МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2010
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2010
Proceedings of the Thirty Ninth Spring Conference of
the Union of Bulgarian Mathematicians
Albena, April 6–10, 2010

ℓ-STABLE FUNCTIONS AND CONSTRAINED OPTIMIZATION*

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The class of ℓ-stable at a point functions defined in [2] and being larger than the class of $C^{1,1}$ functions, it is generalized from scalar to vector functions. Some properties of the ℓ-stable vector functions are proved. It is shown that constrained vector optimization problems with ℓ-stable data admit second-order conditions in terms of directional derivatives, which generalizes the results from [2] and [5].

1. Introduction. We deal with the constrained vector optimization problem

$$\min_C f(x)\,, \quad g(x) \in -K\,, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are given functions, $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ are pointed closed convex cones, and n, m and p are positive integers. When f and g are ℓ-stable functions at a point x^0, then we derive second-order optimality conditions x^0 to be a solution of this problem. The paper generalizes the results of [5] from problems with $C^{1,1}$ data to problems with ℓ-stable data and those of [2] from scalar unconstrained problems to vector constrained problems. Classical second-order conditions assume C^2 data. We call the problem nonsmooth if at least one of the functions f and g is not C^2. Problems with $C^{1,1}$ data are brought in optimization by J.-B. Hiriart-Urruty, J.-J Strodiot, V. Hien Nguen [7] and since then are investigated by many authors especially with respect to second-order optimality conditions in nonsmooth optimization, say J.-P. Penot, X. Q. Yang, D. Klatte, K. Tammer, V. Jeyakumar, D. T. Luc, L. Liu, P. Georgiev, N. Zlateva etc. References can be found e.g. in [2], [6] and [5]. Recall that a function is called $C^{1,1}$ if it is Fréchet differentiable with locally Lipschitz derivative. In Ginchev, Guerraggio, Rocca [6] second-order conditions are established in terms of the Dini second-order directional derivative for $C^{1,1}$ unconstrained problems, both scalar and vector. These conditions are generalized in [5] for constrained problems of type (1). Motivated by [6], Bednařík, Pastor introduce in [2] the class of ℓ-stable at a point functions as a generalization of the class of $C^{1,1}$ functions and show that unconstrained scalar problems with ℓ-stable data admit second-order conditions similar to those of [6]. The ℓ-stable functions need not be differentiable beyond the reference point x^0, while this is not the case with the $C^{1,1}$ (near x^0) functions. For this reason they concord better with the optimality conditions, which do not assume differentiability beyond the reference point. For this reason we consider the ℓ-stable at a point functions as an important class of functions. In the present paper

*2000 Mathematics Subject Classification: 90C29, 90C30, 90C46, 49J52.
Key words: Vector optimization, ℓ-stable functions, second-order conditions.

we generalize the results of [2] to constraint vector problems of type (1) with ℓ-stable data.

2. Basic notions and ℓ-stable functions. For the norm and the dual parity in the considered finite-dimensional spaces we write $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, S stands for the unit sphere in $\mathbb{R}^n$ and $B(x,r)$ for the closed ball with center x and radius r. The polar cone of the cone $M \subset \mathbb{R}^k$ is defined by $M' = \{\zeta \in \mathbb{R}^k \mid \langle\zeta,\, \phi\rangle \geq 0 \text{ for all } \phi \in M\}$ and its second polar cone is $M'' = \{\phi \in \mathbb{R}^k \mid \langle\zeta,\, \phi\rangle \geq 0 \text{ for all } \zeta \in M'\}$. We set $M'[\phi] = \{\zeta \in M' \mid \langle\zeta,\, \phi\rangle = 0\}$. Then $M'[\phi] \subset M'$ and $M[\phi] := (M'[\phi])'$ is a closed convex cone. It can be shown that $M[\phi]$ is the contingent cone [1] of M at ϕ. In this paper we apply the notation $M[\phi]$ for $M = K$ and $\phi = -g(x^0)$; then we deal with the cone $K[-g(x^0)]$. For the closed convex cone M' we define $\Gamma_{M'} = \{\zeta \in M' \mid \|\zeta\| = 1\}$. Given a set $A \subset \mathbb{R}^k$, then the distance from $y \in \mathbb{R}^k$ to A is $d(y,A) = \inf\{\|a-y\| \mid a \in A\}$. The oriented distance from y to A is defined by $D(y,A) = d(y,A) - d(y, \mathbb{R}^k \setminus A)$. It is known [5] that when $M \subset \mathbb{R}^k$ is a closed convex cone, then $D(y,-M) = \sup_{\xi\in\Gamma_{M'}} \langle\xi,\, y\rangle$.

We call the (local) solutions of problem (1) minimizers. A point x^0 satisfying the constraint $x^0 \in -K$ is called feasible for (1). The point x^0 is said to be an i-minimizer (isolated minimizer) of order k for (1), $k > 0$, if x^0 is feasible and there exists a neighbourhood U of x^0 and a constant $A > 0$ such that $D(f(x)-f(x^0), -C) \geq A\,\|x-x^0\|^k$ for $x \in U \cap g^{-1}(-K)$. In [5] through the oriented distance the following characterization is derived: A feasible point x^0 is an i-minimizer of order k for (1) if and only if x^0 is an i-minimizer of order k for the scalar problem

$$\min D(f(x) - f(x^0), -C)\,, \quad g(x) \in -K\,. \tag{2}$$

The lower directional derivative of the function $\varphi : \mathbb{R}^k \to \mathbb{R}$ at the point $x \in \mathbb{R}^n$ in direction $u \in \mathbb{R}^n$ is defined by $\varphi^\ell(x,u) = \liminf_{t\to 0^+} \frac{1}{t}(\varphi(x+tu) - \varphi(x))$. The function φ is called ℓ-stable at x (see [2]) if there exist a neighbourhood U of x and $\kappa > 0$ such that

$$|\varphi^\ell(y,u) - \varphi^\ell(x,u)| \leq \kappa\|y-x\|, \quad \forall y \in U, \quad \forall u \in S\,.$$

The main properties of the ℓ-stable functions are summarized in the next theorem:

Theorem 1 ([2], [3])**.** *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be ℓ-stable at x. Then φ is Lipschitz near x and Fréchet differentiable at x.*

Actually, this result is proved in [2] under the assumption that φ is continuous near x, while in [3] it is shown that the continuity hypothesis is redundant.

We define the ℓ-stability for vector functions as follows:

Definition 1. *Let $M \subset \mathbb{R}^k$ be a closed convex cone. We define the function $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ to be ℓ-stable at $x \in \mathbb{R}^n$ with respect to M if there exist a neighbourhood U of x and $\kappa > 0$ such that*

$$|\Phi^\ell_\zeta(y,u) - \Phi^\ell_\zeta(x,u)| \leq \kappa\,\|y-x\|, \quad \forall\zeta \in \Gamma_{M'}, \quad \forall y \in U, \quad \forall u \in S. \tag{3}$$

Here $\Phi_\zeta : \mathbb{R}^n \to \mathbb{R}$ is defined by $\Phi_\zeta(y) = \langle\Phi(y), \zeta\rangle$. We say that Φ is ℓ-stable at $x \in \mathbb{R}^n$ if it is ℓ-stable at x with respect to some pointed closed convex cone M.

Obviously, each $C^{1,1}$ vector function $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ is ℓ-stable (with respect to any closed convex cone $M \subset \mathbb{R}^k$). Examples in [2] show that the converse is not true even for scalar functions. The following theorem is a generalization of Theorem 1 to vector functions.

Theorem 2. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ be ℓ-stable at x with respect to the pointed closed convex cone M. Then, Φ is Lipschitz near x and Fréchet differentiable at x.*

Proof. We observe first that $\mathbb{R}^k = M' - M'$. Indeed, since M is pointed, which means $M \cap (-M) \subset \{0\}$, we deduce that $\operatorname{int} M' \neq \emptyset$ (otherwise the linear span of the convex cone M' would be a proper linear subspace $L \subset \mathbb{R}^k$ and, hence, its polar L' would be a proper linear subspace contained in M – a contradiction). Therefore, each $\zeta \in \mathbb{R}^k$ admits a representation $\zeta = \alpha_1 \zeta^1 + \alpha_2 \zeta^2$ with $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\zeta^1, \zeta^2 \in \Gamma_{M'}$. By definition the scalar functions $\Phi_{\zeta^i}(\cdot) = \langle \zeta^i, \Phi(\cdot) \rangle$, $i = 1, 2$, are ℓ-stable at x, hence, by Theorem **??** for any $x \in \mathbb{R}^n$ they are Lipschitz near x and Fréchet differentiable at x. Then, $\Phi_\zeta = \alpha_1 \Phi_{\zeta^1} + \alpha_2 \Phi_{\zeta^2}$ implies that Φ_ζ is Lipschitz near x and Fréchet differentiable at x.

Let $e^j = (0, \dots, 1, \dots, 0)$, $j = 1, \dots, k$, be the canonical basis in $\mathbb{R}^k$ (the only unit in e^j is on j-th place). Then,

$$\Phi(\cdot) = \sum_{j=1}^{k} \langle e^j, \Phi(\cdot) \rangle \, e^j = \sum_{j=1}^{k} \Phi_{e^j}(\cdot) \, e^j .$$

Now, the Lipschitz continuity of Φ near x and the Fréchet differentiability of Φ at x follows on the basis of Theorem 1 from those of Φ_{e^j}, $j = 1, \dots, k$. $\square$

In the proof of Theorem 2 we have used only that the functions Φ_ζ, $\zeta \in \Gamma_{M'}$, are ℓ-stable at x, which is less restrictive than the condition Φ to be ℓ-stable at x with respect to M.

Definition 2 ([6]). *Given a function $\Phi : \mathbb{R}^n \to \mathbb{R}^k$, possessing a Fréchet derivative $\Phi'(x)$ at $x \in \mathbb{R}^n$ we define the second-order (set-valued) Dini directional derivative of Φ at x in direction $u \in \mathbb{R}^n$ as the Painlevé-Kuratowski limit*

$$\Phi''_u(x) = \operatorname*{Limsup}_{t \to 0^+} \frac{2}{t^2} \left(\Phi(x + tu) - \Phi(x) - \Phi'(x) \, u \right) .$$

Theorem 3. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ be ℓ-stable at x with respect to the pointed closed convex cone M. Then, there exist constants $\delta > 0$ and $\alpha > 0$ such that for all $u, v \in \mathbb{R}^n$ and all $t \in \mathbb{R}$ with $0 < t < \delta / \max(\|u\|, \|v\|)$ it holds*

$$\left\| \frac{2}{t^2} \left(\Phi(x + tu) - \Phi(x) - t \, \Phi'(x) u \right) - \frac{2}{t^2} \left(\Phi(x + tv) - \Phi(x) - t \, \Phi'(x) v \right) \right\| \le \alpha \left(\|u\| + \|v\| \right) \|u - v\| . \tag{4}$$

In particular, if $v = 0$, then $\left\| \frac{2}{t^2} \left(\Phi(x + tu) - \Phi(x) - t \, \Phi'(x) u \right) \right\| \le \alpha \, \|u\|^2$.

Proof. Let $\delta > 0$ be such that inequality (3) be satisfied and Φ be Lipschitz (hence, continuous) on $U = B(x, \delta)$. Fixing u and v consider $t \in (0, \delta / \max(\|u\|, \|v\|))$. For any such t choose $\zeta^t \in \Gamma_{M'}$ so that

$$\begin{aligned} &|\langle \zeta^t, \Phi(x + tu) - \Phi(x + tv) - t \Phi'(x)(u - v) \rangle| \\ &= \sup\nolimits_{\zeta \in \Gamma_{M'}} |\langle \zeta, \Phi(x + tu) - \Phi(x + tv) - t \Phi'(x)(u - v) \rangle| \, . \end{aligned}$$

Put $L = \inf_{\|e\|=1} \sup_{\zeta \in \Gamma_{M'}} |\langle \zeta, e \rangle|$. Since $\operatorname{int} \Gamma_{M'} \neq \emptyset$, we have $L > 0$. In the sequel we use the following result being a corollary of the Diewert Mean Value Theorem [4] for the continuous function Φ_{ζ^t}: There exist $\tau^-, \tau^+ \in [0, 1]$ such that

$$\Phi^\ell_{\zeta^t}(x + ta^-, u - v) \le \frac{\Phi_{\zeta^t}(x + tu) - \Phi_{\zeta^t}(x + tv)}{t} \le \Phi^\ell_{\zeta^t}(x + ta^+, u - v)$$

with $a^- = (1 - \tau^-)u + \tau^- v$ and $a^+ = (1 - \tau^+)u + \tau^+ v$ (and applying this result taking τ to be either τ^- or τ^+). We get the estimations

$$\begin{aligned}
&\left\| \frac{2}{t^2} \left(\Phi(x + tu) - \Phi(x) - t\,\Phi'(x)u\right) - \frac{2}{t^2} \left(\Phi(x + tv) - \Phi(x) - t\,\Phi'(x)v\right) \right\| \\
&\quad \le \frac{2}{Lt^2} \, \| \, \Phi_{\zeta^t}(x + tu) - \Phi_{\zeta^t}(x + tv) - t\langle \zeta^t, \Phi'(x)(u - v)\rangle \, \| \\
&\quad \le \frac{2}{Lt} \, \| \, \Phi^\ell_{\zeta^t}(x + t((1 - \tau)u + \tau v), u - v) - \Phi^\ell_{\zeta^t}(x, u - v) \, \| \\
&\quad \le \frac{2\kappa}{L} \, \|(1 - \tau)u + \tau v\| \, \|u - v\| \le \frac{2\kappa}{L} \left(\|u\| + \|v\|\right) \|u - v\| \,,
\end{aligned}$$

which is (4) with $\alpha = 2\kappa/L$. □

The above proof used the Diewert Mean Value Theorem, which is given below for the sake of completeness.

Theorem 4 (Diewert Mean Value Theorem [4])**.** *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and $\psi : [\alpha, \beta] \to \mathbb{R}$ be lower semicontinuous function. Then there exists $\gamma \in [\alpha, \beta)$ such that*

$$\psi^\ell(\gamma, 1) \ge \frac{\psi(\beta) - \psi(\alpha)}{\beta - \alpha}$$

3. Optimality conditions. While the proofs of the auxiliary results of Theorems 2 and 3 show essential differences with the corresponding results in [5], the proof of Theorem below is similar to that for $C^{1,1}$ functions.

Theorem 5 (Sufficient conditions)**.** *Let x^0 be a feasible point for problem* (1) *with f and g being ℓ-stable at x^0 with respect to the pointed closed convex cones C and K correspondingly. Suppose that for each $u \in \mathbb{R}^n \setminus \{0\}$ one of the following two conditions is satisfied:*

$\mathbb{S}'$: $$(f'(x^0)u,\, g'(x^0)u) \notin -(C \times K[-g(x^0)]) \,,$$

$\mathbb{S}''$: $$\begin{gathered}(f'(x^0)u,\, g'(x^0)u) \in -(C \times K[-g(x^0)] \setminus \operatorname{int} C \times \operatorname{int} K[-g(x^0)]) \\ \text{and } \forall\, (y^0, z^0) \in (f,\, g)''_u(x^0) : \exists\, (\xi^0, \eta^0) : \\ (\xi^0, \eta^0) \in C' \times K'[-g(x^0)] \setminus \{(0, 0)\}, \\ \langle \xi^0,\, f'(x^0)u\rangle + \langle \eta^0,\, g'(x^0)u\rangle = 0,\ \langle \xi^0,\, y^0\rangle + \langle \eta^0,\, z^0\rangle > 0 \,.\end{gathered}$$

Then, x^0 is an i-minimizer of order two for problem (1)*.*

Proof. Suppose that x^0 is not an i-minimizer of order two for (1). We claim that then there exists u^0, for which no one of the conditions $\mathbb{S}'$ and $\mathbb{S}''$ is satisfied. Choose a monotone decreasing sequence $\varepsilon_k \to 0^+$. Then, by assumption there exist sequences $t_k \to 0^+$ and $u^k \in \mathbb{R}^n$, $\|u^k\| = 1$, such that $g(x^0 + t_k u^k) \in -K$ and

$$D(f(x^0 + t_k u^k) - f(x^0), -C) = \max_{\xi \in \Gamma_{C'}} \langle \xi,\, f(x^0 + t_k u^k) - f(x^0)\rangle < \varepsilon_k t_k^2 \,. \tag{5}$$

Passing to a subsequence, due to the compactness of S, we may assume that $u^k \to u^0$. We may assume according to the boundedness claimed in Theorem 3, that $y^{0,k} \to y^0$, $z^{0,k} \to z^0$, where $y^{0,k}$ and $z^{0,k}$ (and similarly y^k and z^k) are defined by

$$y^{0,k} = \frac{2}{t_k^2}\left(f(x^0+t_k u^0) - f(x^0) - t_k\, f'(x^0)u^0\right)\,,$$
$$y^{k} = \frac{2}{t_k^2}\left(f(x^0+t_k u^k) - f(x^0) - t_k\, f'(x^0)u^k\right)\,,$$
$$z^{0,k} = \frac{2}{t_k^2}\left(g(x^0+t_k u^0) - g(x^0) - t_k\, g'(x^0)u^0\right)\,,$$
$$z^{k} = \frac{2}{t_k^2}\left(g(x^0+t_k u^k) - g(x^0) - t_k\, g'(x^0)u^k\right)\,.$$

Now we have $(y^0, z^0) \in (f(x^0), g(x^0))''_u$. We may assume also that $0 < t_k < r$ and both f and g are Lipschitz and satisfy inequality (4) (mutatis mutandis) with constants κ_f and κ_g respectively on $B(x^0, r)$. Now we have $y^k \to y^0$ (and similarly $z^k \to z^0$) obtained on the basis of the estimations from Theorem 3

$$\|y^k - y^0\| \le \|y^k - y^{0,k}\| + \|y^{0,k} - y^0\| \le \alpha_f(\|u^k\| + \|u^0\|)\,\|u^k - u^0\| + \|y^{0,k} - y^0\|\,.$$

We prove that $\mathbb{S}'_p$ is violated at u^0, that is

$$f'(x^0)u^0 \in -C\,, \quad g'(x^0)u^0 \in -K[-g(x^0)]\,. \tag{6}$$

The first inclusion in (6) comes from $D(f'(x^0)u^0), -C) = \max_{\xi\in\Gamma_{C'}}\langle\xi, f'(x^0)u^0\rangle < \varepsilon$, $\forall\varepsilon > 0$. This follows from

$$\langle\xi,\, f'(x^0)u^0\rangle = \left\langle\xi,\, \frac{1}{t_k}\left(f(x^0+t_k u^k) - f(x^0)\right)\right\rangle$$
$$+\left\langle\xi,\, f'(x^0)u^k - \frac{1}{t_k}\left(f(x^0+t_k u^k) - f(x^0)\right)\right\rangle + \langle\xi,\, f'(x^0)(u^0 - u^k)\rangle\,,$$

since each term on the right-hand side can be made arbitrary small uniformly on $\xi \in \Gamma_{C'}$ (the first one is due to (5), the second one is due to the Fréchet differentiability of f at x^0, the third one because $u^k \to u^0$). Similar estimations can be repeated substituting f for g and $\xi \in \Gamma_{C'}$ for $\eta \in K[-g(x^0)]'$, which gives the second inclusion in (6). The only difference occurs with the first estimation, now

$$\left\langle\eta,\, \frac{1}{t_k}\left(g(x^0+t_k u^k) - g(x^0)\right)\right\rangle = \frac{1}{t_k}\,\langle\eta,\, g(x^0+t_k u^k)\rangle \le 0\,,$$

since $g(x^0 + t_k u^k) \in -K$.

We prove that $\mathbb{S}''$ is violated at u^0. To finishe we assume that

$$(f'(x^0)u,\, g'(x^0)u) \in -(C \times K[-g(x^0)] \setminus \operatorname{int} C \times \operatorname{int} K[-g(x^0)])\,,$$

since otherwise the first requirement in condition $\mathbb{S}''$ would not be satisfied. Let (y^0, z^0) be taken as above and suppose that (ξ^0, η^0) can be chosen so that $\mathbb{S}''$ be satisfied. Then,

$$\langle\xi^0,\, y^0\rangle + \langle\eta^0,\, z^0\rangle = \lim_k\left(\langle\xi^0,\, y^k\rangle + \langle\eta^0,\, z^k\rangle\right)$$
$$= \lim_k\Bigg(\frac{2}{t_k^2}\,\langle\xi^0,\, f(x^0+t_k u^k) - f(x^0)\rangle + \frac{2}{t_k^2}\,\langle\eta^0,\, g(x^0+t_k u^k) - g(x^0)\rangle$$
$$- \frac{2}{t_k^2}\left(\langle\xi^0,\, f'(x^0)u^k\rangle + \langle\eta^0,\, g'(x^0)u^k\rangle \le 0\right)\Bigg)$$
$$\le \limsup_k \frac{2}{t_k^2}\, D(f(x^0+t_k u^k) - f(x^0), -C) + \limsup_k \frac{2}{t_k^2}\,\langle\eta^0,\, g(x^0+t_k u^k)\rangle$$
$$\le \limsup_k \frac{2}{t_k^2}\,\varepsilon_k t_k^2 = 0\,,$$

a contradiction.

The hypotheses of Theorem 5 can be relaxed to ℓ-stable f and g, instead of ℓ-stable with respect to C and K, respectively.

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ℓ-УСТОЙЧИВИ ФУНКЦИИ И УСЛОВНА ОПТИМИЗАЦИЯ

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Класът на ℓ-устойчивите в точка функции, дефиниран в [2] и разширяващ класа на $C^{1,1}$ функциите, се обобщава от скаларни за векторни функции. Доказани са някои свойства на ℓ-устойчивите векторни функции. Показано е, че векторни оптимизационни задачи с ограничения допускат условия от втори ред изразени чрез посочни производни, което обобщава резултати от [2] и [5].