МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2013
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2013
Proceedings of the Forty Second Spring Conference
of the Union of Bulgarian Mathematicians
Borovetz, April 2–6, 2013

STABILITY WITH INITIAL DATA DIFFERENCE FOR DELAY DIFFERENTIAL EQUATIONS*

Snezhana Hristova, Iva Pahleva

Stability with initial data difference of nonlinear delay differential equations is introduced and studied. This type of stability generalizes the known in the literature concept of stability. It gives us the opportunity to compare the behavior of two solutions which initial values as well as initial intervals are different. Lyapunov functions as well as comparison results for scalar ordinary differential equations have been employed.

1. Introduction. One of the main problems in the qualitative theory of differential equations is stability of the solutions. Several types of stability have been investigated in the past, applying various methods such as first and second method of Lyapunov. Stability gives us the opportunity to compare the behavior of solutions starting at different points. Often in the real situations it may be impossible to have only a change in the space variable and not also in the initial time or the initial interval. It requires introducing and studying a new generalization of the classical concept of stability which involves the change of both initial time/interval and initial points. Recently, the study of stability with an initial time difference for ordinary differential equations has been initiated and the corresponding theory of differential inequalities has been investigated (see, for example, [1]–[9] and cited therein references).

In the present paper, we initiate the study of stability with initial data difference for delay differential equations. The difficulty arises because there is a significant difference between initial data difference stability and the classical notion of stability. Continuous Lyapunov's functions as well as comparison results for ordinary differential equation with a parameter are employed. The derivative of Lyapunov functions with respect to the given equations and initial time difference is defined in appropriate way.

2. Preliminary notes and results. Consider the initial value problem for the system of nonlinear delay differential equations

$$x'(t) = f(t, x(t), x(t-r)) \quad \text{for} \quad t \geq t_0, \tag{1}$$

$$x(t) = \varphi(t + t_0) \quad \text{for} \quad t \in [-r, 0], \tag{2}$$

where $x \in \mathbb{R}^n$, $f \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $r > 0$ is a fixed constant, $\varphi \in C([-r, 0], \mathbb{R}^n)$, $t_0 \in R_+$. Denote by $x(t; t_0, \varphi)$ the solution of (1), (2).

****2000 Mathematics Subject Classification:** 34K45, 34D20.

Key words: stability, initial data difference, Lyapunov functions, delay differential equations.

Consider also the initial value problem at a different initial data, i.e. let $\tau_0 \in R_+, \tau_0 \neq t_0$, $\psi \in C([-r,0],\mathbb{R}^n)$ and $x(t;\tau_0,\psi)$ be the solution of system (1) with the initial condition

$$x(t) = \psi(t+\tau_0) \quad \text{for} \quad t \in [-r,0]. \tag{3}$$

The solutions of the initial value problems (1), (2) and (1), (3) are defined on different initial intervals, i.e. the first one is defined on $[t_0 - r, t_0]$, the second - on $[\tau_0 - r, \tau_0]$.

Assume for any $\psi \in C([-r,0],\mathbb{R}^n)$ and $\tau_0 \geq 0$ the solution of (1)–(3) exists on $[t_0,\infty)$.

$$K = \{a \in C[\mathbb{R}_+,\mathbb{R}_+] \ : \ a(s) \text{ is strictly increasing and } a(0)=0\};$$
$$CK = \{b \in C[\mathbb{R}_+ \times \mathbb{R}_+,\mathbb{R}_+] \ : \ b(t,.) \in K \text{ for any fixed } t \in [0,\infty)\},$$
$$S(\rho) = \{x \in \mathbb{R}^n : \ \|x\| < \rho\}, \quad \Lambda = C(\mathbb{R}_+ \times \mathbb{R}^n,\mathbb{R}_+),$$
$$\|\phi\|_0 = \max\{\|\phi(s)\| : \ s \in [-r,0]\} \ \text{ for } \phi \in C([-r,0],\mathbb{R}^n).$$

Definition 1. *Let $x^*(t) = x(t;t_0,\varphi)$ be a solution of* (1), (2). *The solution $x^*(t)$ is said to be* stable with initial data difference *if for every $\varepsilon > 0$ there exists a positive $\delta = \delta(t_0,\varphi,\varepsilon) > 0$ such that for any $\psi \in C([-r,0],\mathbb{R}^n)$ and any $\tau_0 \in \mathbb{R}_+$, inequalities $\|\varphi - \psi\|_0 < \delta$ and $|\tau_0 - t_0| < \delta$ imply $\|x(t+\eta;\tau_0,\psi) - x^*(t)\| < \varepsilon$ for $t \geq t_0$ where $\eta = \tau_0 - t_0$.*

The stability with initial data difference gives us an opportunity to compare solutions of delay differential equations which both initial intervals and initial functions are different. In the case of ordinary differential equations stability with initial time difference is studied in [2], [4], [6], [8].

Remark 1. The concept of stability with initial data difference is a generalization of the classical concept of stability of a solution.

We will give a brief overview of both concepts of stability.

Case 1. (*Stability of a solution with respect to a nonzero solution*). Let $x^*(t) = x(t;t_0,\varphi_0)$ be a solution of (1), (2). Study the stability of $x^*(t)$. Consider the solution $\tilde{x}(t) = x(t;t_0,\psi)$ of (1), where $\psi \in C([-r,0],\mathbb{R}^n) : \psi \not\equiv \varphi$. Consider the initial value problem for the delay differential equations

$$\begin{aligned} &z' = \tilde{f}(t,z(t),z(t-r)), \quad t \geq t_0 \\ &z(t) = \phi(t+t_0), \quad t \in [-r,0], \end{aligned} \tag{4}$$

where $\tilde{f}(t,z,y) = f(t,z+x^*(t),y+x^*(t-r)) - f(t,x^*(t),x^*(t-r))$ and $\phi(t) = \psi(t) - \varphi(t)$, $t \in [-r,0]$.

The function $z(t;t_0,\psi_0,\varphi_0) = \tilde{x}(t) - x^*(t)$ is a solution of the IVP (4).

The initial value problem (4) has a zero solution and the study of stability properties of the nonzero solution $x^*(t)$ of (1) is equivalent to the study of stability of zero solution of (4).

Case 2. (*Stability with initial data difference*). Study the stability with initial data difference of $x^*(t)$. Consider the solution $\tilde{X}(t) = x(t;\tau_0,\psi)$ of (1), (3), where both, the initial function $\psi \in C([-r,0],\mathbb{R}^n) : \ \psi \not\equiv \varphi$ and $\tau_0 \neq t_0$. Denote by $\eta = \tau_0 - t_0$ and consider the initial value problem for delay differential equations

$$\begin{aligned} &z' = \tilde{f}(t,z(t),z(t-r)), \quad t \geq t_0 \\ &z(t) = \phi(t+t_0), \quad t \in [-r,0], \end{aligned} \tag{5}$$

where $\tilde{f}(t,z,y) = f(t+\eta,z+x^*(t),y+x^*(t-r)) - f(t,x^*(t),x^*(t-r))$ and $\phi(t) =$

$\psi(t+\eta) - \varphi(t)$, $t \in [-r, 0]$.

Then $z(t; t_0, \tau_0, x_0, y_0) = \tilde{X}(t+\eta) - x^*(t)$ is a solution of (5).

The system (5) has no zero solution since $\tilde{f}(t, 0, 0) = f(t+\eta, x^*(t).x^*(t-r)) - f(t, x^*(t), x^*(t-r)) \not\equiv 0$. Therefore, in this case study of stability with initial data difference of $x^*(t)$ could not be reduced to the study of stability of the zero solution of an appropriate delay differential equation.

In our further investigations we will use the following comparison initial value problem for the scalar ordinary differential equation:

$$u' = g(t, u, \eta), \qquad u(t_0) = u_0 \tag{6}$$

where $u \in \mathbb{R}$, $g(t, 0, \eta) \equiv 0$ for $t \in \mathbb{R}_+, \eta \in [0, \rho)$ is a parameter, $\rho \in (0, \infty)$ is given.

We will assume that for any $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ and any $\eta \in [0, \rho]$ the initial value problem (6) has a solution, defined on $[t_0, \infty)$.

Let $t \geq 0$, $x \in \mathbb{R}^n$, $V(t, x) \in \Lambda$, and $\phi, \psi \in C([-r, 0], \mathbb{R}^n)$. We define a derivative of the function $V(t, x)$ along trajectories of solutions of (1):

$$D^-_{(1)} V(t, \phi(0), \psi(0), \eta) = \limsup_{\varepsilon \to 0-} \frac{1}{\varepsilon} \Bigg\{ V\Bigg(t + \varepsilon, \phi(0) - \psi(0) +$$

$$\varepsilon \Bigg(f(t+\eta, \phi(0), \phi(-r)) - f(t, \psi(0), \psi(-r) \Bigg) \Bigg) - V(t, \phi(0) - \psi(0)) \Bigg\}.$$

Lemma 1 (Comparison result)**.** *Let the following conditions be satisfied:*

1. *Function $g \in C(\mathbb{R}_+ \times \mathbb{R} \times [0, \rho], \mathbb{R}_+)$ where $\rho > 0$ is a fixed number.*

2. *The points $t_0, \tau_0 \in \mathbb{R}_+ : \eta = \tau_0 - t_0 \in (0, \rho)$ and $x^*(t) = x(t; t_0, \varphi)$ and $\tilde{x}(t) = x(t; \tau_0, \psi)$ are solutions of the initial value problems* (1), (2), *and* (1), (3) *defined on $[t_0 - r, T]$ and $[\tau_0 - r, T + \eta]$, respectively.*

3. *The initial value problem* (6) *has a solution for any value of the parameter $\eta \in [0, \rho]$, which is defined for $t \in [t_0, T]$.*

4. *Function $V \in \Lambda$ and for any point $t \geq t_0$ and any $\varphi, \psi \in C([-r, 0], \mathbb{R}^n)$ such that $V(t+s, \varphi(s) - \psi(s)) > V(t, \varphi(0) - \psi(0))$ for $s \in [-r, 0)$ the inequality*

$$D^-_{(1)} V(t, \varphi(0), \psi(0)), \eta) \leq g(t, V(t, \varphi(0) - \psi(0)), \eta)$$

holds.

Then the inequality $\max_{s \in [-r, 0]} V(t_0 + s, \varphi(s) - \psi(s)) \leq u_0$ *implies*

$$V(t, x^*(t) - \tilde{x}(t+\eta)) \leq u^*(t), \qquad t \in [t_0, T].$$

Proof. Let point u_0 be such that $\max_{s \in [-r, 0]} V(t_0 + s, \varphi(s) - \psi(s)) \leq u_0$.

Let $u_n(t, \eta)$ be the maximal solution of the initial value problem

$$u' = g(t, u, \eta) + \frac{1}{n}, \qquad u(t_0) = u_0 + \frac{1}{n}, \tag{7}$$

where n is a natural number and $\eta \in (0, \rho)$ is a parameter. According to condition 4 the solution $u_n(t, \eta)$ is defined for $t \in [t_0, T]$.

Define a function $m(t) \in C([t_0 - r, T], \mathbb{R}_+)$ by $m(t) = V(t, x^*(t) - \tilde{x}(t+\eta))$.

Because of the fact that $u^*(t) = \lim_{n \to \infty} u_n(t, \eta)$ for any fixed $\eta \in [0, \rho]$, it is enough to prove that for any natural number n

$$m(t) \le u_n(t) \quad \text{for } \; t \in [t_0, T]. \tag{8}$$

Note that for any natural number n inequality $m(t_0) \le u_0 < u_n(t_0)$ holds.

Assume that inequality (8) is not true. Let n be a natural number such that there exists a point $\xi > t_0 :\; m(\xi) > u_n(\xi)$. Let $t_n^* = max\{t > t_0 : m(s) < u_n(s) \text{ for } s \in [t_0, t)\}$, $t_n^* < \infty$.

Then $m(t_n^*) = u_n(t_n^*), \quad m(t) < u_n(t)$ for $t \in [t_0, t_n^*)$, where $\delta > 0$ is enough small number and therefore

$$D_- m(t_n^*) \ge u_n'(t_n^*) = g(t, u_n(t_n^*), \eta) + \frac{1}{n} = g(t, m(t_n^*), \eta) + \frac{1}{n}. \tag{9}$$

From $g(t, u) + \dfrac{1}{n} > 0$ on $[t_n^* - r, t_n^*]$ it follows that function $u_n(t)$ is nondecreasing and $m(t_n^*) = u_n(t_n^*) \ge u_n(s) > m(s)$ for $s \in [t_n^* - r, t_n^*)$,i.e. the inequality

$$V(t_n^* + s, x^*(t_n^* + s) - \tilde{x}(t_n^* + s + \eta)) > V(t_n^*, x^*(t_n^*) - \tilde{x}(t_n^* + \eta)) \tag{10}$$

holds for $s \in [-r, 0)$.

Define $\Phi(t) = x^*(t_n^* + t)$ and $\Psi(t) = \tilde{x}(t_n^* + t + \eta)$ for $t \in [-r, 0]$. Therefore $\Phi, \Psi \in C([-r, 0], \mathbb{R}^n)$.

According to (10) we have $V(t_n^* + t, \Phi(t) - \Psi(t)) > V(t_n^*, \Phi(0) - \Psi(0))$ for $t \in [-r, 0)$. We apply condition 4 of Lemma 1 and get $D_- m(t_n^*) = D^-_{(1)} V(t, \Phi(0), \Psi(0)), \eta) \le g(t, m(t_n^*)) < g(t, m(t_n^*), \eta) + \dfrac{1}{n}$ that contradicts (9). $\square$

3. Main Results.

Theorem 1 (Stability with initial data difference). *Let the following conditions hold:*

1. *The function $x^*(t) = x(t; t_0, \varphi)$ is a solution of* (1), (2), *where $\varphi \in C([-r, 0], \mathbb{R}^n)$, $t_0 \in \mathbb{R}_+$.*

2. *There exists a function $V \in \Lambda$ such that*

(i) *for any point $t \in \mathbb{R}_+$ and any functions $\varphi, \psi \in C([-r, 0], \mathbb{R}^n)$ such that $V(t + s, \varphi(s) - \psi(s)) > V(t, \varphi(0) - \psi(0))$ for $s \in [-r, 0)$ the inequality*

$$D^-_{(1)} V(t, \varphi(0), \psi(0)), \eta) \le g(t, V(t, \varphi(0) - \psi(0)), \eta), \tag{11}$$

holds, where $g \in C(\mathbb{R}_+ \times \mathbb{R} \times [0, \rho), \mathbb{R}_+)$, $g(t, 0, \eta) \equiv 0$, $\rho > 0$ is a constant.

(ii) $b(\|x\|) \le V(t, x) \le a(\|x\|), \quad t \in \mathbb{R}_+,\; x \in S(\rho)$, *where $a, b \in K$.*

3. *The equation* (6) *is stable with respect to the parameter.*

Then the solution $x^*(t)$ is stable with initial data difference.

Proof. Let $\varepsilon \in (0, \rho)$ be a positive number and $x^*(t) = x(t; t_0, \varphi)$.

From condition 3 there exist $\Delta_1 = \Delta_1(\varepsilon) > 0$ and $\delta_1 = \delta_1(\varepsilon, t_0) > 0$ such that the inequalities $|\eta| < \Delta_1$ and $|u_0| < \delta_1$ imply

$$|u(t; t_0, v_0)| < b(\varepsilon), \quad t \ge t_0, \tag{12}$$

where $u(t; t_0, v_0)$ is a solution of the equation (6). Assume $\delta_1 < b(\varepsilon)$.

From $a \in K$ there exists $\delta_2 = \delta_2(\varepsilon) > 0$: if $\|x\| < \delta_2$ then $a(\|x\|) < \delta_1$.

Let $\psi \in C([-r, 0], \mathbb{R}^n) :\; \|\varphi - \psi\|_0 < \delta$ and $\tau_0 :\; 0 < \eta = \tau_0 - t_0 < \Delta_1$, where $\delta = \min(\delta_1, \delta_2)$.

We will prove that

$$\|x^*(t) - \tilde{x}(t + \eta)\| < \varepsilon, \quad t \ge t_0, \tag{13}$$

where $\tilde{x}(t) = x(t;\tau_0,\psi)$ is a solution the initial value problem (1), (3).

We note that inequality (13) holds on $[t_0 - r, t_0]$.

Suppose inequality (13) is not true for $t > t_0$. Therefore, there exists a point $t^* > t_0$ such that

$$(14) \qquad \|x^*(t^*) - \tilde{x}(t^* + \eta)\| = \varepsilon, \quad \text{and} \quad \|x^*(t) - \tilde{x}(t + \eta)\| < \varepsilon, \quad t \in [t_0, t^*).$$

From inequality (14) it follows that there exists a point $t_0^* \in [t_0, t^*)$ such that $\|x^*(t) - \tilde{x}(t + \eta)\| < \delta < \delta_2$ for $t \in [t_0 - r, t_0^*]$.

Now let $u_0 = \max\limits_{s\in[-r,0]} V(t_0^* + s, x^*(t_0^* + s) - \tilde{x}(t_0^* + s + \eta))$.

Since $[t_0^* - r, t_0^*] \subset [t_0 - r, t_0^*]$ and $\|x^*(t_0^*) - \tilde{x}(t_0^* + \eta)\| < \delta_2$ we get

$$u_0 < V(t_0^*, x^*(t_0^*) - \tilde{x}(t_0^* + \eta)) \le a(\|x^*(t_0^*) - \tilde{x}(t_0^* + \eta)\|) < \delta_1.$$

Let $u^*(t; t_0, u_0)$ be the maximal solution of the scalar differential equation (6) for the parameter $\eta = \tau_0 - t_0$. Therefore, $u^*(t; t_0, u_0) < \varepsilon$, $[t_0^*, \infty)$.

From the choice of ε follows that $x^*(t) - \tilde{x}(t + \eta) \in S(\rho) \quad t \in [t_0 - r, t^*)$.

From inequalities (12), (14), the choice of the point t^*, and condition (ii) of Theorem 1 we obtain $b(\varepsilon) > u^*(t^*; t_0^*, u_0^*) \ge V(t^*, x^*(t^*) - \tilde{x}(t^* + \eta)) \ge b(\|x^*(t^*) - \tilde{x}(t^* + \eta)\|) = b(\varepsilon)$. The contradiction proves inequality (13). $\square$

REFERENCES

[1] T. Jankowski. Delay integro-differential inequalities with initial time difference and applications. *J. Math. Anal. Appl.*, **291** (2004), 605–624.

[2] V. Lakshmikantham, S. Leela, J. V. Devi. Stability criteria for solutions of differential equations relative to initial time difference, *Int. J. Nonlinear Diff. Eqns*, **5** (1999), 109–114.

[3] A. Li, L. Wei, J. Ye. Exponential and global stability of nonlinear dynamical systems relative to initial time difference. *Appl. Math. Comput.*, **217** (2011), 5923–5929.

[4] A. Li, E. Feng, S. Li. Stability and boundedness criteria for nonlinear differential systems relative to initial time difference and applications. *Nonlinear Anal.: Real World Appl.*, **10** (2009), 1073–1080.

[5] M. D. Shaw, C. Yakar. Stability criteria and slowly growing motions with initial time difference. *Problems of Nonlinear Analysis in Engineering Systems*, **1** (2000), 50–66.

[6] X. Song, A. Li, Zh. Wang. Study on the stability of nonlinear differential equations with initial time difference. *Nonlinear Anal.: Real World Appl.*, **11** (2010), 1304–1311.

[7] X. Song, S. Li, A. Li. Practical stability of nonlinear differential equation with initial time difference. *Appl. Math. Comput.*, **203** (2008), 157–162.

[8] C. Yakar, M. Shaw. A Comparison result and Lyapunov stability criteria with initial time difference. *Dynam. Cont., Discr. Impuls. Sys., series A: Mathematical Analysis*, **12** (2005), 731–737.

[9] C. Yakar, M. D. Shaw. Initial time difference stability in terms of two measures and variational comparison result. *Dynamics of Continuous, Discrete and Impulsive Systems. Series A: Mathematical Analysis*, **15** (2008) No 3, 417–425.

Snezhana Hristova
Iva Pahleva
Faculty of Mathematics and Informatics
Plovdiv University
4000 Plovdiv, Bulgaria
e-mail: snehri@uni-plovdiv.bg
ipahleva@gmail.com

УСТОЙЧИВОСТ С РАЗЛИКА В НАЧАЛНИТЕ ДАННИ НА ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ СЪС ЗАКЪСНЕНИЕ

Снежана Георгиева Христова, Ива Георгиева Пъхлева

Въведено е понятието устойчивост с разлика в началните данни за диференциални уравнения със закъснение. Този тип устойчивост обобщава класическото понятие устойчивост и дава възможност да се сравнява поведението на две решения, които имат както различни начални стойности, така и различни начални интервали. Използвани са функции на Ляпунов и сравнение с повходящи скаларни диференциални уравнения.