МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2013
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2013
Proceedings of the Forty Second Spring Conference
of the Union of Bulgarian Mathematicians
Borovetz, April 2–6, 2013

# ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF BOUNDARY-VALUE PROBLEMS FOR NONLINEAR SYSTEMS WITH DOUBLE SINGULARITY*

**Neli Sirakova**

This paper discusses the asymptotic decomposition of a solution of nonlinear boundary-value problems (BVP) with double singularity and general boundary conditions. We assume that the differential system contains an additional function, which defines the perturbation as double singular.

**1. Introduction.** In this paper we deal with BVP of ordinary differential equations in the form

$$\varepsilon \frac{dx}{dt} = Ax + \varepsilon F(x, t, \varepsilon, f(t, \varepsilon)) + \varphi(t), \quad t \in [a, b], \tag{1}$$

$$l(x) = h, \tag{2}$$

where $\varepsilon$ is a small positive parameter, $0 \leq \varepsilon \ll 1$ , $h \in R^m$.

The following conditions should be observed:

**(C1)** $A$ is a constant $(n \times n)$-matrix. Let $\sigma(A)$ be the spectrum of the matrix $A$ and $\lambda_i \in \sigma(A)$ $\forall i = \overline{1, n}$. We assume that $\lambda_i \neq \lambda_j$, $i \neq j$ and $\operatorname{Re} \lambda_i < 0$. The function $\varphi(t)$ is an $n$-dimensional vector-function of the class $C^{(\infty)}([a, b])$;

**(C2)** The function $F(x, t, \varepsilon, f(t, \varepsilon))$ is an $n$-dimensional vector-function having an arbitrary order continuous partial derivatives with respect to all arguments in the domain $G = D_x \times [a, b] \times [0, \bar{\varepsilon}] \times D_f$, where $D_x \subset R^n$ is a neighborhood of the solution $x^{(0)}(t)$ of the degenerate system $(\varepsilon = 0)$ $Ax^{(0)} + \varphi(t) = 0$; $D_f \subset R^p$ is a bounded and closed domain, $0 < \bar{\varepsilon} \ll 1$. The function $f = f(t, \varepsilon)$ is smooth of arbitrary order with respect to all arguments in the domain $G_1 = [a, b] \times (0, \bar{\varepsilon}]$ and its values belong to $D_f$.

**(C3)** $l$ is a linear, bounded vector functional, $l \in (C\,[a, b] \to R^n, R^m)$, $h \in R^m$.

We assume that the function $f$ contains singular elements (for example, $f = f(\exp(-t/\varepsilon), \sin(t/\varepsilon))$. On one hand, the small parameter $\varepsilon$ is in front of the derivative and on another, $\varepsilon^{-1}$ is involved in the function $f$. Therefore the boundary problem (1), (2) is with double singularity.

The Cauchy problem for nonlinear systems with double singularity was investigated in [6]. In the present work the behavior of the asymptotic expansion of the solution of the problem (1), (2) is studied. The construction of a formal asymptotic solution of

*2010 Mathematics Subject Classification: 34B15, 35F60.
**Key words:** differential equations, boundary value problem, nonlinear systems, double singularity.

the problem (1), (2) is performed in [4] and is based on the boundary function method described in [8].

The papers [2, 3, 4, 7] consider in both cases $m = n$ and $m \neq n$ the asymptotic expansion of the solution of the almost regular BVP

$$\frac{dx}{dt} = A(t)x + \varepsilon F(x, t, \varepsilon, f(t, \varepsilon)) + \varphi(t), \quad l(x) = h, \quad t \in [a, b], \quad 0 < \varepsilon \ll 1,$$

where $f(t, \varepsilon)$ is a singular function.

If $x = (x_1, \cdots, x_n)$, then the standard norm of the vector $x$ is defined by $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$, while the standard norm of a $(m \times n)$-matrix $B = (b_{ij})$ is defined by $\|B\| = \max\limits_{i} \sum\limits_{j=1}^{n} |b_{ij}|$. Furthermore, we define a norm of the linear operator $l$ by $\|l(\psi)\| \leq \bar{b}\|\psi\|, \bar{b} > 0$.

**2. Auxiliary results.** Formally, the asymptotic decomposition of the solution of (1), (2) was obtained in paper [5] by introducing a new parameter. Instead of the problem (1), (2) we consider the boundary problem with two parameters

$$\varepsilon \frac{dz}{dt} = Az + \varepsilon . F(z, t, \varepsilon, f(t, \mu)) + \varphi(t), \quad t \in [a, b], \tag{3}$$

$$l(z) = h.$$

The problem (3) is singularly perturbed with respect to the small parameter $\varepsilon$, and it is possible to use the method of boundary functions [8, 5], i.e. the solution to the border problem is searched in the form

$$z(t, \varepsilon, \mu) = \sum_{k=0}^{\infty} \left[ z^{(k)}(t, \mu) + \Pi_k(\tau, \mu) \right] \varepsilon^k \tag{4}$$

where $\tau = \dfrac{t-a}{\varepsilon} = (t-a)\varepsilon^{-1}$, $\Pi_k(\tau, \mu)$, $k \geq 0$ , $\tau \in [0, +\infty)$ are boundary functions in a right neighborhood of point $t = a$. After the determination of $z^{(k)}(t, \mu)$ and $\Pi_k(\tau, \mu)$ the solution of (1), (2) has the form

$$x(t, \varepsilon) = \sum_{k=0}^{\infty} \left[ z^{(k)}(t, \varepsilon) + \Pi_k(\tau, \varepsilon) \right] \varepsilon^k. \tag{5}$$

According to condition C2 for the function $F$ we assume moreover that it has the representation

$$F(z, t, \varepsilon, f(t, \mu)) = \sum_{k=0}^{\infty} A_k(t, \mu) z^{k+1} \varepsilon^k, \tag{6}$$

where $A_k(t, \mu)$ are $(n \times n)$-matrices with elements having arbitrary order continuous derivatives with respect to $t \in [a, b]$, $\mu \in (0, \bar{\varepsilon}]$.

When setting in [5] the coefficients of the decomposition (5) we substantially utilize the fundamental matrix $\varphi(\tau)$ of the system $\dfrac{dx}{d\tau} = Ax$, $\tau \in [0, \infty)$ and let $U(\tau, s) = \varphi(\tau)\varphi^{-1}(s)$ be the corresponding Cauchy matrix. We introduce the $(m \times n)$ matrix $D(\varepsilon) = l\left(U(\cdot, s)\right) = l\left(\varphi\left(\dfrac{(\cdot) - a}{\varepsilon}\right)\varphi^{-1}(s)\right)$. It has different representations depending on the type of the functional $l$. In this work, as in [5], we assume that the matrix $D(\varepsilon)$ has the form

$$D(\varepsilon) = l(U(\cdot, s)) = D_0 + O\left(\exp\left(-\frac{\lambda}{\varepsilon}\right)\right),\ \lambda > 0$$

where $D_0$ is a $(m \times n)$- matrix with constant elements and $O\left(\exp\left(-\frac{\lambda}{\varepsilon}\right)\right)$ is $(m \times n)$ matrix, whose elements are exponentially small in terms of features $\varepsilon$ and may be ignored because they are smaller than any powers of $\varepsilon$.

The results obtained in [5] are related to the rank of the $(m\times n)$ matrix $D_0$ : $\operatorname{rank} D_0 = n = m$, $\operatorname{rank} D_0 = m < n$ and $\operatorname{rank} D_0 = n_1 < n = m$. In this work we consider only the case $\operatorname{rank} D_0 = n = m$. The results of [5] in this case are presented in Theorem 1.

Moreover, we will use the introduced in [5] indications

(7)
$$\varphi_k(t, \mu, z^{(0)}, \cdots, z^{(k-1)}) = \begin{cases} 0, k = 0, \\ A_0(a, \mu) z^{(0)},\ k = 1, \\ A_0(a, \mu) z^{(k-1)} + g_k(t, \mu, z^{(0)}, \cdots z^{(k-2)}),\ k = 2, 3, \ldots, \end{cases}$$

(8)
$$\psi_k(\tau, \mu, \Pi_0, \cdots, \Pi_{k-1}) = \begin{cases} 0, k = 0, \\ A_0(a, \mu)\Pi_0,\ k = 1, \\ A_0(a, \mu)\Pi_{k-1} + f_k(\tau, \mu, \Pi_0, \ldots, \Pi_{k-2}),\ k = 2, 3, \ldots \end{cases}$$

The functions $g_k$ and $f_k$ have a polynomial character with respect to $z^{(0)}, \ldots, z^{(k-2)}$ and $\Pi_0, \ldots, \Pi_{k-2}$, respectively, with norm-bounded coefficients.

**Theorem 1** ([5])**.** *We assume that conditions $(C1) - (C3)$ hold, and* $\operatorname{rank} D_0 = n = m$*. Then there is a unique solution in the domain $G_1$ the BVP (1), (2) which is continuously differentiable with respect to $t \in [a, b]$ and continuous for $\mu \in (0, \bar{\varepsilon}]$. The series (5) is formally asymptotic series for this solution, where the functions $z^{(k)}(t, \mu), k \geq 0$ have the form*

(9)
$$z^{(k)}(t, \mu) = \begin{cases} -A^{-1}\varphi(t), k = 0, \\ A^{-1}\left(\frac{\partial}{\partial t} z^{(k-1)}(t, \mu) - \varphi_k(t, \mu, z^{(0)}, \ldots, z^{(k-1)})\right), k \geq 1. \end{cases}$$

*for $\mu = \varepsilon$ and the functions $\Pi_\kappa(\tau, \mu)$, $k \geq 0$ have the form*

$$\Pi_0(\tau) = U(\tau, a) D_0^{-1}\bar{h}_0, \tau \in [0, \infty), \quad \overline{h}_0 = h - l(z^{(0)}) = h + l(A^{-1}\varphi(\cdot)),$$

$$\Pi_1(\tau, \mu) = U(\tau, a) D_0^{-1}\bar{h}_1(\mu) + \int_0^\tau U(\tau, s) A_0(a, \mu)\Pi_0(s) ds, \overline{h}_1(\mu)$$
$$= -l(z^{(1)}(\cdot)) - l\left(\int_0^{(\cdot)} U(\cdot, s) A_0(a, \mu)\Pi_0(s) ds\right)$$

$$\Pi_\kappa(\tau, \mu) = U(\tau, a) D_0^{-1}\bar{h}_\kappa(\mu)$$
$$+ \int_0^\tau U(\tau, s)\left[A_0(a, \mu)\Pi_{k-1}(s, \mu) + f_k(s, \mu, \Pi_0, \Pi_1, \ldots, \Pi_{k-2})\right] ds, \quad k \geq 2$$

*for* $\mu = \varepsilon$ *and*

$$\overline{h}_k(\mu) = -lz^{(k)}(\cdot) - l\left(\int_a^{(\cdot)} U(\cdot,s)[A_0(a,\mu)\Pi_{k-1}(s,\mu) + f_k(s,\mu,\Pi_0(s),\ldots,\Pi_{k-2}(s,\mu))]ds\right).$$

The functions $z^{(k)}(t,\mu), k \geq 0$, are bounded, i.e. the following inequalities hold $\left\| z^{(k)}(t,\mu)\right\| \leq N_k$, $\forall t \in [a,b]$ and $\forall \mu \in (0,\bar{\varepsilon}]$, where $N_k$ are positive constants. Then the boundary functions $\Pi_k(\tau,\mu)$, $k \geq 0$ decrease exponentially at $\tau \to \infty$ and $0 < \mu \leq \bar{\varepsilon}$.

**3. Main results.** In the boundary problem (3) we make a change $u(t,\varepsilon,\mu) = z(t,\varepsilon,\mu) - Z_n(t,\varepsilon,\mu)$ where $Z_n(t,\varepsilon,\mu) = \sum_{k=0}^{n}\left[z^{(k)}(t,\mu) + \Pi_k(\tau,\mu)\right]\varepsilon^k$ is the $n$-th partial sum of the series (4). As a result we obtain that $u(t,\varepsilon,\mu)$ satisfies boundary problems

$$\varepsilon\frac{du}{dt} = Au + H_n(u,t,\varepsilon,\mu), \tag{10}$$

$$l(u) = 0,$$

where

$$H_n(u,t,\varepsilon,\mu) = \varepsilon F(u + Z_n, t, \varepsilon, f(t,\mu)) - \sum\nolimits_{k=0}^{n}\varphi_k\varepsilon^k - \sum\nolimits_{k=0}^{n}\psi_k\varepsilon^k. \tag{11}$$

The functions $\varphi_k$ and $\psi_k$ are referred in equations (7) and (8), respectively.

**Lemma 2.** *The following inequality holds:*

$$\|H_n(0,t,\varepsilon,\mu)\| \leq C\varepsilon^{n+1}, C > 0, t \in [a,b], \varepsilon \in [0,\varepsilon_1], \mu \in (0,\varepsilon_1], \quad 0 < \varepsilon_1 < \bar{\varepsilon}.$$

**Proof.** We will determine the type of $\varepsilon F(Z_n,t,\varepsilon,f(t,\mu))$.

On the basis of (6) we present the function $F$ with the powers of $\varepsilon$. Each coefficient in front of the powers of $\varepsilon$ consists of two sums. One of them depends on $t$ and the other on $\tau$. We use that $\tau = \dfrac{t-a}{\varepsilon}$, $\dfrac{d\Pi_k}{dt} = \varepsilon^{-1}\dfrac{d\Pi_k}{d\tau}$ and we expand $A(a+\tau\varepsilon,\mu)$ in a Taylor series in a neighbourhood of the point $(a,\mu)$. We obtain consecutively

$$\varepsilon F(Z_n,t,\varepsilon,f(t,\mu)) = \varepsilon F\left(\sum_{k=0}^{n}(z^{(k)} + \Pi_k)\varepsilon^k, t, \varepsilon, f(t,\mu)\right)$$

$$= \varepsilon\sum_{k=0}^{\infty}\bar{F}^k(t,\tau,\mu,z^{(0)},\ldots,z^{(k-1)},\Pi_0,\ldots,\Pi_{k-1})\varepsilon^k = \sum_{k=0}^{\infty}\bar{F}^k\varepsilon^{k+1} = \sum_{k=1}^{\infty}\bar{F}^{k-1}\varepsilon^k,$$

where

$$\bar{F}^0 = A_0(t,\mu)z^{(0)} + A_0(a,\mu)\Pi_0 = \varphi_1 + \psi_1,$$
$$\bar{F}^1 = A_0(t,\mu)z^{(1)} + g_2(t,\mu,z^{(0)}) + A_0(a,\mu)\Pi_1 + f_2(\tau,\mu,\Pi_0) = \varphi_2 + \psi_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\bar{F}^k = A_0(t,\mu)z^{(k)} + g_{k+1}(t,\mu,z^{(0)},z^{(1)},\ldots,z^{(k-1)}) + A_0(a,\mu)\Pi_k$$
$$+ f_{k+1}(\tau,\mu,\Pi_0,\Pi_2,\ldots,\Pi_{k-1}) = \varphi_{k+1} + \psi_{k+1}.$$

Therefore $\varepsilon F(Z_n,t,\varepsilon,f)$ has the form

$$\varepsilon F(Z_n,t,\varepsilon,f) = \sum_{k=1}^{\infty}\varphi_k\varepsilon^k + \sum_{k=1}^{\infty}\psi_k\varepsilon^k = \sum_{k=1}^{n}\varphi_k\varepsilon^k + \sum_{k=1}^{n}\psi_k\varepsilon^k + \mathrm{O}(\varepsilon^{k+1}). \tag{12}$$

Note that in Theorem 1 the boundedness of $z^{(k)}$ and the exponential decrease of $\Pi_k$

have been proved hence (12) is fulfilled. Then $\varepsilon F(Z_n, t, \varepsilon, f) - \sum_{k=1}^{n} \varphi_k \varepsilon^k - \sum_{k=1}^{n} \psi_k \varepsilon^k = \mathrm{O}(\varepsilon^{n+1})$, i.e.

$$\|H_n(0, t, \varepsilon, \mu)\| \leq \left\|\mathrm{O}(\varepsilon^{n+1})\right\| \leq C\varepsilon^{n+1}. \qquad \square$$

**Lemma 3.** *Let in the neighbourhood of the degenerate solution $\|z^{(0)}\| < \delta$, the following inequality be fulfilled $\|z\| \leq \rho < \delta$ for $t \in [a, b], \varepsilon \in [0, \varepsilon_1], \mu \in (0, \varepsilon_1]$. Then there is a positive constant $\bar{C}$, such that if $\|u'\| \leq \bar{\delta}$ and $\|u''\| \leq \bar{\delta}$, where $0 < \bar{\delta} < \delta$ and $\bar{\delta} + \rho < \delta$, the function $H_n(u, t, \varepsilon, \mu)$ satisfies the inequality*

$$\|\Delta H_n\| = \|H_n(u', t, \varepsilon, \mu) - H_n(u'', t, \varepsilon, \mu)\| \leq \bar{C}\varepsilon\|u' - u''\|.$$

**Proof.** We use the presentation of $H_n$ by (11). Then

$$\begin{aligned}\Delta H_n &= H_n(u', t, \varepsilon, \mu) - H_n(u'', t, \varepsilon, \mu) = \varepsilon F(u' + Z_n, t, \varepsilon, f(t, \mu)) - \sum_{k=0}^{\infty} \varphi_k \varepsilon^k - \sum_{k=0}^{\infty} \psi_k \varepsilon^k \\ &\quad - \left[\varepsilon F(u'' + Z_n, t, \varepsilon, f(t, \mu)) - \sum\nolimits_{k=0}^{\infty} \varphi_k \varepsilon^k - \sum\nolimits_{k=0}^{\infty} \psi_k \varepsilon^k\right] \\ &= \varepsilon F(u' + Z_n, t, \varepsilon, f(t, \mu)) - \varepsilon F(u'' + Z_n, t, \varepsilon, f(t, \mu)) \\ &= \varepsilon \int_0^1 F_x(Z_n + \theta(u' - u'') + u'', t, \varepsilon, f(t, \mu)) d\theta (u' - u'').\end{aligned}$$

The following equality holds

$$\begin{aligned}\|\Delta H_n\| &= \|H_n(u', t, \varepsilon, \mu) - H_n(u'', t, \varepsilon, \mu)\| \\ &= \left\|\varepsilon \int_0^1 F_x(Z_n + \theta(u' - u'') + u'', t, \varepsilon, f) d\theta (u' - u'')\right\|.\end{aligned}$$

Then we can get the estimate

$$\|\Delta H_n\| \leq \varepsilon \int_0^1 \|F_x(Z_n + \theta(u' - u'') + u'', t, \varepsilon, f)\| \, d \, \|\theta(u' - u'')\|.$$

The integrand function $F_x$, in accordance with condition C2, is a continuous function in $G$, where $x = Z_n + \theta(u' - u'') + u''$. Moreover the following is fulfilled $\|x\| \leq \|Z_n + \theta(u' - u'') + u''\| \leq \|Z_n\| + \|\theta u'\| + \|(1 - \theta)u''\| \leq \rho + \theta\bar{\delta} + (1 - \theta)\bar{\delta} = \rho + \bar{\delta} \leq \delta$.

Then there is a positive constant $\bar{C}$, such that the following inequality holds

$$\|F_x(Z_n + \theta(u' - u'') + u'', t, \varepsilon, f\| \leq \bar{C}.$$

For the evaluation of $\|\Delta H_n\|$we get

$$\|\Delta H_n\| = \varepsilon \int_0^1 \|F_x(Z_n + \theta(u' - u'') + u'', t, \varepsilon, f)\| d\theta \, \|(u' - u'')\| \leq \varepsilon \int_0^1 \bar{C} d\theta \, \|(u' - u'')\| .$$

$\square$

Let $W(t, s, \varepsilon)$ is a fundamental system of the solutions of the system $\varepsilon \dfrac{du}{dt} = Au$, $W(s, s, \varepsilon) = E_n$. The following lemmas are fulfilled:

**Lemma 4** ([1, 8])**.** *The fundamental matrix $W(t, s, \varepsilon)$ satisfies the inequality $\|W(t, s, \varepsilon)\| \leq \beta \exp\left(-\alpha \dfrac{t - s}{\varepsilon}\right)$, where $\alpha$ and $\beta$ are positive constants and $0 \leq s \leq t \leq b$.*

**Lemma 5** ([1, 8])**.** *Any decision of the continuous system* $\varepsilon\dot{u} = Au + H_n(u, t, \varepsilon, \mu)$ *is equivalent to the integral equation* $u = W(t, a, \mu)\xi(a, \varepsilon, \mu) + \int_a^t W(t, s, \varepsilon)\frac{1}{\varepsilon}H_n(u, s, \varepsilon, \mu)ds.$

**Lemma 6** ([1, 8])**.** *If* $\varepsilon \to 0$*, the integral* $\int_a^t \left\|\frac{1}{\varepsilon}W(t, s, \varepsilon)\right\|ds$ *is uniformly bounded in the range* $[a, b]$*, i.e. there exists a positive constant* $M$*, such that if* $\varepsilon \to 0$ *and* $t \in [a, b]$ *the inequality holds* $\int_a^t \left\|\frac{1}{\varepsilon}W(t, s, \varepsilon)\right\|ds \leq M$.

**Theorem 7.** *Let the conditions of Theorem* 1 *be satisfied and* $\det R(\varepsilon) \neq 0 \ \forall\varepsilon \in [0, \bar{\varepsilon}]$*, where* $R(\varepsilon) = l(W(\cdot, a, \varepsilon))$ *is an* $(m\times n)$*-matrix . Then there are constants* $\varepsilon^* > 0$*,* $C^* > 0$ *such that the problem* (1)*,* (2) *has a unique solution* $x(t, \varepsilon)$ *and it satisfies the inequality*

$$\|x(t, \varepsilon) - X_n(t, \varepsilon)\| \leq C^*\varepsilon^{n+1} \tag{13}$$

*at* $t \in [a, b]$ *and* $0 < \varepsilon \leq \varepsilon^*$.

**Proof.** To prove that (1), (2) has the only solution satisfying (13) means to prove that the boundary problem (3) has a unique solution satisfying $\|z(t, \varepsilon, \mu) - Z_n(t, \varepsilon, \mu)\| \leq C^*\varepsilon^{n+1}$. Therefore (3) we make the following replacement $u(t, \varepsilon, \mu) = z(t, \varepsilon, \mu) - Z_n(t, \varepsilon, \mu)$ and obtain the boundary problem (10).

To prove the theorem it is sufficient to show that (10) has a unique solution such that $\|u(t, \varepsilon, \mu)\| \leq C^*\varepsilon^{n+1}$. For the system (10) we apply Lemma 5 and consider the equivalent integral equation

$$u(t, \varepsilon, \mu) = W(t, a, \mu)\xi(a, \varepsilon, \mu) + \int_a^t W(t, s, \varepsilon)\frac{1}{\varepsilon}H_n(u, s, \varepsilon, \mu)ds.$$

We substitute this equation in the boundary condition of (10) and under the condition of the theorem $\det R(\varepsilon) \neq 0$ we get $\xi(a, \varepsilon, \mu) = R^{-1}(\varepsilon)b(u, \varepsilon, \mu)$, where

$$b(u, \varepsilon, \mu) = -l\int_a^{(\cdot)} \frac{1}{\varepsilon}W(\cdot, s, \varepsilon)H_n(u, s, \varepsilon, \mu)ds.$$

Thus, for the integral equation we find the presentation

$$u(t, \varepsilon, \mu) = W(t, a, \mu)R^{-1}(\varepsilon)b(u, \varepsilon, \mu) + \int_a^t \frac{1}{\varepsilon}W(t, s, \varepsilon)H_n(u, s, \varepsilon, \mu)ds. \tag{14}$$

For the integral equations (14) we apply the method of successive approximations:

$$\begin{aligned} &u_0(t, \varepsilon, \mu) = 0 \\ &u_k(t, \varepsilon, \mu) = W(t, a, \varepsilon)R^{-1}(\varepsilon)b(u_{k-1}, \varepsilon, \mu) \\ &\qquad + \int_a^t \frac{1}{\varepsilon}W(t, s, \varepsilon)H_n(u_{k-1}(s, \varepsilon, \mu), s, \varepsilon, \mu)ds, \quad k \geq 1. \end{aligned} \tag{15}$$

Given the Lemmas 2, 4, 6, type of $b(u, \varepsilon, \mu)$ and evaluation $\|l(\psi)\| \leq \bar{b}\|\psi\|$, $\bar{b} > 0$, it is easy to identify the inequalities $\|R^{-1}(\varepsilon)\| \leq \bar{M}\ell^{\alpha\frac{t-a}{\varepsilon}}$, $\bar{M} > 0$, $\alpha > 0$, $\|b(u_0, \varepsilon, \mu)\| = \|b(0, \varepsilon, \mu)\| \leq b_1\varepsilon^{n+1}, b_1 > 0$.

Then from (15) we find

$$\|u_1 - u_0\| = \|W(t,a,\varepsilon)R^{-1}(\varepsilon)b(u_0,\varepsilon,\mu) + \int_a^t \frac{1}{\varepsilon} W(t,\varepsilon,s)H_n(u_0(s,\varepsilon,\mu),s,\varepsilon,\mu)ds\|$$
$$\leq \|W(t,a,\varepsilon)\|\|R^{-1}(\varepsilon)\|\|b(0,\varepsilon,\mu)\| + \int_a^t \left\|\frac{1}{\varepsilon}W(t,\varepsilon,s)H_n(0,s,\varepsilon,\mu)\right\|ds$$
$$\leq \bar{C}\varepsilon^{n+1} \leq \frac{\nu}{2}, \quad \upsilon = 2\bar{C}\varepsilon^{n+1}, \quad \bar{C} > 0.$$

With the help of Lemma 3 we obtain that $\|u_2 - u_1\| \leq \bar{C}\varepsilon\|u_1 - u_0\| \leq \frac{1}{2}.\frac{\nu}{2}$, for some $0 < \varepsilon_2 \ll 1$ and $\varepsilon \leq \varepsilon_2 = \frac{1}{2\bar{C}}$. By induction we find that

$$\|u_k - u_{k-1}\| \leq \frac{1}{2^{k-1}} \cdot \frac{\nu}{2}\ \forall k \geq 1,\ \forall t \in [a,b], \varepsilon \in (0,\varepsilon_2],\ \mu \in (0,\varepsilon_2],\ \|u_k\| \leq \delta,\ \|u_{k-1}\| \leq \delta.$$

Then $\|u_k(t,\varepsilon,\mu)\| \leq \sum_{i=1}^{k} \|u_i - u_{i-1}\| \leq \nu$, i.e. $\|u_k(t,\varepsilon,\mu)\| \leq C^*\varepsilon^{n+1}$ with $C^* = 2\bar{C}$, in $t \in [a,b]$, $\varepsilon \in (0,\varepsilon^*]$, $\mu \in (0,\varepsilon^*]$ , $0 < \varepsilon^* < \varepsilon_2$. Therefore, successive approximations $u_k(t,\varepsilon,\mu)$ are uniformly convergent to the solution $u(t,\varepsilon,\mu)$ of the problem (10), i.e. a solution (10) exists and the inequality $\|u(t,\varepsilon,\mu)\| \leq C^*\varepsilon^{n+1}$, $t \in [a,b]$, $\varepsilon \in (0,\varepsilon^*]$, $\mu \in (0,\varepsilon^*]$ holds, which is similar to (13). The uniqueness of the solution follows from the fact that the right side of the differential system (10) is Lipschitz. □

## REFERENCES


[1] L. Flatto, N. Levinson. Periodic Solutions of Singularity Perturbed Systems. *Mathematics* (*Periodic Collection of Translation*), **2** (1958) No 2, 61–68 (in Russian).

[2] L. I. Karandzhulov, N. D. Sirakova. Boundary-Value Problems with Integral Conditions, CP1184. *Applications of Mathematics in Engineering and Economics*, Sozopol 2011, Edited by G. Venkov and V. Pasheva. American Institute of Physics, 978-0-7354-0750-3, 2011, 181–188.

[3] L. I. Karandzhulov, N. D. Sirakova. Asymptotic expansion of solution for almost regular and weekly perturbed system of ordinary differential equations. *Mathematics and education in mathematics*, **41** (2012), 185–190.

[4] L. I. Karandzhulov, N. D. Sirakova. Asymptotic expansion of the solutions for almost regular nonlinear system, PMM: Transactions of scientific works, UlDTU, 2011, 233–238 (in Russian).

[5] L. I. Karandzhulov, N. D. Sirakova. Boundary-value problems for nonlinear systems with double singularity. *Applications of Mathematics in Engineering and Economics*, Sozopol 2012, Edited by G. Venkov and V. Pasheva, American Institute of Physics, 978-0-7354-1111-1, 2012, 247–256.

[6] R. P. Kuzmina. Asymptotic methods for ordinary differential equations, Kluwer Academic Publishers, Dordrecht-Boston, 2000.

[7] N. D. Sirakova, L. I. Karandzhulov. Boundary-Value Problems for Almost Regular and Weakly Perturbed Nonlinear Systems, Mathematical analysis, Differential equations and their applications. *Proceedings Bulgarian-Turkish-Ukrainian scientific conference*, Sunny Beach, September 15-20, 2010, Edited by A. Andreev and L. Karandzhulov, Academic Publishing House "Prof. Marin Drinov" ISBN 978-954-322-454-8, Sofia, 2011, 195–206.

[8] A. B. Vasileva, V. Botuzov. Asymptotic Expansions of the Solutions of Singularly Perturbed Equations, Nauka, Moscow, 1973 (in Russian).

Neli Dimitrova Sirakova
Technical University – Sofia
e-mail: evklid@abv.bg

## АСИМПТОТИЧНО ПОВЕДЕНИЕ НА РЕШЕНИЯТА НА ГРАНИЧНИ ЗАДАЧИ С ДВОЙНА СИНГУЛЯРНОСТ

**Нели Сиракова**

В работата се разглежда асимптотичното поведение на решението на нелинейни гранични задачи с двойна сингулярност и общи гранични условия. Предполагаме, че диференциалната система съдържа допълнителна функция, която определя задачата като двойно сингулярна. При определени условия се доказва асимптотичност на решението на поставената гранична задача.