

COMPLETE INTEGRABILITY OF A NONLINEAR
ELLIPTIC SYSTEM, GENERATING BI-UMBILICAL
FOLIATED SEMI-SYMMETRIC HYPERSURFACES IN \mathbb{R}^4 *

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We find explicitly all bi-umbilical foliated semi-symmetric hypersurfaces in the four-dimensional Euclidean space.

1. Introduction. Let (M^n, g) be an n -dimensional Riemannian manifold. We denote by $T_p M^n$ the tangent space to M^n at a point $p \in M^n$ and by $\mathfrak{X}M^n$ - the algebra of all vector fields on M^n . The associated Levi-Civita connection of the metric g is denoted by ∇ , the Riemannian curvature tensor R is defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$; $X, Y \in \mathfrak{X}M^n$.

A *semi-symmetric space* is a Riemannian manifold (M^n, g) , whose curvature tensor R satisfies the identity $R(X, Y) \cdot R = 0$ for all vector fields $X, Y \in \mathfrak{X}M^n$. (Here $R(X, Y)$ acts as a derivation on R).

According to the classification of Z. Szabó [9, 10] the main class of semi-symmetric spaces is the class of all Riemannian manifolds foliated by Euclidean leaves of codimension two. The foliated semi-symmetric spaces can be considered also as Riemannian manifolds of conullity two [1].

We recall that a Riemannian manifold (M^n, g) is of *conullity two*, if at every point $p \in M^n$ the tangent space $T_p M^n$ can be decomposed in the form $T_p M^n = \Delta_0(p) \oplus \Delta_0^\perp(p)$, where $\dim \Delta_0(p) = n - 2$, $\dim \Delta_0^\perp(p) = 2$ and $\Delta_0(p)$ is the nullity vector space of the curvature tensor R_p , i.e. $\Delta_0(p) = \{X \in T_p M^n \mid R_p(X, Y)Z = 0; Y, Z \in T_p M^n\}$. The $(n - 2)$ -dimensional distribution $\Delta_0 : p \longrightarrow \Delta_0(p)$ is integrable and its integral manifolds are totally geodesic and locally Euclidean. So, (M^n, g) is foliated by Euclidean leaves of codimension two.

The foliated semi-symmetric hypersurfaces in Euclidean space \mathbb{R}^{n+1} are studied in [4] with respect to their second fundamental form. They can be considered as hypersurfaces of type number two, i.e. hypersurfaces whose rank of the second fundamental form is equal to two everywhere. Each foliated semi-symmetric hypersurface M^n in \mathbb{R}^{n+1} is characterized by a second fundamental form $h = \nu_1 \eta_1 \otimes \eta_1 + \nu_2 \eta_2 \otimes \eta_2$, where η_1 and η_2 are unit one-forms; ν_1 and ν_2 are functions on M^n , $\nu_1 \nu_2 \neq 0$. The Euclidean leaves

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of the foliation are the integral submanifolds of the distribution Δ_0 , determined by the one-forms η_1 and η_2 , i.e. $\Delta_0(p) = \{X \in T_p M^n \mid \eta_1(X) = 0, \eta_2(X) = 0\}$, $p \in M^n$.

Let Δ_0^\perp be the two-dimensional geometric distribution, which is orthogonal to the distribution Δ_0 , i.e. to the Euclidean leaves of the foliation of M^n . In case of $\nu_1 = \nu_2$, at each point $p \in M^n$ the shape operator A of M^n has two equal nonzero eigenvalues corresponding to Δ_0^\perp , and an eigenvalue $\nu = 0$ with multiplicity $n - 2$, corresponding to Δ_0 . That is why the foliated semi-symmetric hypersurfaces satisfying the condition $\nu_1 = \nu_2$ are called *bi-umbilical*.

The foliated semi-symmetric hypersurfaces in \mathbb{R}^{n+1} are characterized in [5] by the following

Theorem 1.1. *A hypersurface M^n in Euclidean space \mathbb{R}^{n+1} is locally a foliated semi-symmetric hypersurface if and only if it is the envelope of a two-parameter family of hyperplanes in \mathbb{R}^{n+1} .*

Using the characterization of a foliated semi-symmetric hypersurface as the envelope of a two-parameter family of hyperplanes, each such hypersurface is determined by a pair of a unit vector-valued function $l(u, v)$ and a scalar function $r(u, v)$, defined in a domain $\mathcal{D} \subset \mathbb{R}^2$.

Since the vector fields l_u and l_v are linearly independent, then the vector-valued function $l(u, v)$ determines a two-dimensional surface $M^2 : l = l(u, v)$, $(u, v) \in \mathcal{D}$ in \mathbb{R}^{n+1} . We use the standard denotations $E(u, v) = g(l_u, l_u)$; $F(u, v) = g(l_u, l_v)$; $G(u, v) = g(l_v, l_v)$ for the coefficients of the first fundamental form of M^2 . According to [2] for each point of a regular surface there exists a neighbourhood, in which isothermal parameters can be introduced, i.e. $E(u, v) = G(u, v)$; $F(u, v) = 0$. Since our considerations are local, without loss of generality we may assume that the surface M^2 is parameterized locally by isothermal parameters. Then, the generated foliated semi-symmetric hypersurface M^n is given by

$$(1.1) \quad X(u, v, w^\alpha) = r l + \frac{r_u}{E} l_u + \frac{r_v}{E} l_v + w^\alpha b_\alpha, \quad \alpha = 1, \dots, n - 2,$$

where $(u, v) \in \mathcal{D}$, $w^\alpha \in \mathbb{R}$, $\alpha = 1, \dots, n - 2$, and $b_1(u, v), \dots, b_{n-2}(u, v)$, $(u, v) \in \mathcal{D}$ are $n - 2$ mutually orthogonal unit vectors, orthogonal to $\text{span}\{l, l_u, l_v\}$ [6].

The bi-umbilical foliated semi-symmetric hypersurfaces are characterized analytically in [6] by the following

Theorem 1.2. *Let M^n be a hypersurface in \mathbb{R}^{n+1} which is the envelope of a two-parameter family of hyperplanes, determined by a unit vector-valued function $l(u, v)$, represented by isothermal parameters, and a scalar function $r(u, v)$. Then, M^n is bi-umbilical if and only if $l(u, v)$ and $r(u, v)$ satisfy the equalities*

$$\begin{aligned} l_{uu} - l_{vv} - \frac{E_u}{E} l_u + \frac{E_v}{E} l_v &= 0; \\ 2l_{uv} - \frac{E_v}{E} l_u - \frac{E_u}{E} l_v &= 0; \\ r_{uu} - r_{vv} - \frac{E_u}{E} r_u + \frac{E_v}{E} r_v &= 0; \\ 2r_{uv} - \frac{E_v}{E} r_u - \frac{E_u}{E} r_v &= 0. \end{aligned}$$

So, the bi-umbilical foliated semi-symmetric hypersurfaces are determined by the following system of non-linear equations for the vector-valued function $l(u, v)$:

$$(1.2) \quad \begin{aligned} l_{uu} - l_{vv} - \frac{E_u}{E} l_u + \frac{E_v}{E} l_v &= 0; \\ 2l_{uv} - \frac{E_v}{E} l_u - \frac{E_u}{E} l_v &= 0, \end{aligned}$$

and $l(u, v)$ satisfies the additional conditions

$$(1.3) \quad g(l_u, l_u) = g(l_v, l_v); \quad g(l_u, l_v) = 0; \quad g(l, l) = 1.$$

By means of the identities (1.3) the system (1.2) can be rewritten in the following way:

$$(1.4) \quad Al_{uu} + Bl_{vv} = 0,$$

where $A = \{I - 2\frac{l_u \otimes l_u}{l_u^2}\}$, $B = \{I - 2\frac{l_v \otimes l_v}{l_v^2}\}$. Since $\det A = -1$, the system (1.4) has the normal form

$$(1.5) \quad l_{uu} = -A^{-1}Bl_{vv}.$$

The characteristic matrix $\det(\lambda^2 I + A^{-1}B)$ of (1.5) has no real roots. Indeed, calculating

$$\{\lambda^2 A + B\} = \{(\lambda^2 + 1)I + \frac{(-\lambda l_u + l_v) \otimes (-\lambda l_u + l_v)}{l_u^2}\};$$

$$\det(\lambda^2 I + A^{-1}B) = \det A^{-1} \det(\lambda^2 I + B) = -\det(\lambda^2 A + B) = (\lambda^2 + 1)^4,$$

we get that the roots of the characteristic matrix are $\lambda = \pm i$. Hence, according to the classification of the general systems [8], system (1.5) is a non-linear elliptic system.

Let us note that (1.2) is an elliptic system in non-divergent form and the general theory of calculus of variation can not be applied to (1.2). That is why, for solving this system we use a different method, which is based rather on the differential geometry of surfaces in \mathbb{R}^4 than on the PDE methods. In such a way we find all solutions of (1.2) satisfying (1.3) in the four-dimensional Euclidean space \mathbb{R}^4 .

In Section 2 we consider the solutions of (1.2), (1.3) as two-dimensional surfaces lying on the unit sphere $S^3(1)$ in \mathbb{R}^4 . Using the derivative formulas of these surfaces, in Theorem 2.1 we prove that each solution $l = l(u, v)$ of system (1.2), (1.3) is part of a sphere S^2 in a constant hyperplane \mathbb{R}^3 of \mathbb{R}^4 .

In Section 3 we apply Theorem 2.1 for giving explicitly all bi-umbilical foliated semi-symmetric hypersurfaces in \mathbb{R}^4 .

2. Solvability of the non-linear elliptic system. In this section we consider the system

$$(2.1) \quad \begin{aligned} l_{uu} - l_{vv} - \frac{E_u}{E} l_u + \frac{E_v}{E} l_v &= 0 \\ 2l_{uv} - \frac{E_v}{E} l_u - \frac{E_u}{E} l_v &= 0 \\ g(l_u, l_u) = g(l_v, l_v) &= E \\ g(l_u, l_v) &= 0 \\ g(l, l) &= 1 \end{aligned}$$

for the vector-valued function $l(u, v) = (l^1(u, v), l^2(u, v), l^3(u, v), l^4(u, v))$ in the 4-dimensional Euclidean space \mathbb{R}^4 .

Let $l = l(u, v)$ be a solution of (2.1), defined in a domain $\mathcal{D} \subset \mathbb{R}^2$. We consider the 2-dimensional surface $M^2 : l = l(u, v)$, $(u, v) \in \mathcal{D}$ ($\mathcal{D} \subset \mathbb{R}^2$) in \mathbb{R}^4 . M^2 is a surface lying on the unit sphere $S^3(1)$ in \mathbb{R}^4 , and the parameters (u, v) are isothermal ones.

Theorem 2.1. *Each solution $M^2 : l = l(u, v)$ of system (2.1) lies on a sphere S^2 in a constant hyperplane \mathbb{R}^3 of \mathbb{R}^4 .*

Proof. Let $M^2 : l = l(u, v)$ be a solution of (2.1). Tkem, the tangent space to M^2 at an arbitrary point $p = l(u, v)$ of M^2 is $T_p M^2 = \text{span}\{l_u, l_v\}$. Since the vector fields l, l_u, l_v are orthogonal, there exists a unique (up to a sign) unit vector field $n(u, v)$, such that $\{l_u, l_v, l, n\}$ form an orthogonal frame field in \mathbb{R}^4 . Using (2.1) we obtain the following derivative formulas of M^2 :

$$(2.2) \quad \begin{aligned} l_{uu} &= \frac{E_u}{2E} l_u - \frac{E_v}{2E} l_v - E l + c n; \\ l_{uv} &= \frac{E_v}{2E} l_u + \frac{E_u}{2E} l_v; \\ l_{vv} &= -\frac{E_u}{2E} l_u + \frac{E_v}{2E} l_v - E l + c n, \end{aligned}$$

where $c(u, v) = g(l_{uu}, n) = g(l_{vv}, n)$. The equalities (2.2) imply

$$n_u = -\frac{c}{E} l_u; \quad n_v = -\frac{c}{E} l_v.$$

Since $n_{uv} = n_{vu}$, then $\left(\frac{c}{E}\right)_u = \left(\frac{c}{E}\right)_v = 0$, and hence $\frac{c}{E} = c_0 = \text{const}$, i.e. $c(u, v) = c_0 E(u, v)$.

Let $\left\{x = \frac{l_u}{\sqrt{E}}, y = \frac{l_v}{\sqrt{E}}\right\}$ be an orthonormal tangent frame field of M^2 . From (2.2) we get

$$(2.3) \quad \begin{aligned} \nabla'_x l &= x; & \nabla'_x n &= -\frac{c}{E} x; \\ \nabla'_y l &= y; & \nabla'_y n &= -\frac{c}{E} y. \end{aligned}$$

From (2.2) it follows that the Riemann curvature K of M^2 is expressed as $K = 1 + \frac{c^2}{E^2} = 1 + c_0^2$. Hence, the surface M^2 is of constant Riemann curvature K .

In case of $c_0 = 0$ the normal vector field n is constant and M^2 lies in the constant 3-dimensional subspace $\mathbb{R}^3 = \text{span}\{l_u, l_v, l\}$ of \mathbb{R}^4 . Moreover, M^2 lies on a sphere $S^2(1) = S^3(1) \cap \mathbb{R}^3$.

In case of $c_0 \neq 0$ we consider the following normal vector fields

$$(2.4) \quad n_1 = \frac{1}{\sqrt{c^2 + E^2}}(c l + E n); \quad n_2 = \frac{1}{\sqrt{c^2 + E^2}}(-E l + c n).$$

From (2.3) and (2.4) we get

$$\begin{aligned} \nabla'_x n_1 &= 0; & \nabla'_x n_2 &= -\sqrt{1 + c_0^2} x; \\ \nabla'_y n_1 &= 0; & \nabla'_y n_2 &= -\sqrt{1 + c_0^2} y. \end{aligned}$$

Hence, M^2 lies on a sphere $S^2\left(\frac{1}{\sqrt{1+c_0^2}}\right)$ in the constant 3-dimensional subspace $\mathbb{R}^3 = \text{span}\{l_u, l_v, n_2\}$ of \mathbb{R}^4 . \square

3. Application to the geometric problem. In Section 2 we proved that each solution $l = l(u, v)$ of (2.1) is a 2-dimensional surface $M^2 : l = l(u, v)$ lying on $S^3(1) \cap \mathbb{R}^3$ for some constant hyperplane \mathbb{R}^3 . The standard parametrization of the unit sphere $S^3(1)$ in \mathbb{R}^4 is

$S^3 : z(u, v, \alpha) = \cos \alpha \cos u \cos v e_1 + \cos \alpha \cos u \sin v e_2 + \cos \alpha \sin u e_3 + \sin \alpha e_4$, where $\{e_1, e_2, e_3, e_4\}$ is the standard basis in \mathbb{R}^4 , and $u \in [0; 2\pi)$, $v \in [0; 2\pi)$, $\alpha \in [0; 2\pi)$.

Let \mathbb{R}^3 be the hyperplane in \mathbb{R}^4 , defined by $z^4 = b = \text{const}$. Then, the 2-dimensional sphere $S^2 = S^3(1) \cap \mathbb{R}^3$ is parameterized by

$$(3.1) \quad S^2 : l(u, v) = a \cos u \cos v e_1 + a \cos u \sin v e_2 + a \sin u e_3 + b e_4,$$

where $\cos \alpha = a$, $\sin \alpha = b$. S^2 is a sphere with Gauss curvature $K = 1 + \frac{b^2}{a^2}$, and radius $r = \frac{1}{\sqrt{K}} = a$. From (3.1) we get $E = g(l_u, l_u) = a^2$; $F = g(l_u, l_v) = 0$; $G = g(l_v, l_v) = a^2 \cos^2 u$.

Obviously the parameters (u, v) of S^2 are not isothermal. After the following change of the parameters:

$$u = -\frac{\pi}{2} + 2 \arctan e^x; \quad v = y,$$

we get

$$\sin u = \frac{e^{2x} - 1}{e^{2x} + 1} = \tanh x; \quad \cos u = \frac{2e^x}{e^{2x} + 1} = \frac{1}{\cosh x}.$$

So, we obtain the following parametrization of S^2 :

$$(3.2) \quad S^2 : l(x, y) = \frac{1}{\cosh x} (a \cos y e_1 + a \sin y e_2 + a \sinh x e_3 + b \cosh x e_4).$$

From (3.2) we get $E = G = \frac{a^2}{\cosh^2 x}$, $F = 0$, i.e. the parameters (x, y) are isothermal. In terms of the parameters (x, y) the system (2.1) can be rewritten in the form:

$$(3.3) \quad \begin{aligned} l_{xx} - l_{yy} &= -2 \tanh x l_x; \\ 2 l_{xy} &= -2 \tanh x l_y. \end{aligned}$$

Now the equalities for the scalar function $r(x, y)$ have the form

$$(3.4) \quad \begin{aligned} r_{xx} - r_{yy} &= -2 \tanh x r_x; \\ 2 r_{xy} &= -2 \tanh x r_y. \end{aligned}$$

Each scalar function $r(x, y)$, satisfying (3.4), together with the sphere $S^2 : l = l(x, y)$, defined by (3.2), generate a bi-umbilical foliated semi-symmetric hypersurface M^3 in \mathbb{R}^4 according to the geometric construction, given by (1.1).

Further we are going to find all solutions of (3.4). From the second equality of (3.4) we get $(\cosh x r_y)_x = 0$, which implies $\cosh x r_y = g(y)$ for some smooth function $g(y)$.

Hence, for fixed x we get

$$(3.5) \quad r(x, y) = \frac{1}{\cosh x} \int_0^y g(t) dt + f(x),$$

where $f(x)$ is a smooth function. The first equality of (3.4) implies

$$g'(y) + \int_0^y g(t) dt = \cosh x f''(x) + 2 \sinh x f'(x).$$

Hence

$$g'(y) + \int_0^y g(t) dt = c_4;$$

$$(3.6) \quad \cosh x f''(x) + 2 \sinh x f'(x) = c_4;$$

where $c_4 = \text{const}$. All solutions for $\int_0^y g(t) dt$ are given by

$$(3.7) \quad \int_0^y g(t) dt = c_1 \cos y + c_2 \sin y + c_4, \quad c_1, c_2 = \text{const}.$$

Equality (3.6) is equivalent to the identity $(\cosh^2 x f'(x))' = c_4 (\sinh x)'$, which implies

$$(3.8) \quad f(x) = c_3 \tanh x - \frac{c_4}{\cosh x} + c_0.$$

Now, from equalities (3.5), (3.7) and (3.8) we get that all solutions of (3.4) are given by:

$$(3.9) \quad r(x, y) = c_0 + \frac{1}{\cosh x} (c_1 \cos y + c_2 \sin y + c_3 \sinh x),$$

where c_0, c_1, c_2, c_3 are constants.

Consequently, all bi-umbilical foliated semi-symmetric hypersurfaces in \mathbb{R}^4 are given by the following formula (up to parametrization):

$$(3.10) \quad X(x, y, w) = r l + \frac{\cosh^2 x}{a^2} r_x l_x + \frac{\cosh^2 x}{a^2} r_y l_y + w n.$$

Calculating r_x and r_y from (3.9), and using (3.10) we obtain the coordinate functions of the bi-umbilical foliated semi-symmetric hypersurface $M^3 : X = X(x, y, w)$:

$$X^1(x, y, w) = \frac{\cos y}{\cosh x} \left[a c_0 + b w - \frac{b^2}{a \cosh x} (c_1 \cos y + c_2 \sin y + c_3 \sinh x) \right] + \frac{c_1}{a};$$

$$X^2(x, y, w) = \frac{\sin y}{\cosh x} \left[a c_0 + b w - \frac{b^2}{a \cosh x} (c_1 \cos y + c_2 \sin y + c_3 \sinh x) \right] + \frac{c_2}{a};$$

$$X^3(x, y, w) = \frac{\sinh x}{\cosh x} \left[a c_0 + b w - \frac{b^2}{a \cosh x} (c_1 \cos y + c_2 \sin y + c_3 \sinh x) \right] + \frac{c_3}{a};$$

$$X^4(x, y, w) = -\frac{a}{b} \left[a c_0 + b w - \frac{b^2}{a \cosh x} (c_1 \cos y + c_2 \sin y + c_3 \sinh x) \right] + \frac{c_0}{b}.$$

Let us denote $f(x, y, w) = \frac{a}{b} c_0 + w - \frac{b}{a \cosh x} (c_1 \cos y + c_2 \sin y + c_3 \sinh x)$. We consider the vector-valued function

$$n(x, y) = \frac{1}{\cosh x} (b \cos y e_1 + b \sin y e_2 + b \sinh x e_3 - a \cosh x e_4).$$

Then, each bi-umbilical foliated semi-symmetric hypersurface $M^3 : X = X(x, y, w)$ is parameterized as follows:

$$X(x, y, w) = f(x, y, w) n(x, y) + C,$$

where $C = \frac{c_1}{a} e_1 + \frac{c_2}{a} e_2 + \frac{c_3}{a} e_3 + \frac{c_0}{b} e_4$ is a constant vector in \mathbb{R}^4 .

Let us note that in [3] G. Ganchev studied the integral surfaces of the distribution Δ_0^\perp orthogonal to the Euclidean leaves of the foliation and proved that the integral surfaces of Δ_0^\perp of any bi-umbilical semi-symmetric hypersurface lie on two-dimensional spheres. Conversely, any two-dimensional sphere generates a family of bi-umbilical semi-symmetric hypersurfaces. Here we obtained explicitly all bi-umbilical semi-symmetric hypersurfaces using the approach of studying these hypersurfaces as the envelopes of two-parameter families of hyperplanes and solving the corresponding non-linear elliptic system. This geometric method of solving systems of PDE can also be applied for the investigation of other non-linear systems arising in geometric problems (see [7]).

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**ПЪЛНА ИНТЕГРУЕМОСТ НА ЕДНА НЕЛИНЕЙНА ЕЛИПТИЧНА
СИСТЕМА, ПОРАЖДАЩА БИ-ОМБИЛИЧНИ ФОЛИРАНИ
ПОЛУСИМЕТРИЧНИ ХИПЕРПОВЪРХНИНИ В \mathbb{R}^4**

Николай Кутев, Величка Милушева

Намираме експлицитно всичките би-омбилични фолирани полусиметрични по-
върхнини в четиримерното евклидово пространство \mathbb{R}^4