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NONINEAR INTEGRAL INEQUALITIES INVOLVING MAXIMA*

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This paper deals with some integral inequalities that involve the maximum of the unknown scalar function of one variable. The considered inequalities are generalizations of the classical integral inequality of Bihari. The importance of these integral inequalities is due to their wide applications in qualitative investigations of various properties of solutions of differential equations with “maxima” and it is illustrated by some direct applications.

1. Introduction. In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in the control theory correspond to the maximal deviation of the regulated quantity ([6]) and they are adequately modeled by differential equations with “maxima”([2], [4]). The qualitative investigation of properties the solutions of differential equations with “maxima” requires building of appropriate mathematical techniques. One of the main mathematical tools, employed successfully for studying existence, uniqueness, continuous dependence, comparison, perturbation, boundedness, and stability of solutions of differential and integral equations is the method of integral inequalities. This method is well studied for ordinary differential equation and delay differential equations ([1], the monograph [3] and cited therein references). At the same time there are only few partial results for integral inequalities containing maximum value of the unknown function ([5]).

The purpose of this paper is to establish some new integral inequalities in the case when maxima of the unknown scalar function is involved into the integral. The direct application of the obtained results are illustrated on an differential equation with “maxima”.

2. Main results. Let $t_0,\ T$ be fixed points such that $0 \le t_0 < T \le \infty$.

Theorem 1. *Let the following conditions be satisfied:*

1. *The functions* $\alpha_i \in C^1([t_0,T),\mathbb{R}_+)$ *are nondecreasing and* $\alpha_i(t) \le t$ *on* $[t_0,T)$ *for* $i = 1,\dots,n$.
2. *The functions* $a_i,\ b_i \in C([\alpha_i(t_0),\alpha_i(T)),\mathbb{R}_+)$ *for* $i = 1,\dots,n$.
3. *The function* $\phi \in C([t_0-h,t_0],[k,\infty))$ *where* $h,k = \text{const} \ge 0$.

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4. *The functions* $g_i \in C(\mathbb{R}_+, (0,\infty))$ *are increasing* $(i = 1, \dots, n)$.

5. *The function* $u \in C([A-h, T), \mathbb{R}_+)$ *and satisfies the following inequalities*

$$(1)\quad u(t) \le k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s) g_i\Big(u(s)\Big) + b_i(s) g_i\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\Big] ds \quad \text{for } t \in [t_0, T),$$

$$(2)\quad u(t) \le \phi(t) \qquad \text{for } t \in [A-h, t_0],$$

where $A = \min_{1\le i\le n} \alpha_i(t_0)$.

Then, for $t_0 \le t \le t_1$ *the inequality*

$$(3)\qquad u(t) \le G^{-1}\left(G(k) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s) + b_i(s)\Big] ds\right)$$

holds, where G^{-1} *is the inverse function of*

$$(4)\qquad G(r) = \int_{r_0}^{r} \frac{ds}{q(s)}, \quad r_0 > 0, \quad \text{and} \quad q(t) = \max_{1\le i\le n} g_i(t),$$

$$t_1 = \sup\Big\{\tau \ge t_0 : G(k) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s) + b_i(s)\Big] ds \in \operatorname{Dom}\left(G^{-1}\right) \quad \text{for } t \in [t_0, \tau]\Big\}.$$

Proof. Define a function $z(t) : [A-h, T) \to \mathbb{R}_+$ by the equalities

$$z(t) = \begin{cases} k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s) g_i\big(u(s)\big) + b_i(s) g_i\big(\max_{\xi\in[s-h,s]} u(\xi)\big)\Big] ds, & t \in [t_0, T) \\ k, & t \in [A-h, t_0] \end{cases}.$$

The function $z(t)$ is nondecreasing, $z(t_0) = k$ and the inequality $u(t) \le z(t)$ holds for $t \in [A-h, T)$. Then, from inequality (1) we get for $t \in [t_0, T)$

$$z(t) \le k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s) g_i\Big(z(s)\Big) + b_i(s) g_i\Big(\max_{\xi\in[s-h,\, s]} z(\xi)\Big)\Big] ds$$

$$(5)\qquad \le k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s) + b_i(s)\Big] g_i\Big(z(s)\Big) ds.$$

Define a function $w : [A-h, T) \to [k, +\infty)$ by the equality

$$w(t) = k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s) + b_i(s)\Big] g_i\Big(z(s)\Big) ds.$$

Differentiate the function $w(t)$, use its monotonicity and the definition of the function $q(t)$, and obtain

$$\big(w(t)\big)' = \sum_{i=1}^{n} \Big[a_i\big(\alpha_i(t)\big) + b_i\big(\alpha_i(t)\big)\Big] g_i\Big(z\big(\alpha_i(t)\big)\Big)\big(\alpha_i(t)\big)'$$

$$\leq q\Big(w(t)\Big) \sum_{i=1}^{n} \Big[a_i\big(\alpha_i(t)\big) + b_i\big(\alpha_i(t)\big)\Big]\big(\alpha_i(t)\big)'. \tag{6}$$

From definition (4) and inequality (6) it follows that

$$\frac{d}{dt} G\Big(z(t)\Big) = \frac{\big(z(t)\big)'}{q\Big(z(t)\Big)} \leq \sum_{i=1}^{n} \Big[a_i\big(\alpha_i(t)\big) + b_i\big(\alpha_i(t)\big)\Big]\big(\alpha_i(t)\big)'. \tag{7}$$

Integrate inequality (7) from t_0 to t, $t \in [t_0, T)$, change the variable $\eta = \alpha_i(s)$ $(i = 1, \ldots, n)$, and obtain

$$G\Big(z(t)\Big) \leq G(k) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(\eta) + b_i(\eta)\Big] d\eta. \tag{8}$$

Since G^{-1} is an increasing function, from inequalities (8) and $u(t) \leq z(t)$ we obtain the required inequality (3). $\square$

Theorem 2. *Let the following conditions be fulfilled:*

1. *Conditions* 1, 2, 3, 4 *of Theorem* 1 *are satisfied, with* $k \geq 1$.
2. *The function* $u \in C([A-h, T), [1, +\infty))$ *and satisfies the inequalities*

$$u(t) \leq k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s) u(s) g_i\Big(\log u(s)\Big)$$

$$+ b_i(s) u(s) g_i\Big(\log\Big(\max_{\xi \in [s-h, s]} u(\xi)\Big)\Big)\Big] ds \qquad \textit{for } t \in [t_0, T), \tag{9}$$

$$u(t) \leq \phi(t) \qquad \textit{for } t \in [A-h, t_0]. \tag{10}$$

Then, for $t_0 \leq t \leq t_2$ *the inequality*

$$u(t) \leq \exp\left(G^{-1}\left[G\big(\log k\big) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s) + b_i(s)\Big] ds\right]\right) \tag{11}$$

holds, where the functions $G(u)$, G^{-1} *and* $q(t)$ *are defined by equalities* (4), *and*

$$t_2 = \sup\Big\{\tau \geq t_0 : G\big(\log k\big) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s) + b_i(s)\Big] ds \in \mathrm{Dom}\left(G^{-1}\right) \quad \textit{for } t \in [t_0, \tau]\Big\}.$$

Proof. Define the function $z(t) : [A-h,T) \to [1,+\infty)$ by

$$z(t) = \begin{cases} k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s)u(s)g_i\Big(\log u(s)\Big) \\ \qquad\qquad +b_i(s)u(s)g_i\Big(\log\Big(\max_{\xi\in[s-h,s]} u(\xi)\Big)\Big)\Big] ds, & t \in [t_0,T) \\ k, & t\in[A-h,t_0] \end{cases}.$$

Function $z(t)$ is nondecreasing, $z(t_0) = k$ and the inequality $u(t) \le z(t)$ holds for $t \in [A-h,T)$. Then from inequality (9) we get for $t \in [t_0,T)$

$$\begin{aligned} z(t) &\le k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s)z(s)g_i\Big(\log z(s)\Big) \\ &\quad +b_i(s)z(s)g_i\Big(\log\Big(\max_{\xi\in[s-h,s]} z(\xi)\Big)\Big)\Big] ds \\ &\le k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s)+b_i(s)\Big] z(s)g_i\Big(\log z(s)\Big)ds. \qquad (12) \end{aligned}$$

Let us define a function $w : [A-h,T) \to [k,+\infty)$ by the equality

$$w(t) = k + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s)+b_i(s)\Big] z(s)g_i\Big(\log z(s)\Big)ds.$$

Differentiate the function $w(t)$, use its monotonicity and the definition of function $q(t)$, and get

$$\begin{aligned} \big(w(t)\big)' &= \sum_{i=1}^{n} \Big[a_i\big(\alpha_i(t)\big)+b_i\big(\alpha_i(t)\big)\Big] z(t)g_i\Big(\log z(t)\Big)\big(\alpha_i(t)\big)' \\ &\le w(t)q\Big(\log w(t)\Big)\sum_{i=1}^{n} \Big[a_i\big(\alpha_i(t)\big)+b_i\big(\alpha_i(t)\big)\Big]\big(\alpha_i(t)\big)'. \qquad (13) \end{aligned}$$

From inequality (13) it follows that

$$\frac{d}{dt}G\Big(\log z(t)\Big) = \frac{\big(z(t)\big)'}{z(t)q\Big(\log z(t)\Big)} \le \sum_{i=1}^{n} \Big[a_i\big(\alpha_i(t)\big)+b_i\big(\alpha_i(t)\big)\Big]\big(\alpha_i(t)\big)'. \qquad (14)$$

Integrate (14) from t_0 to t, $t \in [t_0,T)$, change the variable $\eta = \alpha_i(s)$, $(i = 1,\dots,n)$, and get

$$G\Big(\log z(t)\Big) \le G(\log k) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(\eta)+b_i(\eta)\Big]d\eta. \qquad (15)$$

Since G^{-1} is an increasing function, from inequality (15) it follows that for $t \in [t_0,T)$ and $i = 1,\dots,n$ the following inequality

$$z(t) \le \exp\Bigg(G^{-1}\Big[G(\log k) + \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \Big[a_i(s)+b_i(s)\Big]ds\Big]\Bigg) \qquad (16)$$

holds.

From inequalities (16) and $u(t) \le z(t)$ we obtain the required inequality (11). $\square$

3. Applications. We apply some of the above inequalities to obtain bounds of the solutions of the following nonlinear differential equation with "maxima"

$$x' = f\Big(t,\ x(t),\ \max_{s\in[\beta(t),\alpha(t)]} x(s)\Big) \quad \text{for} \quad t \ge t_0 \tag{17}$$

with initial condition

$$x(t) = \varphi(t - t_0) \qquad \text{for} \quad t \in [t_0 - h, t_0], \tag{18}$$

where $x \in \mathbb{R},\ \varphi : [-h, 0] \to \mathbb{R},\ f : [0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h > 0$ is a constant.

Let the following conditions be satisfied:

(**H1**). The functions $\alpha,\ \beta \in C([t_0,\infty), \mathbb{R}_+),\ \alpha(t)$ is an increasing function, $\beta(t) \le \alpha(t) \le t$ and there exists a constant $h > 0 :\ 0 < \alpha(t) - \beta(t) \le h$ for $t \ge t_0$.

(**H2**). The function $f \in C([t_0,\infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\Big|f(t,\ x,\ y)\Big| \le K(t)\big|x\big|^p + L(t)\big|y\big|^p$ for $t \ge t_0,\ x,\ y \in \mathbb{R}$, where $K(t),\ L(t) \in C([t_0,\infty), \mathbb{R}_+)$ and $p = \text{const} < 1$.

(**H3**). The function $\varphi \in C([-h, 0], \mathbb{R})$ and $\max_{t\in[-h,0]} \big|\varphi(t)\big| \ge \big|\varphi(0)\big|$.

(**H4**). The solution $x(t; t_0, \varphi)$ of initial value problem (17), (18) is defined for $t \ge t_0 - h$.

The solution $x(t) = x(t; t_0, \varphi)$ for $t \ge t_0$ satisfies the inequalities

$$|x(t)| = \left|\varphi(0) + \int_{t_0}^{t} f\Big(s, x(s),\ \max_{s\in[\beta(t),\alpha(t)]} x(s)\Big)ds\right|$$

$$\le |\varphi(0)| + \int_{t_0}^{t} K(s)\big|x(s)\big|^p ds + \int_{t_0}^{t} L(s)\Big(\max_{\xi\in[\beta(t),\alpha(t)]} \big|x(\xi)\big|\Big)^p ds \tag{19}$$

$$\le \big|\varphi(0)\big| + \int_{t_0}^{t} K(s)\Big(u(s)\Big)^p ds + \int_{\alpha(t_0)}^{\alpha(t)} L(\alpha^{-1}(\eta))(\alpha^{-1}(\eta))'\Big(\max_{\xi\in[\eta-h,\eta]} u(\xi)\Big)^p d\eta. \tag{20}$$

Note that the conditions of Theorem 1 are satisfied for $k = |\varphi(0)|$, $n = 2$, $\alpha_1(t) \equiv t$, $\alpha_2(t) \equiv \alpha(t)$, $a_1(t) \equiv K(t)$, $b_1(t) = a_2(t) \equiv 0$, $b_2(t) = L(\alpha^{-1}(t))(\alpha^{-1}(t))'$, $g_1(u) \equiv g_2(u) = u^p$, $G(u) = \frac{u^{1-p}}{1-p}$, $G^{-1}(u) = \sqrt[1-p]{(1-p)u}$, $\text{Dom}(G^{-1}) = R_+$ and $t_1 = +\infty$.

According to Theorem 1, for $t \ge t_0$ we obtain the following bound for the solution of the initial value problem (17), (18)

$$\big|x(t; t_0, \varphi)\big| \le \sqrt[1-p]{|\varphi(0)|^{1-p} + (1-p)\int_{t_0}^{t} K(s)ds + (1-p)\int_{\alpha(t_0)}^{\alpha(t)} L(s)ds}.$$

Note that in the case $p = 1$, i.e. when $\Big|f(t,\ x,\ y)\Big| \le K|x| + L|y|$ for $t \ge t_0,\ x,\ y \in \mathbb{R}$, where $K, L > 0$, we have $G(u) = \ln u$, $G^{-1}(u) = e^u$ and from Theorem 1 we obtain the following bound $\big|x(t; t_0, \varphi)\big| \le |\varphi(0)|e^{K(t-t_0)+L(\alpha(t)-\alpha(t_0))}$ for the solution. If the right hand-side of the equation (17) is Lipshitz, then we obtain the uniqueness of the solution.

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НЕЛИНЕЙНИ ИНТЕГРАЛНИ НЕРАВЕНСТВА СЪДЪРЖАЩИ МАКСИМУМИ

Снежана Христова, Кремена Стефанова, Лозанка Тренкова

В статията се изучават някои интегрални неравенства, които съдържат максимума на неизвестната функция на една променлива. Разглежданите неравенства са обобщения на класическото неравенство на Бихари. Значимостта на тези интегрални неравенства се дълже на широкото им приложение при качественото изследванене на различни свойства на решенията на диференциални уравнения с "максимум" и е илюстрирано с някои директни приложения.