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2D APPORTIONMENT METHODS*

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The paper deals with 2D proportional (or bi-proportional) electoral systems in which the number of party mandates is determined at a nation wide level while the personification of mandates is done through regional party lists. In addition, the number of mandates in each region is preliminary determined proportionally to the population.

Variants of such systems have been used in seven parliamentary elections in Bulgaria during the period 1990–2009. These systems as well as new improved 2D apportionment methods are considered. Results from numerical simulations with data from the 2009 Bulgarian parliamentary elections are given.

Introduction and notation. There were seven parliamentary elections in Bulgaria during the period 1990–2009 as well as two elections for members of the European Parliament (2007 and 2009). Several variants of proportional electoral systems (PES) have been used in these elections.

The Bulgarian PES acts nationwide but the party mandates are personified by 31 regional party lists. This is a *2D system with constraints on the numbers of regional party list mandates*, or a *bi-proportional apportionment system*. Such systems have been studied in [1, 5, 9, 10]. Although an integer optimization problem may be formulated and solved for such PES, this approach meats difficulties due to possible non-uniqueness of the solution.

In this paper we define 2D PES intended to realize the Bulgarian type electoral process and present results from numerical simulations with real data. Such numerical experiments are possible due to the good practice to maintain a detailed information about the election results since 1991, see [3].

In what follows we shall use the notations: $\mathbb{N}_k$ – the set of natural numbers $\{k, k+1, \dots\}$; $\mathbb{Z}$ – the set of integers; $\mathbb{Q}$ and $\mathbb{Q}_+$ – the sets of rational and non-negative rational numbers; $\mathbb{R}$ and $\mathbb{R}_+$ – the sets of real and non-negative real numbers; $[x] \in \mathbb{Z}$ – the entire part of $x \geq 0$; $\|u\| = |u_1|+|u_2|+\cdots+|u_n|$ – the 1-norm of the vector $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$; $\preceq$ – the component-wise partial order relation in $\mathbb{R}^n$ ($u \preceq v$ whenever $u_j \leq v_j$ for $j = 1, 2, \dots, n$).

1D methods for PES. Two types of methods for PES are widely used in electoral practice: *Highest averages methods* and *Largest remainder methods*. In this section

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we consider these methods in their standard 1D versions. 2D generalizations of these methods are considered later on.

Suppose that we have to allocate $M > 1$ seats, or mandates, among $n > 1$ parties $\Pi_1, \Pi_2, \ldots, \Pi_n$ with votes $v_1, v_2, \ldots, v_n$. Let $s_j \geq 0$ be the number of seats assigned to party Π_j by the electoral system, and set $v = (v_1, v_2, \ldots, v_n) \in \mathbb{N}_1^n$, $s = (s_1, s_2, \ldots, s_n) \in \mathbb{N}_0^n$, where $\|s\| = M$.

The numbers s_j must be approximately proportional to v_j, i.e. $s_j/v_j \approx M/\|v\|$, $j = 1, 2, \ldots, n$. A PES method is realized by a function $f = (f_1, f_2, \ldots, f_n) : \mathbb{N}_1^n \times \mathbb{N}_2 \to \mathbb{N}_0^n$ such that $s = f(v, M)$ and $\|f(v, S)\| = M$. Usually f is determined by an algorithm which transforms (v, M) into s. Such an algorithm includes a random choice mechanism. The algorithm works also when v is a positive non-integer vector. Thus, we may consider f as a function mapping $\mathbb{R}_+^n \times \mathbb{N}_2 \to \mathbb{N}_0^n$.

Denote $\sigma = Mv/\|v\| = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{Q}_+^n$, $\|\sigma\| = M$. The next three conditions for PES are accepted as desirable.

C1 If a party Π_i has more votes than party Π_j then it not have less seats, i.e. $(v_i - v_j)(s_i - s_j) \geq 0$, $i, j = 1, 2, \ldots, n$.

C2 If more seats are allocated, then no party loose a seat, i.e. $f(v, M) \preceq f(v, M+k)$, $k \in \mathbb{N}_1$. The violation of this condition is known as *Alabama paradox*. If the inequality $f(v, M) \preceq f(v, M + k)$ is fulfilled for $k = 1, 2, \ldots, l$ and is violated for $k = l + 1$ for some $l \in \mathbb{N}_1$, then we have an *Alabama paradox of l-th order*, ([7]).

C3 The *fair share* conditions $[\sigma_j] \leq s_j \leq [\sigma_j] + 1$, $j = 1, 2, \ldots, n$ are satisfied.

The following important result [2] has been stated in 1982.

Theorem 1. *There is no PES that satisfies all three conditions* **C1–C3**.

Denote by $v^* = v/\|v\|$, $s^* = s/\|s\|$ the vectors of *relative votes* and *relative seats*, where $\|v^*\| = \|s^*\| = 1$. Then, we may define a *measure for proportionality* $\mu_p(v, M) = \|s^* - v^*\|_p$, where $\| \cdot \|_p$, $p \geq 1$, in the Hölder p-norm.

Let M positive numbers $1 = q_1 < q_2 < \cdots < q_M$ be given. Then, the sequence (q_k) defines a plain *Highest averages method* as follows: 1) the parties are ordered so as $v_1 \geq v_2 \geq \cdots \geq v_n$ (if there are equal number of votes the ordering is by ties); 2) the matrix $(h_{kj}) \in \mathbb{N}_1^{M \times n}$ is constructed, where $h_{kj} = v_j/q_k$; 3) at each step the largest element $h_{pr} = \max\{h_{kj} : k = 1, 2, \ldots, M,\ j = 1, 2, \ldots, n\}$ gives one seat to party Π_r and is afterwards excluded from consideration; 4) if there are different elements of equal magnitude, then first obtains a seat the element at a higher row, and if the equal elements are at the same row, then first obtains a seat the more left element.

When $q_k = k$, we have the *D'Hondt method* which satisfies conditions **C1** and **C2** but may violate condition **C3**. The result produced by this method is denoted as $s = \mathbf{D}(v, M)$. The D'Hondt method has been used in the Bulgarian parliamentary elections in 1990 (as a part of a mixed electoral system), 1991, 1994, 1997, 2001 and 2005.

Let a number $a \geq 0$ be given. A plain *Largest remainder method* with a parameter a acts as follows: 1) the vector $\tau = (\tau_1, \tau_2, \ldots, \tau_n) = (M + a)v^* \in \mathbb{Q}_+^n$ is calculated; 2) each party Π_j obtains $[\tau_j]$ initial seats and if $a = 0$ and $\tau = [\tau]$, then all M seats are allocated; 3) if $\tau \neq [\tau]$, then there are seats for additional allocation (if $a = 0$, then the number of seats for additional allocation is $\|\sigma - [\sigma]\| \in \mathbb{N}_1$); 4) first gets one additional seat to the party Π_k with maximum remainder $\tau_k - [\tau_k] = \max\{\tau_j - [\tau_j] : j = 1, 2, \ldots, n\}$; 5) the second seat goes to the party with the next largest remainder, etc. until all additional seats are allocated.

For $a = 0$ we have the *Hare-Niemayer method* which satisfies conditions **C1** and **C3** but may violate condition **C2**. The result produced by this method is denoted by $s = \mathbf{H}(v, M)$. It has been used in Bulgaria in 2007 and 2009 (twice).

2D methods for PES – the Bulgarian case. The Bulgarian PES is a 2D system which acts as follows. First, the country is divided into $m = 31$ electoral regions $R_1, R_2, \ldots, R_m$. In each region R_i a number of seats r_i is preassigned proportionally to its population p_i so that $r = (r_1, r_2, \ldots, r_m) = \mathbf{H}(p, M)$, $p = (p_1, p_2, \ldots, p_m)$.

The *data* for distribution of seats among party lists Π_{ij} is the matrix $V = (v_{ij}) \in \mathbb{R}_+^{m\times n}$ of votes, where $v_{ij} \geq 1$ is the number of votes for the party list Π_{ij} of Π_j in the region R_i. The *result* is the matrix $S = (s_{ij})$ with elements $s_{ij} \geq 0$, where s_{ij} is the number of seats assigned to Π_{ij}. The matrix S satisfies the restrictions $\sum_{i=1}^m s_{ij} = s_j$, $\sum_{j=1}^n s_{ij} = r_i$. There is also voting abroad the corresponding results being accounted in different ways.

Optimal seat distributions. A possible criterion for optimality of the seat distribution S is the difference between the maximum and minimum prices of seats for party lists Π_{ij} with $s_{ij} \geq 1$, (e.g. [6])

$$P(S) = \max_{i,j} \frac{v_{ij}}{s_{ij}} - \min_{i,j} \frac{v_{ij}}{s_{ij}} \to \min .$$

The solution to this problem may not be unique.

The next two criteria reflect the "inter-party fairness"

$$R_1(S) = \max_{s_{ij}\geq 1, s_{kj}\geq 1} \frac{v_{ij}s_{kj}}{s_{ij}v_{kj}}, \ R_2(S) = \max_{s_{ij}=0, s_{kj}\geq 1} \frac{v_{ij}s_{kj}}{v_{kj}}.$$

Another criterion is based on the quadratic measure,

$$Q_\lambda(S) = \sum_{i=1}^m \sum_{j=1}^n \lambda \left(\frac{v_{ij}}{\|V_{i\bullet}\|} - \frac{s_{ij}}{r_i} \right)^2 + (1-\lambda) \left(\frac{v_{ij}}{\|V_{\bullet j}\|} - \frac{s_{ij}}{s_j} \right)^2, \ 0 \leq \lambda \leq 1,$$

where $V_{i\bullet}$ and $V_{\bullet j}$ are the i-th row and the j-th column of the matrix V, correspondingly. It was shown that the solution of a modification of this integer optimization problem is unique [6] (up to ties).

Without the requirement that S is an integer matrix, the necessary and sufficient conditions $\partial Q_\lambda(S)/\partial s_{ij} = 0$ for minimum of $Q_\lambda(S)$ in S lead to explicit expressions $s_{ij}^0 = \beta_{ij}(\lambda) v_{ij}$, $i = 1, 2, \ldots,$ $j = 1, 2, \ldots, n$,where

$$\beta_{ij}(\lambda) = \frac{r_i s_j \left(\lambda s_j \|V_{\bullet j}\| + (1-\lambda) r_i \|V_{i\bullet}\|\right)}{\left(\lambda s_j^2 + (1-\lambda) r_i^2\right) \|V_{i\bullet}\| \, \|V_{\bullet j}\|}.$$

Distribution of votes from abroad. In the existing Central Election Commission (CEC) algorithm the votes from abroad are taken into account only in the determination of the total number of party seats. Afterwards, the algorithm uses the profile of votes cast in the country.

Here we propose another scheme to account the votes from abroad. Let $w = (w_1, w_2, \ldots, w_n)$ be the vector of votes cast abroad, where w_j is the number of votes for party Π_j. Then the vector of party seats is computed from $s = \mathbf{H}(M, u)$, $u = v + w$,

$v = (\|V_{\bullet 1}\|, \|V_{\bullet 2}\|, \ldots, \|V_{\bullet n}\|)$.

For any $\gamma \in [0,1]$ we introduce the matrix $U(\gamma) = (u_{ij}(\gamma))$ of (generally fractional) votes $u_{ij}(\gamma) = v_{ij}\,(1 + \gamma w_j/\|V_{\bullet j}\|)$. In the CEC algorithms for 2005 and 2009 elections it was assumed $\gamma = 0$. In our experiments we have used $\gamma \in \{0, 1/2, 1\}$.

The $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$-algorithm. For $\alpha, \beta, \gamma \in [0,1]$ we define the (α, β, γ)-algorithm using the matrix $U = U(\gamma)$ as input data. The algorithm proceeds as follows.

A1 The lists Π_{ij} of any party Π_j compete for the s_j party seats using the HN method. As a result we get the vector $t_j = (t_{1j}, t_{2j}, \ldots, t_{nj}) = \mathbf{H}(V_j, s_j)$ and the initial seat matrix is $T = (t_{ij})$.

A2 If $\|T_{i\bullet}\| = r_i$ (the region R_i has exact number of mandates), then each list Π_{ij} obtains $s_{ij} = t_{ij}$ seats and the region R_i is excluded from further distribution.

A3 If $\|T_{i\bullet}\| > r_i$ for some i (the region R_i has a surplus of mandates) then a seat is taken from the list Π_{ij} with at least one seat and with *least future price* $u_{ij}/(t_{tj} - \alpha)$ of one seat. If all party lists in R_i with at least one seat currently have exactly one seat, a seat is taken from the list with least votes.

A4 If $\|T_{i\bullet}\| < r_i$ for some i (there are regions with shortage of mandates) then a seat is given to the list Π_{kl} with the *largest future price* $u_{kl}/(t_{kl} + \beta)$ of one seat among all regions with shortage of mandates.

We stress that in 2009 elections the CEC used the $(0, 1, 0)$-algorithm. This algorithm, however, is far behind according to different criteria. In previous elections CEC used a different algorithm based on the D'Hondt method with $s = \mathbf{D}(u, M)$. Then, in step **A1** the vectors t_j are computed as $\mathbf{D}(V_j, s_j)$. Next, the algorithm applies step **A2** and starts working with regions with a shortage of mandates, i.e. in a sense steps **A3** and **A4** are interchanged. The algorithm maximizes the minimum regional price of one seat but allows very high prices of some seats (in 2005 the prices of seats for one party varied from 62 206 to 6 895 [3]).

Several other algorithms have been designed which eventually produce close results.

The *modified* (α, β, γ)*–algorithm* (denoted also as $(\alpha, \beta, \gamma, \ \mathrm{mod}\ )$-algorithm) uses input data $U(\gamma)$ but instead of starting from regions with surplus of mandates (step **A3**), it starts with regions with shortage of mandates (step **A4**).

Another interesting seat allocation method is called the *2D HN method* and acts as follows:

B1 The matrix Σ with elements $\sigma_{ij} = M u_{ij}/\|u\|$ is constructed, where $u = (u_1, u_2, \ldots, u_n)$, $u_j = \|U_{\bullet j}\|$. Each list Π_{ij} initially obtains $[\sigma_{ij}]$ seats.

B2 If the sum of seats in a region R_i is equal to r_i, then each list Π_{ij}, $j = 1, 2, \ldots, n$, obtains $[\sigma_{ij}]$ seats and region R_i is excluded from further distribution of seats.

B3 If the sum of seats in a region R_i exceeds r_i, then a seat is taken from the list Π_{ij} with the least price of one seat.

B4 If the sum of seats for party Π_j is equal to s_j then each list Π_{ij}, $i = 1, 2, \ldots, m$, obtains $[\sigma_{ij}]$ seats and party Π_j is excluded form further distribution of seats.

B5 If the sum of seats for a given party Π_j exceeds s_j, then a seat is taken from the list Π_{ij} with the least price of one seat.

B6 If there are additional seats for allocation after steps **B1–B5**, then the list Π_{ij} with the largest remainder $\sigma_{ij} - [\sigma_{ij}]$ among all remaining lists takes the first additional seat. The list with the second largest remainder takes the second seat, etc.

A *modified 2D HN method* is similar to the 2D HN method considered above. At step B1 it starts with initial number of seats $\left[s_{ij}^0\right]$ instead of $[\sigma_{ij}]$.

Another method is the *2D D'Hondt method* which acts as follows: 1) First gets one seat the party list Π_{ij} with the largest number of votes v_{ij}. The number v_{ij} is then divided by 2. 2) Second takes a seat the party list with the second largest number of votes. If this number had been divided to k ,then it is set to its initial value divided to $k+1$. 3) If a party Π_j (or a region R_i) has obtained the preliminary determined number of seats s_j (or r_i), then it is excluded from further distribution of seats.

Monotonicity and Alabama type paradoxes. Consider the 2D apportionment $S = (s_{ij}) \in \mathbb{N}_0^{m\times n}$ corresponding to a given vote matrix $V = (v_{ij}) \in \mathbb{Q}_+^{m\times n}$, a seat vector $s \in \mathbb{N}_0^n$ and a region vector $r \in \mathbb{N}_1^m$, such that $\|s\| = \|r\| = M$.

The apportionment S is called *party–wise monotone*, *region–wise monotone* and *completely monotone* if $(v_{ij}-v_{kj})(s_{ij}-s_{kj}) \geq 0$, $(v_{ij}-v_{il})(s_{ij}-s_{il}) \geq 0$, $(v_{ij}-v_{kl})(s_{ij}-s_{kl}) \geq 0$ for $i,k = 1,2,\ldots,m$, $j,l = 1,2,\ldots,n$, respectively. The total number of violations of party and region monotonicity conditions is the *measure of discordance* of the allocation S and is denoted by Δ.

The following result has been stated in [5].

Theorem 2. *Let $m > 2$ or $n > 2$. Then, there exist data V, s, r for which any S is neither party–wise nor region–wide monotone. In particular, S is not completely monotone.*

It is interesting that even for $m = n = 2$ there is data V, s, r for which a party (or regional) monotone allocation S does not exist.

Example 3. *Let $m = n = 2$, $M = 2$, $s = r = [1\ \ 1]$T and* $V = \begin{bmatrix} x & x \\ x+1 & x+1 \end{bmatrix}$, *where $x \in \mathbb{N}_1$. Then, there are two admissible apportionments* $S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $S_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. *Moreover, S_k is not monotone for party Π_k, $k = 1,2$.*

All methods (with exception of 2D D'Hondt methods) admit the Alabama paradox in its 2D version. Let a 2D method produces the matrix $S = S(V,M)$. For $N > M$ the 2D Alabama paradox means that the matrix $S(V,N) - S(V,M)$ may have negative elements (eventually equal to -1). The number of these elements is the *number of Alabama paradoxes with base M*. Similarly, if $N < M$, then the number of negative elements $S(V,M) - S(V,N)$ is the *number of Alabama paradoxes with base M*.

Numerical results. Numerical results using the official data for the 2009 parliamentary elections in Bulgaria are given at the table below using the methods considered. The minimum element in each column of the table is given in ( ) brackets, while the maximum one – in [ ] brackets. The values of Q are computed for $\lambda = 1/2$.

First we consider the hypothetic results without the votes from abroad (here $\gamma = 0$ by definition).

No	α	β	Q	P	Δ	R_1	R_2
1	0	0	0.9695	24 667	168	[18.5748]	[10.9390]
2	0	1/2	0.9903	24 045	179	11.3763	6.6997
3	0	1	1.0238	24 045	199	11.3763	6.6997
4	1/2	0	(0.8820)	(22 321)	78	(6.0342)	2.5260
5	1/2	1/2	0.9443	23 026	124	8.7238	4.0177
6	1/2	1	0.9752	23 557	132	8.7238	4.0177
7	1	0	1.0415	31 765	52	9.0418	(1.8061)
8	1	1/2	1.0508	[33 528]	47	16.3306	3.2620
9	1	1	1.0595	[33 528]	(42)	16.3306	3.0613
10	2D-HN	–	0.9235	22 778	130	7.7997	4.5933
11	2D-HN-mod	–	0.9427	24 667	165	[18.5748]	[10.9390]
12	2D-D	–	[1.1901]	26 888	[229]	11.3763	6.6997

Method (1/2,0,0) under no. 4 in the table is one of the best relative to the combination of the five criteria Q, P, Δ, R_1 and $R_2 2$.

Next we present the results taking into account the votes from abroad.

No	α	β	γ	Q	P	Δ	R_1	R_2
1	0	0	0	0.9554	25 806	132	7.7997	5.0116
2	0	0	1/2	0.9563	26 549	131	11.3763	6.6997
3	0	0	1	1.0199	30 455	149	8.0685	4.7517
4	0	1/2	0	0.9679	26 056	146	8.7238	5.1376
5	0	1/2	1/2	0.9739	26 056	151	8.7238	5.1376
6	0	1/2	1	1.0213	31 119	162	11.3763	6.6997
7	0	1	0	0.9842	26 056	159	8.7238	5.1376
8	0	1	1/2	1.0000	26 056	158	8.7238	5.1376
9	0	1	1	1.0456	30 627	164	8.7238	5.1376
10	1/2	0	0	(0.8764)	(23 500)	55	6.0342	2.3759
11	1/2	0	1/2	0.8892	24824	64	8.4223	3.3163
12	1/2	0	1	0.9018	28 591	77	5.4848	2.5014
13	1/2	1/2	0	0.9170	26056	89	8.7238	3.9785
14	1/2	1/2	1/2	0.9392	25 806	97	8.4223	3.5570
15	1/2	1/2	1	0.9299	29 839	112	8.7238	3.9785
16	1/2	1	0	0.9336	26 056	98	8.7238	3.9785
17	1/2	1	1/2	1.0041	26 056	133	8.7238	3.9785
18	1/2	1	1	0.9537	29 839	118	8.7238	3.9785
19	1	0	0	1.0852	38 864	64	12.6201	(0.9982)
20	1	0	1/2	1.0450	32 158	59	6.0342	1.0018
21	1	0	1	1.0450	30 058	(53)	(4.5974)	1.0018
22	1	1/2	0	1.0849	[41 052]	69	[36.4811]	6.8386
23	1	1/2	1/2	1.0135	35 669	73	24.3463	[7.2870]
24	1	1/2	1	1.0135	31 758	55	4.6846	1.3378
25	1	1	0	1.1004	39 624	63	16.3306	3.0613

26	1	1	1/2	1.0415	35 669	63	24.3463	[7.2870]
27	1	1	1	1.0271	35 669	54	24.3463	[7.2870]
28	2D-HN	–	0	0.9484	26 549	136	11.3763	6.6997
29	2D-HN	–	1/2	0.9720	26 056	123	8.7238	5.1376
30	2D-HN	–	1	0.9511	25 806	100	7.7997	4.5933
31	2D-HN-mod	–	0	0.9150	26 549	123	11.3763	6.6997
32	2D-HN-mod	–	1/2	0.9161	26 549	129	11.3763	6.6997
33	2D-HN-mod	–	1	0.9373	26 549	156	11.3763	6.6997
34	2D-D	–	0	[1.2310]	26 400	[218]	8.7238	5.1376
35	2D-D	–	1/2	1.1760	26 680	212	8.7238	5.1376
36	2D-D	–	1	1.1438	27 452	217	11.3763	6.6997

Again Method (1/2,0,0) under no. 10 in the table is one of the best relative to all five criteria Q, P, Δ, R_1 and R_2.

Similar results have been obtained with data from previous elections using the MATLAB codes developed.

Conclusions. In this paper we have considered the mathematical aspects of the proportional electoral systems used in Bulgaria, also [8]. These systems act proportionally at a nationwide level but the party mandates are distributed by 31 regional party lists. This leads to a problem with constraints similar to these in the integer transport optimization problem. Several algorithms have been used to solve the problem since a standard optimization procedure may not be politically acceptable due to possible non-uniqueness of the solution. A numerical comparison of the methods proposed shows that the (α, β, γ)–method with $\alpha = 1/2$ and $\beta = \gamma = 0$ is the best relative to a combination of five quantitative criteria.

MATLAB codes for simulation of 2D apportionment algorithms have been developed and are available upon request from the authors.

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МЕТОДИ ЗА ДВУМЕРНО ПРОПОРЦИОНАЛНО РАЗПРЕДЕЛЕНИЕ

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В работата се разглеждат двумерни пропорционални изборни системи, при които броят на партийните мандати се определя на национално ниво, а персонификацията на мандатите става чрез регионални партийни листи. При това, броят на мандатите във всеки район се определя пропорционално на населението. Предложени са нови подобрени методи за двумерно разпределение и са представени резултати от числени пресмятания с данните от парламентарните избори през 2009 г.