a litle more than a centure and a half ago, appeared the selebrate memoir **Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse**, *Monatsb. der Königl. Preuss. Akad. der Wissen. zu Berlin aus dem Jahr* 1859 (1860), 671-680 of B. RIEMANN (1826 - 1866). In E. BOMBIERI'S survey **Problem of the Millenium: the Riemann Hypothesis**, this memoir is qualified as "epoch-making" as well as "really astonished for the novelty of ideas included". In **Sieben Milleniums-Probleme. I**, Internat. Math. Nachr., Nr 184 (2000) 29-36, M. DRMOTA calls it "bahnbrechend für die analytische Zahlentheorie" thus repeating JÖRG BRÜDERN, **Primzahlverteilung**, Vorlesung im Wintersemester 1991/92, Mathematisches Institut Göttingen. These "estimates" and other similar ones confirmed the words of E. TITCHMARSH at the begining of Chapter X of his **The theory of Riemann zeta-function**, Oxford 1951: "The memoir, in which Riemann considered the zeta-function became famous thanks to the great number of ideas included in it. Many of them has been worked afterwards, and some of them are not exausted even till now".

RIEMANN'S memoir is devoted to the function $\pi(x)$ defined as the number of prime numbers less or equal to the real and positive number x. This is really the fact, but the "main role" in it is played by the already mentioned zeta-function. In order to make clear the basic idea of RIEMANN, let us quote a part of the second section of his memoir:

"Bei dieser Untersuchungen diente mir als Ausgangspunkt die von EULER gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\dfrac{1}{p^s}} = \sum \frac{1}{n^s} \tag{1.1}$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergieren, darstellt wird, bezeichne ich durch $\zeta(s)$".

Namely RIEMANN was the first who considered the function denoted by him by $\zeta(s)$ as a function of a complex variable and this leaded him to a number of remarkable discoveries.

After RIEMANN the notation $\zeta(s)$ remained unchanged but the notion "zeta-function" got a very wide revalence. As RIEMANN has pointed out, by the equality

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.2}$$

is defined a holomorphic function of the complex variable $s = \sigma + it$ provided $\sigma > 1$. An easy computation leads to an integral represenation of $\zeta(s)$ by means of the function $\pi(x)$. Indeed, from (1.1) it follows that $\zeta(s) \neq 0$ in the half-plane $\operatorname{Re} s > 1$ and, moreover, that

$$\log\zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right) = -\sum_{n=2}^{\infty}\{\pi(n) - \pi(n-1)\}\log\left(1 - \frac{1}{n^s}\right)$$

$$= -\sum_{n=2}^{\infty}\pi(n)\left\{\log\left(1 - \frac{1}{n^s}\right) - \log\left(1 - \frac{1}{(n+1)^s}\right)\right\} = \sum_{n=2}^{\infty}\pi(n)\int_n^{n+1}\frac{s\,dx}{x(x^s-1)},$$

i.e.

$$\log\zeta(s) = s\int_2^{\infty}\frac{\pi(x)\,dx}{x(x^s-1)}, \quad s = \sigma + it, \quad \sigma > 1.$$

Much more deeper is the formula expressing $\pi(x)$ in terms of the so-called non-trivial zeros of the meromorphic function obtained by the analytical continuation of the function $\zeta(s)$ in the whole complex plane. The discovery of this formula is one of the main achievements of Riemann included in the memoire under consideration.

There are many ways to prove that $\zeta(s)$ can be analytically contunued on the left of the line $\operatorname{Re} s = 1$. The idea of that one which we are going to sketch here belongs to Riemann. It is based on the equalities

$$\int_0^{\infty}\exp(-\pi n^2 x)x^{s/2-1}\,dx = \frac{\Gamma(s/2)}{\pi^{s/2}n^s}, \quad s = \sigma + it, \sigma > 0, \quad n = 1, 2, 3, \ldots,$$

which are corollaries of the well-known integral representation of $\Gamma(s)$ in the half-plane $\sigma > 0$, namely

$$\Gamma(s) = \int_0^{\infty} x^{s-1}\exp(-x)\,dx.$$

If $\sigma > 1$, then

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^{\infty}\psi(x)x^{s/2-1}\,dx, \tag{1.3}$$

where

$$\psi(x) = \sum_{n=1}^{\infty}\exp(-\pi n^2 x), \quad x > 0. \tag{1.4}$$

In order to justify the validity of (1.3), Riemann used the function

$$\theta(x) = \sum_{n=-\infty}^{\infty}\exp(-\pi n^2 x), \quad x > 0. \tag{1.5}$$

From the inequalities $\exp(-\pi n^2 x) \leq \exp(-\pi n x), x > 0, n = 1, 2, 3, \ldots$ it follows that $\psi(x) = O(\exp(-\pi x))$ when $x \to \infty$. Since $\theta(x) = 2\psi(x) + 1, \theta(x) = O(1)$ when $x \to \infty$. Then, the relations

$$\theta(x) = x^{-1/2}\theta(1/x) \tag{1.6}$$

and $\psi(x) = (1/2)(\theta(x) - 1)$ imply $\psi(x) = O(x^{-1/2})$ when $x \to 0$. Since $\sigma > 1$, the integral on the right-hand side of (1.3) is absolutely convergent. The series defining the function (1.4) is uniformly convergent on every compact subset of the ray $(0,\infty)$. Hence,

$$\int_0^\infty \psi(x)x^{s/2-1}\,dx = \int_0^\infty \left(\sum_{n=1}^\infty \exp(-\pi n^2 x)\right) x^{s/2-1}\,dx$$

$$= \sum_{n=1}^\infty \int_0^\infty \exp(-\pi n^2 x)x^{-s/2-1}\,dx = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Further, using $2\psi(x)+1 = x^{-1/2}(2\psi(x^{-1})+1)$, from (1.3) after some computations Riemann came to the representation

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \psi(x)(x^{s/2-1}+x^{(1-s)/2-1})\,dx, \quad \operatorname{Re} s > 1. \tag{1.7}$$

The integral in the last equality is uniformly convergent on every compact subset of the complex plane. That means the left-hand side of (1.7) admits an analytic continuation in the whole complex plane except the points 0 and 1. The zero point is a pole of the function $\Gamma(s/2)$, the point $s = 1$ is a simple pole of $\zeta(s)$ with residu equal to one. The other poles of $\Gamma(s/2)$ are at the points $-2, -4, -6, \dots$ and all they are simple. The equality (1.7) shows that all they are regular points for the already contunued function $\zeta(s)$. More precisely, these points are simple zeros of $\zeta(s)$. They are called trivial. Moreover, there are no other zeros of $\zeta(s)$ in the half-plane $\operatorname{Re} s < 0$. Since $\zeta(s) \neq 0$ when $\operatorname{Re} s > 1$, all other possible zeros of $\zeta(s)$ are in the closed strip $0 \leq \operatorname{Re} s \leq 1$ named critical strip. The zeros of $\zeta(s)$ in the crirtical strip are called non-trivial.

The right-hand side of (1.7) does not change if we replace s by $1-s$. This leads immediately to the relation

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s) \tag{1.8}$$

usually refered as a functional equation for $\zeta(s)$.

Let the function $\xi(s)$ be defined by the equality

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s). \tag{1.9}$$

It is clear that $\xi(s)$ is an entire function. Further, the functional equation (1.8) yields that

$$\xi(s) = \xi(1-s). \tag{1.10}$$

It is clear also that all the zeros of $\xi(s)$ coinside with the non-trivial zeros of $\zeta(s)$, i.e. all they are in the critical strip. Moreover, since $(s+(1-s))/2 = 1/2$, from (1.10) it follows that the zeros of $\xi(s)$ are symmetrically situated with respect to the point $1/2$. It is known that the entire function $\xi(s)$ is real, i.e. $\xi(s)$ is a real number if and only if s is real and, hence, the euality $\overline{\xi(\overline{s})} = \xi(s)$ holds for every $s \in \mathbb{C}$. That means: if ρ is a zero of $\xi(s)$, then so does $\overline{\rho}$ as well as $1-\overline{\rho}$. If $\rho = \sigma + it$, then $1-\overline{\rho} = 1-\sigma+it$, i.e. ρ and $1-\overline{\rho}$ are symmetrcally situated with respect to the line $\operatorname{Re} s = 1/2$.

In his memoir Riemann claimed that the function $\zeta(s)$ has infinitely many zeros in the critical strip and moreover, that the following "explicit" formula

$$\pi(x) = li(x) + \sum_{\rho\in\mathcal{N}, \operatorname{Im}\rho>0} (li(x^\rho) + li(x^{1-\rho})) \tag{1.11}$$

$$+ \int_x^\infty \frac{dt}{(t^2-1)\log t} - \log 2, \quad x \geq 2.$$

holds for the function $\pi(x)$, where li is the integral logarithm and $\mathcal{N}$ is the set of zeros of $\zeta(s)$ in the critical strip. The validity of the above formula was really confirmed by H. von Mangoldt in 1894.

Let, further, the function $\Xi(z)$ be defined by

$$\Xi(z) = \xi\left(\frac{1}{2} + iz\right), \quad z \in \mathbb{C}. \tag{1.12}$$

Then (1.10) immediately yields

$$\Xi(z) = \Xi(-z), \tag{1.13}$$

i.e. the functional equation for $\zeta(s)$ is equivalent to the fact that $\Xi(z)$ is an even function. Moreover, it is clear that all the possible zeros of $\Xi(z)$ are in the strip $|\operatorname{Im} z| \le 1/2$.

From (1.7), (1.9) and (1.12) it follows that

$$\Xi(z) = \frac{1}{2} - \left(z^2 + \frac{1}{4}\right) \int_1^\infty \psi(x) x^{-3/4} \cos\left(\frac{z}{2}\log x\right)\, dx.$$

Integrating by parts and using the equality $4\psi'(1) + \psi(1) = -1/2$, Riemann got that

$$\Xi(z) = 4\int_1^\infty \left\{x^{3/2}\psi'(x)\right\}' x^{-1/4} \cos\left(\frac{z}{2}\log x\right)\, dx. \tag{1.14}$$

Based on the last representation he made his remarkable comment about the zeros of the function $\Xi(z)$ namely: "... es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Bewis zu wünschen, ich habe indess die Aufsuch desselben nach einigen flüchtigen vergeblichen Versuche vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchungen entberlich schien".

The above words of Riemann has born the conjecture, usually called *Riemann's hypothesis*, that all the zeros of the function $\Xi(z)$ are real. It is equivalent to the conjecture that all the non-trivial zeros of the function $\zeta(s)$ are on the line $\operatorname{Re} s = 1/2$. In spite of the efforts made in the last 150 years, this conjecture is neither proved, nor disproved.

Remarks

1. Euler's identity (1.1), considered by him only when s is a real number, is equivalent to the fundamental theorem of Arithmetic to say that if $p_1 < p_2 < p_3 < \dots$ are the prime numbers, then every natural number $n \ge 2$ has a unique representation of the kind $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k}$, where $\alpha_j, 1 \le j \le k$ are non-negative integers.

2. The series in the right-hand sides of the equalities

$$\eta(s) = \sum_{n=1}^\infty \frac{(-1)^n}{n^s}, \quad L(s) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^s},$$

as Dirichlet series, are uniformly convergent on every compact subset of the half-plane $\operatorname{Re} s > 0$. The holomorphic functions defined by them satisfy the functional equations

$$(2^{s-1} - 1)\eta(1-s) = -(2^s - 1)\pi^{-s} \cos\frac{\pi s}{2} \Gamma(s)\eta(s),$$

$$L(1-s) = 2^s \pi^{-s} \sin\frac{\pi s}{2} \Gamma(s) L(s).$$

corresondingly.The first of them is equivalent to the functional equation for $\zeta(s)$. The second is contained in a paper of L. Euler published in 1749 and "checked" by him only for some real values of s.

In fact, both functions $\eta(s)$ and $L(s)$ are analytically contunuable as entire functions. Moreover, $\eta(s) = (1 - 2^{1-s})\zeta(s)$ (G.H. Hardy, **Divergent series**, Oxford, 1949, Chapter II, **2.2**).

3. The function $\pi(x)$ and more precissely, its growth as $x \to \infty$ has been an object of investigation of many mathematicians. About the year 1800 it was already numerically cleared that

$$\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty, \tag{1.15}$$

i.e that $\pi(x)x^{-1}\log x \to 1$ when $x \to \infty$. At the same time K. Gauss proposed that

$$\pi(x) \sim \int_2^x \frac{dt}{\log t}, \quad x \to \infty, \tag{1.16}$$

so that it is not surprising that the first member of Riemann's explicit formula (1.11) is just $li(x)$. Let us point out that the validity of Gauss' asymptotic formla was proved independently by J. Hadamard and J. de la Vallée Poussin at the end of 19th centery. It is well-known that (1.15) and (1.16) are equivalent, but it seems that (1.15) is much more popular. Each one of them is called *asymptotic low of prime numbers distribution* or, briefly, *asymptotic low* (*prime number theorem* as well as *Primzahlsatz* are also used). It is well-known that its validity is equivalent to the fact that $\zeta(1+it) \neq 0$ for every real $t \neq 0$. Namely the last property of Riemann's zeta-function has been established by J. Hadamard and J. de la Vallée Poissin.

The asymptotics of the function $\pi^*(x) = \pi(x) - li(x)$ as $x \to \infty$ is still an open problem. Riemann's explicit formula shows that its behaviour is strongly connected with the zero-distribution of $\zeta(s)$ in the crirtical strip. Indeed, all the results known till now confirmed that the asymptotics of the function $\pi^*(x)$ depends on the absence of zeros of $\zeta(s)$ in subregions of the critical strip.

In 1899 J. de la Vallée Pousin proved that $\zeta(s)$ has no zeros in the region defined by the inequality $\sigma > 1 - A(\log(|t| + 2))^{-1}$ and as a corollary he got that $\pi^*(x) = O(x\exp(-a(\log x)^{1/2}))$. In 1922 J.E. Litlewood proved that $\zeta(s) \neq 0$ if $\sigma > 1 - A\log(\log t)(\log t)^{-1}, t \geq t_0 > 0$ and thus obtained that

$$\pi^*(x) = O(x\exp(-a(\log x \log\log x)^{1/2}))$$

(here and below A and a denote positive constants different in different cases). A sharpening of Littlewood's results is given in 1936 by N.G. Tchudakov. He proved that $\zeta(s) \neq 0$ when $\sigma > 1 - A(\log t)^{-3/4}(\log\log t)^{-3/4}$, provided t is suficiently large, and as a corollary that $\pi^*(x) = O(x\exp(-a(\log x)^{-4/7}(\log\log x)^{-3/7}))$.

In 1958 I.M. Vinogradov and N.M. Korobov proved (independently) that $\zeta(s) \neq 0$ when $\sigma > 1 - A(\log(|t| + 3))^{-1/3}(\log\log(|t| + 3))^{-2/3}$. A corollary of this result is that

$$\pi^*(x) = O(x\exp(-a(\log x)^{3/5})(\log\log x)^{-1/5})$$

as $x \to \infty$. It seems the last asymptotic estimate is the best one known till now.

In 1901 H. von Koch proved that if Riemann's hypothesis is true, then $\pi^*(x) = O(x^{1/2}\log x)$ as $x \to \infty$. He proved also that if $\pi^*(x) = O(x^{\theta+\varepsilon})$ for some fixed $\theta \in [1/2, 1)$ and arbitrary positive ε when $x \to \infty$, then $\zeta(s) \neq 0$ for $\sigma > \theta$. Since $\log x = O(x^\varepsilon)$ for every positive ε when $x \to \infty$, it follows that the validity of the estimate $\pi^*(x) = O(x^{1/2}\log x), x \to \infty$ leads to the validity of Riemann's hypothesis.

4. The denotations for the function (1.9) and (1.12) are assumed by E. Titchmarsh.

The entire function defined by the right-hand side of (1.9) appeared first in RIEMANN'S memoir. Replacing s by $1/2+it$, he defined in fact the function $\Xi(t)$, denoting it by $\xi(t)$.

From (1.14) it follows that

$$\Xi(z) = 2\int_1^\infty \Psi(x)\cos((z/2)\log x)\,dx,$$

where

$$\Psi(x) = \{3\psi'(x) + 2x\psi''(x)\}x^{1/4}. \tag{1.17}$$

Then, after substituting x by $\exp 2u$, $-\infty < u < \infty$, one gets

$$\Xi(z) = 2\int_0^\infty \Phi(u)\cos zu\,du, \tag{1.18}$$

where

$$\Phi(u) = 2\Psi(\exp 2u)\exp 2u, \quad -\infty < u < \infty, \tag{1.19}$$

i.e.

$$\Phi(u) = 2\sum_{n=1}^\infty (2\pi^2 n^4 \exp(9u/2) - 3\pi n^2 \exp(5u/2))\exp(-\pi n^2 \exp 2u). \tag{1.20}$$

It is not quite obvious that the function $\Phi(u)$ is even, but this is really the fact. Indeed, (1.17), (1.19) and the relation $x = \exp 2u$ yield

$$\Phi(-u) = 2\{3\psi'(x^{-1}) + x^{-1}\psi''(x^{-1})\}x^{-5/4}$$

Further, using that $2\psi(x)+1 = x^{-1/2}\{2\psi(x^{-1})+1\}, 0 < x < \infty$, after some computation one gets

$$2\{3\psi'(x^{-1}) + x^{-1}\psi''(x^{-1})\}x^{-5/4} = 2\{s\psi'(x) + x\psi''(x)\}x^{5/4}, 0 < x < \infty,$$

i.e. $\Phi(-u) = \Phi(u), 0 < u < \infty$. Then, (1.18) can be written as

$$\Xi(z) = \int_{-\infty}^\infty \Phi(u)\exp(izu)\,du. \tag{1.21}$$

5. It was already mentioned that RIEMAN'S hypothesis is neither proved nor disproved. But it is well-known that infinitely many zeros of the function $\zeta(s)$ are located on the line $\operatorname{Re} s = 1/2$ which is equivalent to the existing of infinitely many real zeros of RIEMAN'S ξ-function. The first proof was given by H.G. HARDY **Sur les zéros de la fonction $\xi(s)$ de Riemann**, *C. R.* 153 (1914), 1012-1014.

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ХИПОТЕЗАТА НА РИМАН

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Тази година се навършват 150 години от публикуването на знаменития мемоар **Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse** (Върху броя на простите числа по-малки от дадена "величина") *Monatsb. der Königl. Akad. der Wissen. zu Berlin aus dem Jahr* 1959 (1960), 671–680 на Б. Риман (1826–1866).

Отправна точка на изследванията на Риман е дължимото на Ойлер тъждество

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s}, \quad s > 1,$$

където p "пробягва" простите, а n – естествените числа.

Главното внимание в мемоара е отделено на функцията

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

която за пръв път е третирана като функция на комплексната променлива $s = \sigma + it$ именно от Риман.

С (1) функцията $\zeta(s)$ е дефинирана когато $\sigma > 1$. Както установява Риман, тя е аналитично продължима в цялата комплексна равнина като мероморфна функция с единствен (прост) полюс в точката $s = 1$. Това е следствие от функционалното уравнение

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)$$

получено от Риман. От него следва, че точките $-2, -4, -6, \ldots$ са прости нули на "продължената" ζ-функция, наречени тривиални. Тъй като $\zeta(s) \neq 0$ когато $|\operatorname{Re} s| > 1$, $s \neq -2, -4, -6, \ldots$ други нули може да има в ивицата $0 \leq \operatorname{Re} s \leq 1$. Всъщност, както е доказано по-късно, върху правите линии $\operatorname{Re} s = 0$ и $\operatorname{Re} s = 1$ няма нули на $\zeta(s)$.

Нека $\mathcal{N}$ е множеството на нетривиалните нули на $\zeta(s)$, т.е. тези, които са в ивицата $0 < \operatorname{Re} s < 1$ и $\pi(x)$ е броят на простите числа ненадминаващи $x > 0$. Риман приема че $\zeta(s)$ има безбройно много нетривиални нули и освен това твърди че за $\pi(x)$ е валидна следната формула

$$\pi(x) = li(x) + \sum_{\rho \in \mathcal{N}, \operatorname{Im} \rho > 0} (li(x^\rho) + li(x^{1-\rho})) + \int_x^\infty \frac{dt}{(t^2-1)\log t} - \log 2, \quad x \geq 2, \tag{2}$$

където li е интегралният логаритъм.

Функцията $\xi(s)$, дефинирана с равенството

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

е холоморфна в цялата комплексна равнина, т.е. тя е цяла функция. Освен това, множеството на нулите ѝ съвпада с множеството на нетривиалните нули на $\zeta(s)$, т.е. на тези за които $0 < \operatorname{Re} s < 1$. С равенството

$$\Xi(z) = \xi\left(\frac{1}{2} + iz\right)$$

е дефинирана също цяла функция, нулите на която са в ивицата $|\operatorname{Im} z| < \frac{1}{2}$.

На Риман се дължи хипотезата, че функцията $\Xi(z)$ има само реални нули, която е еквивалентна с хипотезата, че нетривиалните нули на $\zeta(s)$ са върху правата линия $\operatorname{Re} s = \frac{1}{2}$. Тази хипотеза нито е потвърдена, нито е опровергната досега.

За аналитичната теория на числата от фундаментално значение е асимптотичното поведение на функцията $\pi(x)$ когато $x \to \infty$. Около 1800 г е установено емпирично, че $\pi(x) \sim x(\log x)^{-1}, x \to \infty$, което е еквивалентно с

$$\pi(x) \sim li(x), \quad x \to \infty. \tag{3}$$

Поведението обаче на функцията $\pi^*(x) = \pi(x) - li(x)$ когато $x \to \infty$ е все още загадка за съвременната математическа наука.

В предложения обзор главното внимание върху мемоара на Риман е отделено на извода на функционалното уравнение за функцията $\zeta(s)$, което, както изглежда, е довело до хипотезата за нетривиалните ѝ нули.

Упоменато е, че формулата (2) за $\pi(x)$, както и валидността на съотношението (3), е доказана в края на 19-и век и, че безбройно много от нулите на $\zeta(s)$ са върху правата линия $\operatorname{Re} s = \frac{1}{2}$ е установено в началото на миналия век. Проследени са усилията resp. резултатите от тях за уточняване на разпределението на нетривиалните нули на $\zeta(s)$, както и влиянието им върху асимптотиката на функцията $\pi^*(x)$.