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FACTORIZATIONS OF SOME GROUPS OF LIE TYPE OF LIE RANK 4*

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In this paper we consider simple groups G which can be represented as a product of two their proper non-Abelian simple subgroups A and B. The representation $G = AB$ is called a (simple) factorization of G. Here we suppose that G is a simple group of Lie type of Lie rank 4 over the finite field $GF(q)$ except the orthogonal groups $P\Omega_8^+(q)$ and determine all the factorizations of G.

1. Introduction. Let G be a finite (simple) group. We are interested in the factorizations of G as a product of two simple subgroups. In previous papers (see [1], [3], [4], and [5]) we have determined all the factorizations of the finite simple groups of Lie type of Lie rank 3. The present paper continues this investigation for the finite simple groups of Lie type of Lie rank 4. The following result is proved:

Theorem. *Let G be a group of Lie type of Lie rank 4 over the finite field $GF(q)$ except the orthogonal groups $P\Omega_8^+(q)$, and let $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then, one of the following holds:*

(1) $G = F_4(q)$, $q = 2^n$, $A \cong PSp_8(q)$, $B \cong {}^3D_4(q)$;

(2) $G = U_8(q)$, $q \not\equiv -1 \pmod 7$, $A \cong U_7(q)$, $B \cong PSp_8(q)$;

(3) $G = P\Omega_{10}^-(q)$, $q = 2^n$, *and* $n \not\equiv 2 \pmod 4$, $A \cong U_5(q)$, $B \cong PSp_8(q)$;

(4) $G = P\Omega_{10}^-(q)$, q *odd*, $q \not\equiv -1 \pmod 5$, $A \cong U_5(q)$, $B \cong \Omega_9(q)$.

Throughout this paper we freely use the notation and basic information on the finite (simple) classical groups given in [8]. Especially, $L_n(q)$, $U_n(q)$ stand for $PSL_n(q)$, respectively $PSU_n(q)$. The list of all finite simple groups of Lie type of Lie rank 4 is the following: $A_4(q) \equiv L_5(q)$, $PSp_8(q)$, ${}^2A_7(q) \equiv U_8(q)$, ${}^2A_8(q) \equiv U_9(q)$, $B_4(q) \equiv \Omega_9(q)$($q$ is odd), $D_4(q) \equiv P\Omega_8^+(q)$, ${}^2D_5(q) \equiv P\Omega_{10}^-(q)$, $F_4(q)$, ${}^2E_6(q)$.

The factorizations of all the finite simple groups as a product of two maximal subgroups (so called maximal factorizations) have been determined in [9]. Particularly, an explicit list of the maximal factorizations of the groups of Lie type of Lie rank 4 can be derived from [9]. We make use of this result here. It follows immediately that the groups $U_9(q)$ have no factorizations; the linear groups $L_5(q)$ have no factorizations too (this is indicated in [1]). The complete list of all the factorizations (for any subgroups A, B) of the groups $F_4(q)$ and ${}^2E_6(q)$ appeared in [7]. This gives (1) in the theorem; the existence of this factorization is also proved there.

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Note that some of the possible cases of factorizations of the symplectic groups $PSp_8(q)$ have been already treated in our paper [2]. As for the orthogonal groups $P\Omega_8^+(q)$, we have excluded, they possess many classes of isomorphic maximal factorizations and accordingly a lot of possible cases of simple factorizations should be considered; this will be done in a forthcoming paper. Many of the maximal factorizations of finite classical groups of the list above, involve (maximal) subgroups which are stabilizers of totally singular, or nonsingular subspaces of the vector space on which the corresponding groups act naturally. Following the notation in [8], we denote by P_k the subgroup stabilizing a k - dimensional totally singular subspace, and N_i^ϵ(where $\epsilon = \pm$ or blank) means the stabilizer of a nonsingular i - subspace of an appropriate type.

2. Proof of the Theorem. We treat the remaining members of the above list of finite groups of Lie type case by case. If G is such a group, we suppose that $G = AB$ for some proper non-Abelian simple subgroups A and B of G.

a) Let $G = \Omega_9(q)$, where q is a power of an odd prime. There exists a single class of maximal factorizations $G = \overline{A}\ \overline{B}$ in which $\overline{A} = N_1^- \cong \Omega_8^-(q).2$ and $\overline{B} = P_4 \cong [q^{10}] : \frac{1}{2}GL_4(q)$. Using the implicitly given in [9] list of the subgroups of G, by order considerations we only come to the following possibilities: $A \cong \Omega_8^-(q)$ and $B \cong L_4(q)$, or $PSp_4(q)$ (both in a $SL_4(q)$ subgroup of P_4). However, standard linear algebra yields that any E_8 subgroup of $SL_4(q)$, q odd, contains the central involution of $SL_4(q)$. Since the 2 - rank of B is at least 3, B can not be a subgroup of $SL_4(q)$, a contradiction.

b) Let $G = PSp_8(q)$. It has been shown in [6] that $PSp_8(2)$ does not have any factorizations, so we may assume that $q > 2$. Again using the list of the maximal factorizations of G, by order considerations we restrict them to the following possible cases of simple factorizations:

1) $A \cong P\Omega_8^\epsilon(q)$ (in a maximal $O_8^\epsilon(q)$ subgroup of G), $B \cong PSp_4(q^2)$ (in a maximal $PSp_4(q^2).2$ subgroup of G), where q is even and $\epsilon = \pm$;

2) $A \cong P\Omega_8^\epsilon(q)$ (as in case 1)), $B \cong L_2(q^4)$(in $PSp_4(q^2)$ subgroup as in case 1)), where q is even and $\epsilon = \pm$;

3) $A \cong PSp_6(q)$ (in P_1 or in $P\Omega_8^\epsilon(q)$ subgroup as in case 1)), $B \cong PSp_4(q^2)$ (as in case 1)), q is even;

4) $A \cong \Omega_8^-(q)$(in a maximal $O_8^-(q)$ subgroup of G), $B \cong L_4(q)$ (in P_4), and q is even;

5) $A \cong \Omega_8^-(q)$(as in case 4)), $B \cong PSp_4(q)$ (in $PSp_4(q^2)$ subgroup as in case 1), in $L_4(q)$ subgroup as in case 4), or in a maximal $PSp_4(q) wr S_2$ subgroup of G), q is even.

We have already proved in [2] that in case 1) and in case 3) (if A is a subgroup of P_1) no factorizations arise. It follows immediately that the case 2) and the rest of the case 3) there are no possibilities for factorizations too. As for the case 5) we have to exclude B to be a subgroup of $PSp_4(q^2)$ subgroup of G.

Let us deal with case 4) in which $|A \cap B| = q^2(q^3-1)(q^2-1)$. The subgroup structure of $L_4(q)$ leads to only one possibility for $A \cap B$ to be a subgroup of a maximal subgroup of B isomorphic to $[q^3] : GL_3(q)$. It means that in $L_3(q)$ there should exist a proper subgroup of order divisible by $(q^3-1)(q+1)/(3, q-1)$ which is a contradiction to the subgroup structure of the group $L_3(q)$. So this case is eliminated. Nonexistence of the factorization in 4) (and the yet mentioned) reduces the case 5) to only one possibility in which B has to be a subgroup of a maximal $PSp_4(q) wr S_2$ subgroup of G. In this case let us denote by A_1 the subgroup $O_8^-(q)$ of G containing A, and let $B_1 \cong PSp_4(q) \times$

$PSp_4(q)$ be the subgroup (in a maximal subgroup $PSp_4(q)wrS_2$ of G) which contains the subgroup B. It could directly be verified that $A_1 \cap B_1 \cong O_4^+(q) \times O_4^-(q)$. Now, if $B \cong PSp_4(q)$ is an arbitrary subgroup of B_1, it follows that $B \cap A_1 = B \cap (A_1 \cap B_1)$, and $|B \cap (A_1 \cap B_1)| \geq |B| \,.\, |A_1 \cap B_1| \,/\, |B_1| = 4(q^2-1)$. Thus, $|A_1 \cap B| \geq 4(q^2-1)$ which yields $|A \cap B| \geq 2(q^2-1)$. Order considerations imply $G \neq AB$.

We have proved that $G = PSp_8(q)$ does not have any simple factorizations.

c) Let $G = U_8(q)$. Only the following cases of possible simple factorizations have to be taken into consideration:

1) $A \cong U_7(q)$, $B \cong PSp_8(q)$;

2) $A \cong U_7(q)$, $B \cong P\Omega_8^-(q)$(in $PSp_8(q)$ subgroup of G), q is even;

3) $A \cong U_7(q)$, $B \cong L_4(q^2)$;

4) $A \cong U_7(q)$, $B \cong PSp_4(q^2)$(in $L_4(q^2)$ subgroup of G, or in $PSp_8(q)$ subgroup of G).

Here $A \cong U_7(q)$ is a subgroup of $N_1 \cong GU_7(q)/(7, q+1)$, and, therefore, such a group A exists if and only if $(7, q+1) = 1$. Case 1) contains the factorization listed in *(2)* of the theorem. It remains to show that this factorization actually exists. Proposition 3.3 in [10] claims the factorization $SU_8(q) = SU_7(q).Sp_8(q)$ with natural embeddings of $SU_7(q)$ and $Sp_8(q)$ in $SU_8(q)$; the interception group is isomorphic to $Sp_6(q)$ naturally embedded in $SU_8(q)$ too. Factoring out by $Z(SU_8(q))$, we obtain the factorization in *(2)*, as $SU_7(q) \equiv U_7(q)$ (by the condition $(7, q+1) = 1$).

In case 2) (here q is even) B is a subgroup of one $B_1(\cong PSp_8(q))$ subgroup of G. From $G = AB$ we have $G = AB_1$ (a covering factorization of $G = AB$). G has a single conjugacy class of subgroups isomorphic to $U_7(q)$, and a single conjugacy class of subgroups isomorphic to $PSp_8(q)$). It follows (from case 1)) that $A \cap B_1 \cong PSp_6(q)$. Hence, by the orders, the factorization $B_1 = (A \cap B_1).B$ occurs. This is a contradiction to the fact that $PSp_8(q)$ does not have any simple factorizations.

In case 3) $|A \cap B| = q^5(q^6-1)(q^4-1).(8, q+1)/(4, q^2-1)$. A subgroup of $B \cong L_4(q^2)$ of such order could be contained only in a maximal subgroup of B isomorphic to $\{[q^6] : GL_3(q^2)\}/(4, q^2-1)$. Hence, $L_3(q^2)$ contains a proper subgroup of order divisible by $(q^6-1)(q^2+1)/\{2.(3, q^2-1)\}$, which contradicts to the subgroup structure of $L_3(q^2)$.

At last, in case 4) we can reckon (as the case 3) have been excluded) that B is a subgroup of a $B_1 \cong PSp_8(q)$ subgroup of G. Similarly, as in case 2) we have $G = AB_1$ and $B_1 = (A \cap B_1).B$. Here we can suppose that q is odd (otherwise $A \cap B_1 \cong PSp_6(q)$ and B_1 must have a simple factorization, a contradiction). Now, the list of the maximal factorizations of $PSp_8(q)$ leads to only one option: $A \cap B_1 < P_1$. This possibility has already been excluded in [2].

d) Let $G = P\Omega_{10}^-(q)$. The list of maximal factorizations of G is given in [9]. This leads, by order considerations, to the following options:

1) $A \cong A_{12}$, $B \cong \Omega_8^-(q)$ (in P_1), $q = 2$;

2) $A \cong U_5(q)$, $B \cong \Omega_9(q)$ (in N_1), $q \not\equiv -1 \pmod 5$;

3) $A \cong U_5(q)$, $B \cong \Omega_8^-(q)$ (in P_1, or in $\Omega_9(q)$ subgroup in N_1), $q \not\equiv -1 \pmod 5$.

In case 1) $|A \cap B| = 2.3^3.5.7$ and since $A \cap B$ is a subgroup of $A \cap P_1 \cong (A_8 \times A_4) : 2$ (see [9], p.115), we conclude that in A exists a subgroup of order divisible by $3^2.5.7$ and dividing $2.3^3.5.7$ which is impossible.

To discuss the next possibilities we need the following two realizations of the group $A \cong U_5(q)$ ($q \not\equiv -1 \pmod 5$) in G. Let V be the natural 5-dimensional unitary space over the finite field $GF(q^2)$ on which some group $\widetilde{A} \cong U_5(q)$ acts, and let $[\cdot,\cdot]$ be an unitary form on V preserved by $\widetilde{A}$. There is an element $\omega \in GF(q^2)$ such that $1, \omega$ form a basis of $GF(q^2)$ over the subfield $GF(q)$, and $\omega + \omega^q = 1$, $\omega^2 = p_0 + p_1\omega$ where p_0, $p_1 \in GF(q)$. The first realization applies if q is even. There is a basis $\{d, e_1, e_2, f_1, f_2\}$ of V over $GF(q^2)$, called a standard unitary basis, such that $[e_i, e_j] = [f_i, f_j] = 0$, $[e_i, f_j] = \delta_{ij}$, $[d, e_i] = [d, f_i] = 0$, $[d, d] = 1$ for $i, j = 1, 2$. The group $\widetilde{A}$ has the following (matrix) realization with respect to that basis:

$$\widetilde{A} = \left\{ \widetilde{X} \in SL_5(q^2) | \, \overline{\widetilde{X}}^t T \widetilde{X} = T, \; T = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & E_{2\times 2} \\ \hline 0 & E_{2\times 2} & 0 \end{array} \right) \right\},$$

where $\overline{\widetilde{X}}$ stands for the matrix $\widetilde{X}$ in which each entry is replaced by its q^{th} power.

V is 10-dimensional vector space over the subfield $GF(q)$ of the field $GF(q^2)$ with basis $\{d, e_1, e_2, f_1, f_2, \omega d, \omega e_1, \omega e_2, \omega f_1, \omega f_2\}$. The quadratic form $Q(u) = [u, u]$ defined in this space has as its associated a symmetric bilinear form $(\cdot, \cdot)$ ($(u, v) = Q(u + v) - Q(u) - Q(v)$), and it is directly checked that this basis is a standard basis of an orthogonal space of type O^-. Further, let $\widetilde{X} \in \widetilde{A}$, $\widetilde{X} = (\widetilde{x}^0_{ij} + \widetilde{x}^1_{ij}.\omega)^5_1 = \widetilde{X}_0 + \widetilde{X}_1.\omega$ with $\widetilde{X}_0 = (\widetilde{x}^0_{ij})^5_1$, $\widetilde{X}_1 = (\widetilde{x}^1_{ij})^5_1$, $\widetilde{x}^l_{ij} \in GF(q)$ ($l = 0, 1$; $i, j = 1, 2, 3, 4, 5$). With respect to the last basis the following matrices form a subgroup A of G, isomorphic to $U_5(q)$:

$$X = \begin{pmatrix} \widetilde{X}_0 & p_0\widetilde{X}_1 \\ \widetilde{X}_1 & \widetilde{X}_0 + p_1\widetilde{X}_1 \end{pmatrix}.$$

The isomorphism is given by the map $\sigma : \widetilde{X} \mapsto X$.

The second realization does not depend on q. In this case we choose an orthonormal basis $\{v_i\}^5_{i=1}$ of V over $GF(q^2)$, and, hence, $\widetilde{A}$ has the following matrix representation:

$$\widetilde{A} = \left\{ \widetilde{X} \in SL_5(q^2) | \, \overline{\widetilde{X}}^t \widetilde{X} = E \right\}.$$

$\{v_i, \omega v_i\}^5_{i=1}$ is a basis of V as an orthogonal space of type O^- over the field $GF(q)$ (the quadratic form Q and its associated bilinear form $(\cdot, \cdot)$ have already been defined). With respect to that basis the same matrices X as above form a subgroup A of G isomorphic to $U_5(q)$. The isomorphism map is σ.

Case 2) splits into factorizations (3) and (4) of the theorem according to q even or q odd, respectively (recall the well known isomorpism $\Omega_9(q) \cong PSp_8(q)$ for q even). It remains to prove the actual existence of these factorizations. Here we use the second realization of A in G and take $N_1 = G_{\langle v_1 \rangle}$. Direct calculations show that $A \cap N_1 \cong \sigma^{-1}(A \cap N_1) = S : H$, where

$$S = \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & X \end{array} \right) | X \in SL_4(q^2), \overline{X}^t X = E \right\} \cong SU_4(q),$$

$$H = \left\langle \left(\begin{array}{c|c} -E_{2\times 2} & 0 \\ \hline 0 & E_{3\times 3} \end{array} \right) \right\rangle \cong Z_{(2,q-1)}.$$

Thus, $A \cap N_1 \cong SU_4(q) : (2, q - 1)$. If q is even, or $q \equiv -1 \pmod 4$, then $N_1 \cong \Omega_9(q)$ and so $B = N_1$. Now, by order considerations, it follows the factorization in (3) or the factorization in (4) (with $q \equiv -1 \pmod 4$) of the theorem. If $q \equiv 1 \pmod 4$, then $N_1 = B. \langle r_{v_1}.r_{v_2} \rangle$, where r_{v_1} and r_{v_2} are the reflections in v_1 and v_2, respectively (see [8], Proposition 4.1.6, case $m_1 = 1$). It is directly checked that

$$\sigma\left(\begin{array}{c|c} -E_{2\times 2} & 0 \\ \hline 0 & E_{3\times 3} \end{array}\right) = \widehat{r_{v_1}}.\widehat{r_{v_2}},$$

where $\widehat{r_{v_1}}$ and $\widehat{r_{v_2}}$ are the matrices of the reflections r_{v_1} and r_{v_2} with respect to the basis $\{v_i, \omega v_i\}_{i=1}^5$ considered above. Hence, $A \cap B \cong SU_4(q)$ and again $G = AB$.

Discussing case 3), let us first suppose that B is a subgroup of $\Omega_9(q)$, a subgroup of G which is contained in N_1. Here we have $|A \cap B| = q^2(q^3+1)(q^2-1).(4, q^5+1)/(4, q^4+1)$. Moreover, $A \cap B$ is contained (as a proper subgroup) in the stabilizer in A of one-dimensional nonsingular subspace (over the field $GF(q^2)$) which is isomorphic to $GU_4(q)$. It follows immediately that in $U_4(q)$ exists a proper subgroup of order divisible by $q^2(q^2-q+1)(q^2-1)/(2, q-1)$ which is impossible if $q > 2$. If $q = 2$, from the shown in the previous case, we obtain the factorization $PSp_8(2) = U_4(2)\Omega_8^-(2)$, but $PSp_8(2)$ has no simple factorization, a contradiction.

Finally, let B is a subgroup in P_1. This time $A \cap B$ is a proper subgroup of the stabilizer in A of one-dimensional singular subspace (over the field $GF(q^2)$) with the following structure: $[q^7] : (SU_3(q).[q^2-1])$. It follows that $U_3(q)$ has a proper subgroup of order divisible by $(q^3+1)/(3, q+1)$ which is possible if and only if $q = 2$. So $A \cong U_5(2)$, $B \cong \Omega_8^-(2)$, and B is a subgroup of $P_1 \cong [2^8] : \Omega_8^-(2)$. Since there is just one conjugacy class of subgroups $U_5(2)$ in G, we can consider A in its first realization given above. It is convenient to regard $P_1 = G_{\langle \omega f_2 \rangle}$. For a standard subgroup $B(\cong \Omega_8^-(2))$ in P_1 direct computations show that $A \cap B \cong SU_3(2)$ naturally embedded in G. So, by the orders, it follows that $G \neq AB$. Now, let $B_1 \cong \Omega_8^-(2)$ be any subgroup of P_1 different from the standard subgroup B. On one hand, obviously, $|B \cap B_1| \geq |B|^2 / |P_1| = |B| / 2^8$ and, hence, $[B : B \cap B_1] \leq 2^8$. On the other hand, $B \cap B_1$ is contained in a (maximal) subgroup of B stabilizing one - dimensional isotropic subspace. Such a stabilizer has the following structure: $[2^6] : U_4(2)$. So either $B \cap B_1 = [2^6] : U_4(2)$, or $B \cap B_1$ is a subgroup of index 2 in $[2^6] : U_4(2)$. Now, it could be seen directly that the naturally embedded in G subgroup $A \cap B \cong SU_3(2)$ is contained in $B \cap B_1^g$ (for some $g \in B$). Hence, $G \neq AB$ in any case.

This completes our considerations. The theorem is proved.

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ФАКТОРИЗАЦИИ НА НЯКОИ ГРУПИ ОТ ЛИЕВ ТИП И ЛИЕВ РАНГ 4

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В настоящата работа се разглеждат крайни прости групи G , които могат да се представят като произведение на две свои собствени неабелеви прости подгрупи A и B. Всяко такова представяне $G = AB$ е прието да се нарича факторизация на G, а тъй като множителите A и B са избрани да бъдат прости подгрупи на G, то разглежданите факторизации са известни още като прости факторизации на G. Тук се предполага, че G е проста група от лиев тип и лиев ранг 4 над крайно поле $GF(q)$. В случай, че G не е група от серията на ортогоналните групи $P\Omega_8^+(q)$ са определени всички възможни прости факторизации на G.

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Ключови думи: крайни прости групи, групи от лиев тип, факторизации на групи.