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A PROBLEM ON INFINITE SERIES: DIFFICULT OR EASY?

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We show that the difficulty of a problem or its solution depends on how we formulate it, no matter that the essential mathematical ideas that lie beneath it are one and the same. Thus, such a problem can be considered at an initial stage of the course material.

In this note we present three problems based on similar mathematical ideas that can be considered easy or difficult for students depending at what stage of Mathematical Analysis course they are presented. Thus we have different means for their solutions, one – based on the basic definitions and intuitive understanding of the notions, hence, more natural but more difficult; the other – based on the theorems and results, proved later in the course. They present us with an opportunity to introduce and relate in a “spiral” way basic ideas and, thus, deepening the students’ understanding.

The following is a standard problem in the course of Mathematical Analysis [1]:

Problem 1. Find the sum of Leibniz’s series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$.

Solution 1. A standard solution involves considering the Taylor expansion of the logarithmic function $\ln(1+x)$, its convergence at $x = 1$ and, hence, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \ln 2$.

This problem can be posed after considering the Taylor expansion of elementary functions. The following solution is possible only after introducing the notion of a definite integral and theorems about term by term integration of a series.

Solution 2. We obtain this solution if we consider the definite integral $\int_0^1 \frac{1}{x+1} dx$.

We expand $\frac{1}{x+1}$ as a geometric progression $\frac{1}{x+1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$, integrate term by term and then evaluate the result $-x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ at $x = 0$ and $x = 1$ (checking that all the required results apply).

We notice that Solution 2 is also a solution to the following problem:

Problem 2. Prove that $\int_0^1 \frac{1}{x+1} dx = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

Both of the above are more or less standard problems with known, hence, standard solutions. The second one uses the Leibniz-Newton formula for calculation of a definite

integral. Hence, Problem 2 can be considered only after introduction of this formula in the Mathematical Analysis course.

However, if we reformulate Problem 2, as shown below, its solution becomes neither straightforward, nor standard. On the other hand, the solution we give can be understood right after the introduction the notion of a definite integral. This approach can be used as a natural transition from the student's own intuition about "area" to the Riemann definition of an integral.

So, let us consider:

Problem 3. *Prove that*

$$(1) \quad \int_0^1 \frac{1}{x+1} dx = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

without using Leibniz-Newton formula for calculation of the definite integral to the left.

Solution 3. Let $f(x) = \frac{1}{x+1}$ (Figure 1).

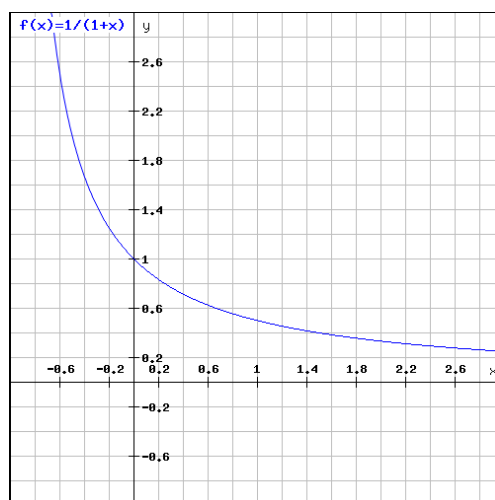


Fig. 1. The graph of the $f(x) = \frac{1}{x+1}$

Let $A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$. We know that by Leibniz's criterion the series defining A is a convergent.

We also know that $\int_0^1 \frac{1}{x+1} dx$, defined as a limit of integral sums, is equal to the shaded area in Figure 2.

We interpret each partial sum of A as a certain area approximation of Area 1.

Then we prove that the integral $\int_0^1 \frac{1}{x+1} dx$ is the limit of the partial sums of the series A .

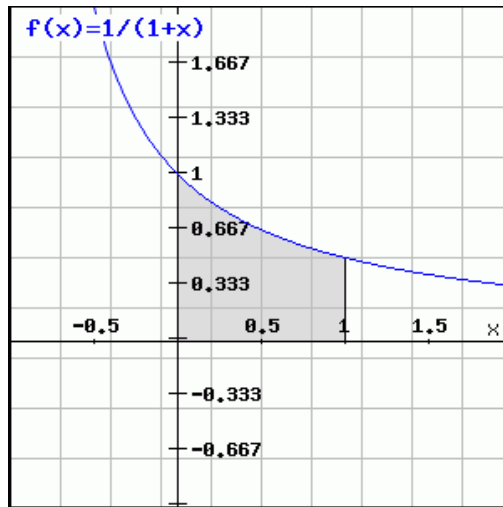


Fig. 2. Area 1 representing $\int_0^1 \frac{1}{x+1} dx$

We approximate Area 1 using disjoint rectangles, the union of whose bases cover the interval $[0, 1]$, i.e. some special Riemann Sums for $\int_0^1 \frac{1}{x+1} dx$.

Let us begin by approximating the area using just one rectangle $- [0, 1] \times [0, 1]$ (L_1). Note that its upper-left corner lies on the graph, i.e. our sample point is the left-hand end of the interval of division.

The area $S(L_1)$ is equal to 1.

The next approximation is with a rectangle R_1 with a base $[0, 1]$ whose upper-right corner lies on the graph, i.e. the sample point is on the right-hand end of the interval of division:

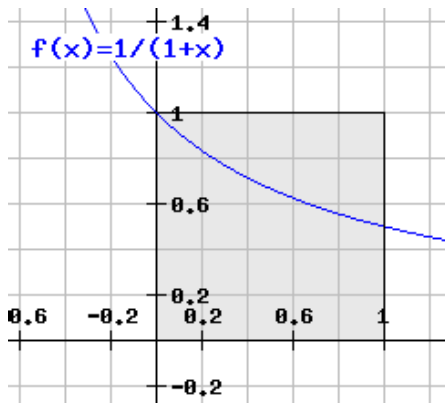


Fig. 3. Rectangle L_1

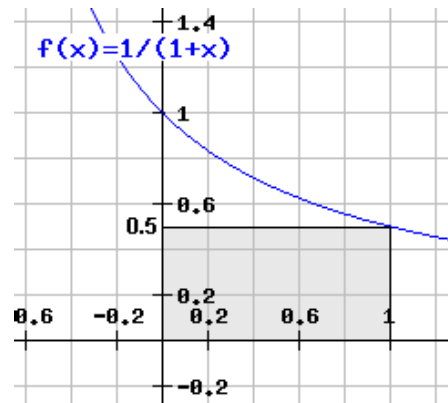


Fig. 4. Rectangle R_1

$S(R_1)$ is equal to $1/2$.

One can think of $S(L_1)$ and $S(R_1)$ as respectively a left-handed and a right-handed Riemann sum for $\int_0^1 \frac{1}{x+1} dx$, with trivial division of $[0, 1]$ of subintervals.

On the other hand, $S(L_1)$ and $S(R_1)$ are respectively equal to the first and the second partial sum of series A, i.e.: $1, 1 - 1/2 = 1/2$.

Now, let L_2 be $[0, 1/2] \times [0, F(0)] \cup [1/2, 1] \times [0, F(1/2)]$.

We go on with constructions as in Figure 5 ((a) and (b)).

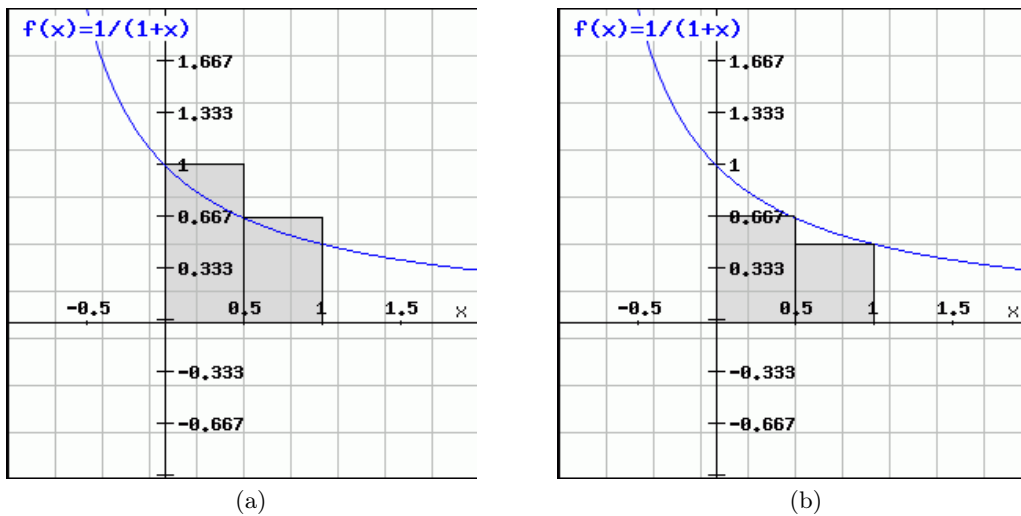


Fig. 5

We use left-hand points in Figure 5 (a) and right-hand points on the Figure 5 (b) – as sample points at which we evaluate $f(x)$.

Let the Riemann Sum to the left be $S(L_2)$ and the Riemann Sum to the right be $S(R_2)$. We have that $S(L_2)$ is $5/6$, and $S(R_2)$ is $7/12$.

The third partial sum of A – S_3 is: $1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, like $S(L_2)$, and S_4 is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$, like $S(R_2)$.

We continue with this process and for a given n and a division σ_n of $[0, 1]$ into n equal sub-intervals. Let $S(L_n)$ and $S(R_n)$ be the Riemann Sums for $\int_0^1 \frac{1}{x+1} dx$ with left-hand end-points and right-hand end-points, respectively.

We prove by induction that

$$(2) \quad S(L_n) = S_{2n-1} \quad \text{and}$$

$$(3) \quad S(R_n) = S_{2n}.$$

We have shown that they are true for $n = 1$ and $n = 2$.

Suppose (2) is true for an arbitrary positive integer n . Let us prove that it is true for $n + 1$.

$$\text{Let } S_{2n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n-2} + \frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2n+1}.$$

Further,

$$\begin{aligned} S(L_n) &= \frac{1}{n} \left(1 + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) \right) = \\ &= \frac{1}{n} \left(1 + \frac{1}{\frac{1}{n}+1} + \frac{1}{\frac{2}{n}+1} + \cdots + \frac{1}{\frac{n-1}{n}+1} \right) = \\ &= \frac{1}{n} \left(1 + \frac{n}{n+1} + \frac{n}{n+2} + \cdots + \frac{n}{2n-1} \right) = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1}, \end{aligned}$$

$$\text{i.e. } S(L_n) = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1}.$$

By the inductive hypothesis $S(L_n) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n-2} + \frac{1}{2n-1}$

So, we have that:

$$(4) \quad \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n-2} + \frac{1}{2n-1}.$$

On the other hand, $S(L_{n+1}) = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} + \frac{1}{2n+1}$.

Let us note that:

$$S(L_{n+1}) = S(L_n) - \frac{1}{n} + \frac{1}{2n} + \frac{1}{2n+1}$$

But, from (4) we know that $S(L_n) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n-2} + \frac{1}{2n-1}$.

So $S(L_{n+1}) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n-2} + \frac{1}{2n-1} - \frac{1}{n} + \frac{1}{2n} + \frac{1}{2n+1}$. Hence,
 $S(L_{n+1}) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n-2} + \frac{1}{2n-1} - \frac{1}{2n} + \frac{1}{2n+1}$. \square

Let us now prove (3). We have already proved that

$$S(L_n) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n-2} + \frac{1}{2n-1}.$$

We use this to prove that $S(R_n) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n}$. Indeed,

$$\begin{aligned} S(L_n) &= \frac{1}{n} \left(1 + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) \right) \\ S(R_n) &= \frac{1}{n} \left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) + f(1) \right). \end{aligned}$$

So,

$$S(R_n) = S(L_n) + \frac{1}{n} (f(1)) - \frac{1}{n} = S(L_n) + \frac{1}{2n} - \frac{1}{n} = S(L_n) - \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n}.$$

\square

Since the integral $\int_0^1 \frac{1}{x+1} dx$ is a limit of some particular sequence of its Riemann sums (e.g. $S(L_n)$) and the sum of the series defining A is a limit of its partial sums, we have that $\int_0^1 \frac{1}{x+1} dx = \lim_{n \rightarrow \infty} S(R_n) = \lim_{n \rightarrow \infty} S(L_n) = \lim_{n \rightarrow \infty} S_{2n-1} = \lim_{n \rightarrow \infty} S_{2n} = \lim_{n \rightarrow \infty} S_n = A$. So we have proved (1).

In that way we see that different formulation of basically one and the same mathematical problem can make it achievable at different stages of a course development as well as make its solution easy or difficult.

REFERENCES

- [1] M. SPIVAK. Calculus, Cambridge University Press, 1994.

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ЕДИН ПРОБЛЕМ ЗА БЕЗКРАЙНИ РЕДОВЕ: ТРУДЕН ИЛИ ЛЕСЕН?

Петра Стайнова

Разглежда се един стандартен въпрос за намиране на сумата на реда на Лайбниц. Дават се две различни решения, използващи знания от различен материал. След преформулиране на проблема се оказва, че решението му е възможно с директно използване само на основни понятия. По този начин е показано, че въз основа на една и съща базисна математическа идея могат да бъдат формулирани задачи, допускащи както стандартни, така и нестандартни решения.