МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2010
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2010
Proceedings of the Thirty Ninth Spring Conference of
the Union of Bulgarian Mathematicians
Albena, April 6–10, 2010

NONLOCAL BOUNDARY VALUE PROBLEMS FOR TWO-DIMENSIONAL POTENTIAL EQUATION ON A RECTANGLE*

Ivan Dimovski[1], **Yulian Tsankov**[2]

Let Φ and Ψ be "arbitrary" linear functionals on $C^1[0,a]$ and $C^1[0,b]$, respectively. The class of BVPs $u_{xx}+u_{yy}=F(x,y)$, $0<x<a$, $0<y<b$, $u(x,0)=0$, $u(0,y)=0$, $\Phi_\xi\{u(\xi,y)\}=g(y)$, $\Psi_\eta\{u(x,\eta)\}=f(x)$ is considered. An extension of Duhamel principle, known for evolution equations, is proposed. An operational calculus approach for explicit solution of these problems is developed. A classical example of such BVP is the Bitsadze – Samarskii problem.

1. Introductions. Let Φ be a linear functional on $C^1[0,\,a]$ and Ψ be a linear functional on $C^1[0,\,b]$. Then they have Stieltjes type representations:

$$\Phi\{f\}=Af(a)+\int\limits_0^a f'(t)d\alpha(t),\ f\ \in C^1[0,a] \tag{1}$$

and

$$\Psi\{g\}=Bg(b)+\int\limits_0^b g'(t)d\beta(t),\ g\ \in C^1[0,b] \tag{2}$$

where α and β are function with bounded variation, A and B being constants.

We consider the potential equation

$$u_{xx}+u_{yy}=F(x,y) \tag{3}$$

on the rectangle $G=\{(x,y):0<x<a,\,0<y<b\}$ with local BV conditions

$$u(x,0)=\varphi(x)\ \text{ and }\ u(0,y)=\psi(x) \tag{4}$$

and nonlocal BV conditions

$$\Phi_\xi\{u(\xi,y)\}=g(y),\Psi_\eta\{u(x,\eta)\}=f(x) \tag{5}$$

with some mild smoothness requirements for the given functions F, φ, ψ, f and g.

---

* **2000 Mathematics Subject Classification:** 44A35, 35L20, 35J05, 35J25

Key words: nonlocal BVP, right-inverse operator, extended Duamel principle, generalized solution, convolution, multiplier, multipliers fraction

[1] Partially supported by Project ID_09_0129 "ITMSFA" with Nat. Sci. Fund. Ministery of Educ. Youth and Sci, Bulgaria.

[2] Partially supported by Grand N 132 of NSF of Bulgaria.

The only restrictions on the functionals Φ and Ψ are the requirements $\Phi_\xi\{\xi\} \neq 0$ and $\Psi_\eta\{\eta\} \neq 0$. They are connected with the approach chosen and may be ousted by means of some technical involvements. For the sake of some normalization of the functionals Φ and Ψ, we assume

$$\Phi_\xi\{\xi\} = 1 \quad \text{and} \quad \Psi_\eta\{\eta\} = 1. \tag{6}$$

We consider the spaces $C(G)$ and $C^1(G)$ of the continuous and smooth functions on $G = [0, a] \times [0, b]$, respectively.

Further, we introduce the right inverse operators L_x and L_y of $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ on $C([0, a] \times [0, b])$ as the solutions $v(x, y) = L_x u(x, y)$ and $w(x, y) = L_y u(x, y)$ of the elementary BVPs

$$\frac{\partial^2 v}{\partial x^2} = u(x, y),\ v(0, y) = 0,\ \Phi_\xi\{v(\xi, y)\} = 0 \tag{7}$$

and

$$\frac{\partial^2 w}{\partial y^2} = u(x, y),\ w(x, 0) = 0,\ \Psi_\eta\{w(x, \eta)\} = 0. \tag{8}$$

The operators L_x and L_y have the explicit representations:

$$L_x\{u(x, y)\} = \int_0^x (x - \xi)u(\xi, y)d\xi - x\Phi_\xi\left\{\int_0^\xi (\xi - \eta)u(\eta, y)d\eta\right\}, \tag{9}$$

$$L_y\{u(x, y)\} = \int_0^y (y - \eta)u(x, \eta)d\eta - y\Psi_\eta\left\{\int_0^\eta (\eta - \varsigma)u(x, \varsigma)d\varsigma\right\}. \tag{10}$$

2. Convolutions. One of the authors had found a convolution $(f_1 \overset{x}{*} f_2)(x)$ in $C[0, a]$ and a convolution $(g_1 \overset{y}{*} g_2)(y)$ in $C[0, b]$ such that the operators L_x and L_y are the convolution operator $\{x\}\overset{x}{*}$ and $\{y\}\overset{y}{*}$, correspondingly.

Theorem 1 [1]**.** *The operations*

$$(f_1 \overset{x}{*} f_2)(x) = -\frac{1}{2}\Phi_\xi\left\{\int_0^\xi h(x, \eta)d\eta\right\}, \tag{11}$$

$$(g_1 \overset{y}{*} g_2)(y) = -\frac{1}{2}\Psi_\eta\left\{\int_0^\eta k(y, \varsigma)d\varsigma\right\}, \tag{12}$$

where

$$h(x, \eta) = \int_x^\eta f_1(\eta + x - \varsigma)f_2(\varsigma)d\varsigma - \int_{-x}^\eta f_1(|\eta - x - \varsigma|)f_2(|\varsigma|)sgn(\varsigma(\eta - x - \varsigma))d\varsigma \tag{13}$$

$$k(y, \eta) = \int_y^\eta g_1(\eta + y - \varsigma)g_2(\varsigma)d\varsigma - \int_{-y}^\eta g_1(|\eta - y - \varsigma|)g_2(|\varsigma|)sgn(\varsigma(\eta - y - \varsigma))d\varsigma \tag{14}$$

are bilinear, commutative and associative operations on $C([0,a])$ *and* $C([0,b])$*, respectively, such that the representations*

$$L_x f(x) = \{x\} \overset{x}{*} f(x) \tag{15}$$

and

$$L_y g(y) = \{y\} \overset{y}{*} g(y) \tag{16}$$

hold.

For a proof see [1].

By means of (11) and (12) a two-dimensional convolution in $C([0,a]\times[0,b])$ can be defined

Theorem 2 [7]**.** *The operation*

$$(u*v)(x,y) = \frac{1}{4}\tilde{\Phi}_\xi \tilde{\Psi}_\eta \{h(x,y,\xi,\eta)\}, \tag{17}$$

where

$$\tilde{\Phi}_\xi\{f(\xi)\} = \Phi_\xi\left\{\int\limits_0^\xi f(\sigma)d\sigma\right\}, \quad \tilde{\Psi}_\eta\{f(\eta)\} = \Psi_\eta\left\{\int\limits_0^\eta f(\tau)d\tau\right\}$$

with

$$\begin{aligned}
&h(x,y,\xi,\eta) = \int\limits_x^\xi \int\limits_y^\eta u(\xi+x-\sigma,\eta+y-\tau)v(\sigma,\tau)d\sigma d\tau - \\
&- \int_{-x}^{\xi}\int_{y}^{\eta} u(|\xi-x-\sigma|,\eta+y-\tau)v(|\sigma|,\tau)\, sgn\,(\xi-x-\sigma)\sigma d\sigma\, d\tau - \\
&- \int_{x}^{\xi}\int_{-y}^{\eta} u(\xi+x-\sigma,|\eta-y-\tau|)v(\sigma,|\tau|)\, sgn\,(\eta-y-\tau)\tau d\sigma\, d\tau + \\
&+ \int_{-x}^{\xi}\int_{-y}^{\eta} u(|\xi-x-\sigma|,|\eta-y-\tau|)v(|\sigma|,|\tau|)\, sgn\,(\xi-x-\sigma)(\eta-y-\tau)\sigma\tau d\sigma\, d\tau
\end{aligned}$$

is a bilinear, commutative and associative operation in $C(G)$ *such that*

$$L_x\{u(x,y)\} = \{x\}\overset{x}{*}\{u(x,y)\},\ L_y\{u(x,y)\} = \{y\}\overset{y}{*}\{u(x,y)\} \tag{18}$$

$$L_x L_y\{u(x,y)\} = \{xy\}*\{u(x,y)\}. \tag{19}$$

The linear space $C = C(G)$ equipped with the multiplication (17) is a commutative Banach algebra $(C, *)$.

Further, we introduce the algebra $\mathfrak{M}$ of the multipliers of $(C, *)$. Let us remind the definition of a multiplier of $(C, *)$.

Definition 1. (See [3]) *A mapping* $M : C \to C$ *is said to by a multiplier of the convolutional algebra* $(C, *)$ *iff the relation*

$$M(u*v) = (Mu)*v \tag{20}$$

holds for all $u,\ v \in C$.

As it is shown in Larsen [2], each such mapping for our convolution (17) is automatically linear and continuous. That's why, further we consider each multiplier of $(C, *)$ as a continuous linear operator.

If $f \in C[0,a]$ and $g \in C[0,b]$, then the convolutional operators $f \overset{x}{*}$ and $g \overset{y}{*}$ defined in C by $(f \overset{x}{*})u = f \overset{x}{*} u$, $(g \overset{y}{*})u = g \overset{y}{*} u$ are multipliers of $(C, *)$ (See Dimovski and Spiridonova [7]). Of course, the operator $\{F(x,y)\}*$ is also multiplier of $(C, *)$.

Further, we use the notations

$$[f]_y = \{f(x)\} \overset{x}{*} \text{ and } [g]_x = \{g(y)\} \overset{y}{*}. \tag{21}$$

3. A two-dimensional operational calculus. In $\mathfrak{M}$ there are elements which are non-divisors of 0. Indeed, such elements are the multipliers $\{x\} \overset{x}{*}$ and $\{y\} \overset{y}{*}$, i.e. the operators L_x and L_y.

Denote by $\mathfrak{N}$ the set of the non-zero non-divisors of the zero on $\mathfrak{M}$. The set $\mathfrak{N}$ is a multiplicative subset of $\mathfrak{M}$, i.e. such that p, $q \in \mathfrak{N}$ implies $pq \in \mathfrak{N}$.

Further, we consider multipliers fractions of the form $\dfrac{M}{N}$ with $M \in \mathfrak{M}$ and $N \in \mathfrak{N}$. They are introduced in a standard manner, using the well-known method of "localisation" from the general algebra [4].

Denote by $\mathcal{M}$ the set $\mathfrak{N}^{-1}\mathfrak{M}$ of multipliers fractions. We consider it as a commutative ring containing the basic field ($\mathbb{R}$ or $\mathbb{C}$), the algebras $(C[0,a], \overset{x}{*})$, $(C[0,b], \overset{y}{*})$, $(C, *)$ and $\mathfrak{M}$, due to the embeddings

$$\begin{aligned}
\mathbb{C} \to \mathcal{M} \text{ or } \mathbb{C} \quad &\to \quad \mathcal{M}: \quad \alpha \mapsto \frac{\alpha\, L_x}{L_x},\\
(C[0,a], \overset{x}{*}) \quad &\to \quad \mathcal{M}: \quad f(x) \mapsto \frac{(L_x f) \overset{x}{*}}{L_x},\\
(C[0,b], \overset{y}{*}) \quad &\to \quad \mathcal{M}: \quad g \mapsto \frac{(L_y g) \overset{y}{*}}{L_y},\\
(C([0,a] \times [0,b]), *) \quad &\to \quad \mathcal{M}: \quad u \mapsto \frac{(L_x L_y u)*}{L_x L_y}.
\end{aligned}$$

Further, we consider all numbers, functions, multiplier and multipliers fractions as elements of a *single algebraic system*: the ring $\mathcal{M}$ of the multipliers fractions.

4. Explicit solution of nonlocal BVPs for the potential equation. We consider the following boundary value problem:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y), \quad 0 < x < a, \ 0 < y < b\\
&u(x,0) = u(0,y) = 0\\
&\Phi_\xi\{u(\xi,y)\} = g(y), \qquad \Psi_\eta\{u(x,\eta)\} = f(x)
\end{aligned} \tag{22}$$

Definition 2. *A function $u(x,y) \in C^1([0,a]\times[0,b])$ is said to be a generalised solution of (22) iff $u(x,y)$ satisfies the integral relation*

$$L_x u + L_y u = L_x f(x).y + L_y g(y).x + L_x L_y F(x,y) \tag{23}$$

Formally, (23) could be obtained from the equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = F(x, y)$ applying to it the operator $L_x L_y$ and taking into account the boundary value conditions.

Lemma 1. *If $u(x,y) \in C^1([0,a] \times [0,b])$ satisfy (23), then $u(x,y)$ satisfies the boundary value conditions:*

$$\begin{aligned}&u(x,0) = u(0,y) = 0,\\ &\Phi_\xi\{u(\xi,y)\} = g(y), \qquad \Psi_\eta\{u(x,\eta)\} = f(x)\end{aligned}$$

Proof. Let us consider (23). For $y = 0$ we find $L_x u(x,0) = 0$. Next we apply the operator $\dfrac{\partial^2}{\partial x^2}$ and find $u(x,0) = 0$. For $x = 0$ we find $L_y u(0,y) = 0$. Applying $\dfrac{\partial^2}{\partial y^2}$, we get $u(0,y) = 0$. If apply Ψ to (23), we obtain $L_x \Psi_\eta\{u(x,\eta)\} = L_x f(x)$. Then, applying $\dfrac{\partial^2}{\partial x^2}$ we obtain $\Psi_\eta\{u(x,\eta)\} = f(x)$. At last, applying Φ to (23), we get $L_y \Phi_\xi\{u(\xi,y)\} = L_y g(y)$ and, hence, $\Phi_\xi\{u(\xi,y)\} = g(y)$. $\square$

Lemma 2. *If $u(x,y) \in C^2([0,a] \times [0,b])$ satisfy (23) then it is a classical solution of* (22).

Proof. Applying the operator $\dfrac{\partial^4}{\partial x^2 \partial y^2}$ to (23), we get $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = F(x,y)$. The fulfilment of the boundary value conditions follows from Lemma 1. $\square$

Lemma 3. *If $u \in C^2(G)$, then it holds:*

$$L_x \left\{ \frac{\partial^2 u}{\partial x^2} \right\} = u(x,y) + (x\Phi_\xi\{1\} - 1)u(0,y) - x\Phi_\xi\{u(\xi,y)\} \tag{24}$$

and

$$L_y \left\{ \frac{\partial^2 u}{\partial y^2} \right\} = u(x,y) + (y\Psi_\eta\{1\} - 1)u(x,0) - y\Psi_\eta\{u(x,\eta)\}. \tag{25}$$

For a proof, see [2].

Most important for our considerations are the algebraic inverses $S_x = \dfrac{1}{L_x}$ and $S_y = \dfrac{1}{L_y}$ of the multipliers L_x and L_y, correspondingly.

Lemma 4. *If $u \in C^2([0,a] \times ([0,b])$, then*

$$u_{xx} = S_x u + S_x\{(x\Phi_\xi\{1\} - 1)u(0,y)\} - [\Phi_\xi\{u(\xi,y)\}]_x, \tag{26}$$

$$u_{yy} = S_y u + S_y\{(y\Psi_\eta\{1\} - 1)u(x,0)\} - [\Psi_\eta\{u(x,\eta)\}]_y. \tag{27}$$

Proof. By multiplication of (24) and (25) by S_x and S_y , correspondingly. $\square$

Using the boundary value conditions of (22), the equation $u_{xx} + u_{yy} = F(x,y)$ can be reduced to a single algebraic equation in $\mathcal{M}$. Indeed, by (26) and (27) we find

$$u_{xx} = S_x u - [g(y)]_x \tag{28}$$

$$u_{yy} = S_y u - [f(x)]_y \tag{29}$$

and the equation $u_{xx} + u_{yy} = F(x,\, y)$ takes the algebraic form:

$$(S_x + S_y)u = F(x, y) + [g(y)]_x + [f(x)]_y.$$

If $S_x + S_y$ is non-divisor of zero, then the last equation has the following formal solution in $\mathcal{M}$:

$$u = \frac{1}{(S_x + S_y)}\{F(x, y)\} + \frac{1}{(S_x + S_y)}[f(x)]_y + \frac{1}{(S_x + S_y)}[g(y)]_x.$$

The requirement $S_x + S_y$ to be a non-divisor of 0 in $\mathcal{M}$ is equivalent to a theorem for uniqueness of the solution of (22). Therefore, our next task is to study the uniqueness for problem (22). In the direct algebraic approach we are following, this problem reduces to the purely algebraic requirement the elements $S_x + S_y$ of $\mathcal{M}$ to be non-divisors of zero in $\mathcal{M}$.

To this end we consider the following two eigenvalue problems:

(30) $$u''(x) + \lambda^2 u(x) = 0,\ x \in (0, a),\ u(0) = 0,\ \Phi_\xi\{u(\xi)\} = 0 \ \text{ in } \ C[0, a],$$

(31) $$v''(y) + \mu^2 v(y) = 0,\ y \in (0, b),\ v(0) = 0,\ \Psi_\eta\{v(\eta)\} = 0 \ \text{ in } \ C[0, b].$$

Let λ_n and μ_m be the eigenvalues of (30) and (31) for $n,\, m \in \mathbb{N}$, correspondingly.

Lemma 5. *If there exists a dispersion relation of the form $\lambda_n^2 + \mu_m^2 = 0$ for some n, $m \in \mathbb{N}$, then $S_x + S_y$ is a divisor of zero in $\mathcal{M}$.*

Proof. Let for some $n,\, m \in \mathbb{N}$ we have $\lambda_n^2 + \mu_m^2 = 0$. Then

$$(S_x + S_y)\sin\lambda_n x \sin\mu_m y = -(\lambda_n^2 + \mu_m^2)\sin\lambda_n x \sin\mu_m y = 0. \qquad \square$$

Theorem 3. *Let $a \in \operatorname{supp}\Phi$. If $\lambda_n^2 + \mu_m^2 \neq 0$ for all $n,\, m \in \mathbb{N}$, then $S_x + S_y$ is a non-divisor of zero in $\mathcal{M}$.*

Proof. Assume the contrary. It is easy to see, that $S_x + S_y$ is a divisor of zero in $\mathcal{M}$ iff there is a function $u \in C^2(G)$, $u \neq 0$, such that $(S_x + S_y)u = 0$. This relation is equivalent to

(32) $$(L_x + L_y)u = 0$$

Let λ_n be an arbitrary eigenvalue of (30). Then, λ_n is a zero of the sine-indicatrix $E(\lambda) = \Phi_\xi\left\{\dfrac{\sin\lambda\xi}{\lambda}\right\}$ of the functional Φ. Let χ_n be the multiplicity of λ_n as a zero of $E(\lambda)$. To λ_n it corresponds the finite sequence of the eigenfunction $\sin\lambda_n x$ and $\chi_n - 1$ associated eigenfunctions

$$\varphi_{n,s}(x) = \left(L_x + \frac{1}{\lambda_n^2}\right)^s \varphi_{n,0}, 0 \leq s \leq \chi_n - 1,$$

where

$$\varphi_{n,0}(x) = \frac{1}{\pi\, i}\int\limits_{\Gamma_n} \frac{\sin\lambda\, x}{\lambda\, E(\lambda)} d\lambda$$

(see Dimovski and Petrova [5], p. 94). Note that $\varphi_{n,\chi_n-1}(x) = \alpha_n \sin\lambda_n x$ with some $\alpha_n \neq 0$.

The corresponding χ_n-dimensional eigenspace is

$$E_{\lambda_n}^{(\chi_n)} = \operatorname{span}\{\varphi_{n,s}(x),\ s = 0, 1, \ldots, \chi_n - 1\}$$

The spectral projector $P_{\lambda_n} : C \to \mathrm{E}_{\lambda_n}^{(\chi_n)}$ is given by $P_{\lambda_n}\{f\} = f \overset{x}{*} \varphi_n$.

According to a theorem of N. Bozhinov [8] in the case $a \in \operatorname{supp} \Phi$, the projectors P_{λ_n} form a total system, i.e. a system for which $P_{\lambda_n}\{f\} = 0$, $\forall n \in \mathbb{N}$ implies $f \equiv 0$. For a simple proof of Bozhinov's theorem in our case, see [5] p. 97–98.

Denote $u_n(x, y) = u(x, y) \overset{x}{*} \varphi_n(x)$. From $(L_x + L_y) \quad u = 0$ it follows

$$(L_x + L_y)u_n = 0. \tag{33}$$

We show that (33) has only the trivial solution $u_n = 0$ in $E_{\lambda_n}^{(\chi_n)}$. Assume that there exists a nonzero solution u_n of (33), i.e. of the form

$$u_n(x, y) = A_{n,k}(y)\varphi_{n,k}(x) + A_{n,k+1}(y)\varphi_{n,k+1}(x) + \cdots + A_{n,\chi_n-1}(y)\varphi_{n,\chi_n-1}(x) \tag{34}$$

with $A_{n,k}(y) \neq 0$ for some k , $0 \leq k \leq \chi_n - 1$. We apply the operator $\left(L_x + \dfrac{1}{\lambda_n^2}\right)^{\chi_n-k-1}$ to (33) and obtain

$$(L_x + L_y)A_{n,\chi_n-1}(y)\,\varphi_{n,\chi_n-1}(x) = 0,$$

since $\left(L_x + \dfrac{1}{\lambda_n^2}\right)^s \varphi_{n,\,0} = 0$, for $s \geq \chi_n$.

But $\varphi_{n,\chi_n-1}(x) = \alpha_n \sin \lambda_n x$ with $\alpha_n \neq 0$. Denote $A_{n,\chi_n-1}(y) = A_n(y)$ and consider $(L_x + L_y)A_n(y) \sin \lambda_n x = 0$ as an equation for $A_n(y)$. It is equivalent to the BVP

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}(A_n(y) \sin \lambda_n x) + \frac{\partial^2}{\partial y^2}(A_n(y) \sin \lambda_n x) = 0\\ &A_n(0) = 0, \quad \Psi_\eta\{A_n(\eta)\} = 0,\end{aligned}$$

which reduces to

$$A_n''(y) - \lambda_n^2 A_n(y) = 0, \quad A_n(0) = 0 \ \text{ and } \ \Psi_\eta\{A_n(\eta)\} = 0.$$

From this equation it follows that $-\lambda_n^2$ is an eigenvalue μ_m^2 of problem (31). Hence, $\lambda_n^2 + \mu_m^2 = 0$ which is a contradiction. Hence, $u_n(x, y) \equiv 0$ for all $n \in \mathbb{N}$. By N. Bozhinov's theorem it follows that $u(x, y) \equiv 0$. Thus, we proved, that $S_x + S_y$ is a non-divisor of 0 in $\mathcal{M}$.

4.1. Let us consider BVP (22) for $f(x) = L_x\{x\} = \dfrac{1}{S_x^2}$ and $g(y) = F(x,\, y) \quad \equiv 0$. We assume that there exists a generalized solution of this problem and denote it by $U(x,\, y)$. It has the following algebraic representation:

$$U = \frac{1}{(S_x + S_y)} L_x\{x\} = \frac{1}{(S_x + S_y)} L_x^2 = \frac{1}{(S_x + S_y)S_x^2}.$$

Then, there exists also the solution of problem (22) for arbitrary $f(x)$, $g(y)$ and $F(x,\, y)$ and it can by represented in the form:

$$\begin{aligned}u &= \frac{1}{(S_x + S_y)}\{F(x, y)\} + \frac{1}{(S_x + S_y)}[f(x)]_y + \frac{1}{(S_x + S_y)}[g(y)]_x =\\ &= S_x^2 \left[\frac{1}{(S_x + S_y)S_x^2} F(x, y) + \frac{1}{(S_x + S_y)S_x^2}[f(x)]_y + \frac{1}{(S_x + S_y)S_x^2}[g(y)]_x\right]\end{aligned}$$

$$u = \frac{\partial^4}{\partial x^4}\left[U \overset{F}{*} (x, y) + U \overset{x}{*} f(x) + U \overset{y}{*} g(y)\right]$$

provided the denoted derivative exists.

4.2. Let us consider BVP (22) for $F(x, y) = xy = L_x L_y = \dfrac{1}{S_x S_y}$ and $g(y) = f(x) \equiv$

0. We denote the solution of this problem by $W(x,\ y)$. Then, we have an algebraic representation of this solution:

$$W = \frac{1}{(S_x+S_y)}L_xL_y = \frac{1}{(S_x+S_y)S_xS_y}$$

The solution of problem (22) for arbitrary $f(x)$, $g(y)$ and $F(x,\ y)$ can by represented in the form:

$$u = S_xS_y\left[\frac{1}{S_xS_y(S_x+S_y)}[f(x)]_y + \frac{1}{S_xS_y(S_x+S_y)}[g(y)]_x + \frac{1}{S_xS_y(S_x+S_y)}\{F(x,y)\}\right]$$

$$u = \frac{\partial^4}{\partial x^2\partial y^2}\left[W \overset{x}{*} f(x) + W \overset{y}{*} g(y) + W * F(x,y)\right]$$

In order the denoted derivative to exist, some smoothness condition on the functions f, g and F should be imposed. Here we not enter into details, but we illustrate these conditions on the example of Bitsadze-Samarskii's problem:

$$\text{(35)}\qquad \begin{aligned} &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,, \quad 0<x<1, \quad 0<y<1 \\ &u(x,0) = u(0,y) = 0, \quad u(x,1) = f(x) \\ &u(1,y) - u\left(\frac{1}{2}, y\right) = 0. \end{aligned}$$

This is the special case of boundary value problem (22) when

$$\Phi_\xi\{u(\xi,y)\} = 2(u(1,y) - u(\frac{1}{2},y)) \ \text{ and } \ \Psi_\eta\{u(x,\eta)\} = u(x,\ 1).$$

Following the approach outlined above, we can find ([4], p 175) that the solution $U(x,y)$ of (35) for $f(x) = L_x\{x\} = \dfrac{x^3}{6} - \dfrac{7x}{24}$ is

$$U(x,y) = \sum_{n=1}^{\infty}\frac{\operatorname{sh}4n\pi y\sin 4n\pi x}{32\pi^3n^3\operatorname{sh}4n\pi} + \sum_{n=1}^{\infty}\frac{9\operatorname{sh}\frac{2}{3}(2n-1)\pi y\sin\frac{2}{3}(2n-1)\pi x}{4\pi^3(2n-1)^3\cos\frac{2}{3}(1+n)\pi\operatorname{sh}\frac{2}{3}(2n-1)\pi}$$

Then, for $f\in C^2[0,\ 1]$, with $f(0) = f(1) - f\left(\dfrac{1}{2}\right) = 0$, we obtain

$$\begin{aligned} u(x,y) = &\int_x^{\frac{1}{2}} U_x(\frac{1}{2}+x-\xi,y)f''(\xi)d\xi - \int_{-x}^{\frac{1}{2}} U_x(\frac{1}{2}-x-\xi,y)f''(|\xi|)\operatorname{sgn}\xi\, d\xi - \\ &- \int_x^1 U_x(1+x-\xi,y)f''(\xi)\,d\xi + \int_{-x}^1 U_x(1-x-\xi,y)f''(|\xi|)\operatorname{sgn}\xi\,d\xi \end{aligned}$$

as a generalized solution of (35). It can be shown that it is a classical solution too, if $f\in C^4[0,\ 1]$ and additionally, $f''(0) = f''(1) - f''\left(\dfrac{1}{2}\right) = 0$ (see this volume, p. 114).

REFERENCES

[1] I. H. Dimovski. Two new convolutions for linear right inverse operators of $\dfrac{d^2}{dx^2}$. *C. R. Acad. Bulgare Sci.*, **29** (1976) No 1, 25–28.
[2] I. H. Dimovski. Nonlocal boundary value problems. In: *Mathematics and Math. Education*, **31** (2009), 31–40.
[3] R. Larsen. An Introduction in the theory of multipliers. Springer, Berlin–New York–Heidelberg, 1971.
[4] S. Lang. Algebra. Addison–Wesley, Reading, Mass. 1965.
[5] I. H. Dimovski, R. I. Petrova. Finite integral transforms for nonlocal boundary value problems. In: Generalized functions and convergence (Katowice, 1988), World Sci. Publ., Teaneck, NJ, 1990, 89–103.
[6] I. H. Dimovski. Convolutional Calculus. Kluwer, Dordrecht, 1990.
[7] I. H. Dimovski, M. Spiridonova. Computational approach to nonlocal boundary value problems by multivariate operational calculus. *Math. Sci. Res. J.*, **9** (2005), No 12, 315–329.
[8] N. S. Bozhinov. On the theorems of uniqueness and completeness on eigen- and associated eigenfunctions of the nonlocal Sturm-Liouville operator on a finite interval. *Diferenzial'nye Uravneya*, **26** (1990), No 5, 741–453 (in Russian).

Ivan Dimovski
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113, Sofia, Bulgaria
e-mail: dimovski@math.bas.bg

Yulian Tsankov
Faculty of Mathematics and Informatics
St. Kliment Ohridsky University of Sofia
5, J. Bourchier Blvd
1164 Sofia, Bulgaria
e-mail: ucankov@fmi.uni-sofia.bg

НЕЛОКАЛНА ГРАНИЧНА ЗАДАЧА ЗА ДВУМЕРНОТО УРАВНЕНИЕ НА ПОТЕНЦИАЛА ВЪРХУ ПРАВОЪГЪЛНИК

Иван Димовски, Юлиан Цанков

Нека Φ и Ψ са произволни линейни функционали съответно върху $C^1[0,a]$ и $C^1[0,b]$. Разгледан е класът от гранични задачи $u_{xx}+u_{yy}=F(x,y)$, $0<x<a$, $0<y<b$, $u(x,0)=0$, $u(0,y)=0$, $\Phi_\xi\{u(\xi,y)\}=g(y)$, $\Psi_\eta\{u(x,\eta)\}=f(x)$. Предложено е разширение на принципа на Дюамел. За намиране на явно решение на нелокални гранични задачи от този тип е развито операционно смятане основано върху некласическа двумерна конволюция. Пример от такъв тип е задачата на Бицадзе-Самарски.