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NILPOTENT ELEMENTS OF THE ENDOMORPHISM SEMIRING OF A FINITE CHAIN AND CATALAN NUMBERS*

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In this paper we aim to describe a generalization of the nilpotent endomorphisms of a finite chain and to consider the role of Catalan numbers for the semirings of such endomorphisms.

1. Introduction. The nilpotent elements in finite transformation semigroups are well studied, see [2], [3] and [6]. In [11] we show that some semigroups are actually semirings. In [9] we find some results in the endomorphism semiring of a finite chain for the nilpotent endomorphisms with fixed point 0. By similar reasonings in [7] Szigeti transfers some linear algebra theorems to lattices. The purpose of this paper is to prove analogous results for endomorphisms with a certain fixed point.

2. Preliminaries. An algebra $R = (R, +, \cdot)$ with two binary operations $+$ and $\cdot$ on R, is called a semiring if: **1.** $(R, +)$ is a commutative semigroup, **2.** $(R, \cdot)$ is a semigroup, **3.** both distributive laws hold $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ for any $x, y, z \in R$.

Facts concerning semirings can be found in [4].

For a finite chain $\mathcal{C}_n = (\{0, 1, \ldots, n-1\}, \vee)$ the endomorphisms form a semiring with respect to the addition and multiplication defined by:

$$h = f + g \text{ when } h(x) = f(x) \vee g(x), \quad h = f \cdot g \text{ when } h(x) = f(g(x)) \text{ for all } x \in \mathcal{C}_n.$$

This semiring is called an *endomorphism semirimg* and is denoted by $\widehat{\mathcal{E}}_{\mathcal{C}_n}$, see [10]. If $\alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}$ such that $f(k) = i_k$ for any $k \in \mathcal{C}_n$ we denote α as an ordered n-tuple $\wr\, i_0, i_1, i_2, \ldots, i_{n-1}\,\wr$. For any $k \in \mathcal{C}_n$ the set $\mathcal{E}^{(k)}_{\mathcal{C}_n} = \{\alpha | \alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n}, \alpha(k) = k\}$ is a subsemiring of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$. In particular, for $k = 0$, semiring $\mathcal{E}^{(0)}_{\mathcal{C}_n}$ has zero element. An element $\alpha \in \mathcal{E}^{(0)}_{\mathcal{C}_n}$ is called nilpotent if $\alpha^m = 0$ for some positive integer m.

The *Catalan sequence* is the sequence $C_0, C_1, \ldots, C_n, \ldots$, where $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$, $n = 0, 1, \ldots$, see [1].

Proposition 2.1 ([9]). *The set $\mathcal{N}_n$ of nilpotent elements of semiring $\mathcal{E}^{(0)}_{\mathcal{C}_n}$ consists of endomorphisms α such that $\alpha(k) < k$ for every $k \in \mathcal{C}_n$, $k \neq 0$. This set is a subsemiring of $\mathcal{E}^{(0)}_{\mathcal{C}_n}$ of order $C_{n-1} = \dfrac{1}{n}\dbinom{2n-1}{n-1}$.*

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If $\alpha(k) \le k$ for every $k \in \mathcal{C}_n$, the endomorphism $\alpha \in \mathcal{E}^{(0)}_{\mathcal{C}_n}$ is called, see [9], *over nilpotent endomorphism.* The subset of $\mathcal{E}^{(0)}_{\mathcal{C}_n}$ consisting of all the nilpotent endomorphisms, is denoted by $\mathcal{ON}_n$.

Proposition 2.2 ([9]). *The set $\mathcal{ON}_n$ is a subsemiring of semiring $\mathcal{E}^{(0)}_{\mathcal{C}_n}$ of order* $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$.

3. Nilpotent endomorphisms. For $k \in \mathcal{C}_n$ constant endomorphism $\wr\, k\, k\, \ldots\, k \,\wr$, see [5], has a remarkable role in semiring $\mathcal{E}^{(k)}_{\mathcal{C}_n}$. It is easy to verify, see [8], that this endomorphism is the multiplicatively absorbing element of the semiring. This implies the idea of studying endomorphisms similar to usual nilpotent elements.

For every constant endomorphism $\kappa_k = \wr\, k\, k\, \ldots\, k \,\wr$ the elements of

$$\mathcal{N}_n^{[k]} = \{\alpha \mid \alpha \in \widehat{\mathcal{E}}_{\mathcal{C}_n},\ \alpha^{n_k} = \kappa_k \text{ for some natural number } n_k\}$$

are called *k-nilpotent endomorphisms.*

We need the following lemma to clarify the structure of $\mathcal{N}_n^{[k]}$.

Lemma 3.1. *For any natural n, $n \ge 2$, and $k \in \{1, \ldots, n-2\}$, the k-nilpotent endomorphisms are maps of the type*

$$\wr\, i_0, \ldots, i_{k-1}, k, i_{k+1}, \ldots, i_{n-1} \,\wr,$$

where $i_r > r$ for $r = 0, \ldots, k-1$ and $i_s < s$ for $s = k+1, \ldots, n-1$.

Proof. Let $\alpha \in \mathcal{N}_n^{[k]}$, where n and k are fixed.

Suppose that $\alpha(k) \le k-1$. Then $\alpha(k-1) \le \alpha(k) \le k-1$. Using $\alpha(k-1) \le k-1$ we observe by induction that $\alpha^m(k-1) \le k-1$ for every natural m, which is a contradiction of the assumption that $\alpha \in \mathcal{N}_n^{[k]}$. Thus $\alpha(k) \ge \alpha(k-1) \ge k$. If we suppose that $\alpha(k) > k$ by induction it follows that $\alpha^m(k) > k$ for every natural m, which is a contradiction of $\alpha \in \mathcal{N}_n^{(k)}$. Hence, $\alpha(k) = k$.

Suppose that for some $r \le k-1$ it follows $\alpha(r) \le r$. Then, by induction, we receive that $\alpha^m(r) \le r \le k-1$ for every natural m, which is a contradiction to the assumption that $\alpha \in \mathcal{N}_n^{[k]}$. Hence, $\alpha(r) > r$ for every $r \le k-1$.

Suppose that for some $s \ge k+1$ we have $\alpha(s) \ge s$. Now, by induction, we find that $\alpha^m(s) \ge s \ge k+1$ for every natural m, which is a contradiction to $\alpha \in \mathcal{N}_n^{[k]}$. Hence, $\alpha(s) < s$ for every $s \ge k+1$ and this completes the proof. $\square$

The next definition appears in [9]. An element x of semiring R with an additively absorbing element a is called a *maxpotent* if $x^n = a$ for some positive integer n. For semiring $\widehat{\mathcal{E}}_{\mathcal{C}_n}$ the subset of maxpotent elements is $\mathcal{N}_n^{[n-1]}$. This set is nonempty because the absorbing element $\wr\, n-1, n-1, \ldots, n-1 \,\wr$ of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$ belongs to $\mathcal{N}_n^{[n-1]}$.

Lemma 3.2. *The set $\mathcal{N}_n^{[n-1]}$ of maxpotent elements of semiring $\widehat{\mathcal{E}}_{\mathcal{C}_n}$ consists of endomorphisms α such that $\alpha(i) > i$ for every $i \in \mathcal{C}_n$, $i < n-1$.*

Proof. We use similar reasonings to the ones in the second part of the last proof. Let $\alpha \in \mathcal{N}_n^{[n-1]}$. Suppose that for some $i < n-1$ it follows that $\alpha(i) \le i$. Then, by induction, we obtain that $\alpha^m(i) \le i < n-1$ for every natural m, which is a contradiction to the assumption that $\alpha \in \mathcal{N}_n^{[n-1]}$. Hence, $\alpha(i) > i$ for every $i < n-1$. $\square$

Theorem 3.3. *For any natural n, $n \geq 2$, and $k \in \{0, 1, \dots, n-2, n-1\}$ the set of k– nilpotent endomorphisms $\mathcal{N}_n^{[k]}$ is a subsemiring of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$. When $k > 0$ there is not zero element in semiring $\mathcal{N}_n^{([k])}$. The order of semiring $\mathcal{N}_n^{[k]}$ is $\left|\mathcal{N}_n^{[k]}\right| = C_k.C_{n-k-1}$, where C_k is the k-th Catalan number.*

Proof. Let $k = 0$ and $\alpha \in \mathcal{N}_n^{[0]}$. If we suppose that $\alpha(0) > 0$, by induction, we find $\alpha^m(0) > 0$ for every natural m, which is a contradiction to the assumption that $\alpha \in \mathcal{N}_n^{(0)}$. Hence the 0-nilpotent endomorphisms belong to semiring $\mathcal{E}_{\mathcal{C}_n}^{(0)}$. But in this semiring the endomorphism $\wr\, 0\, 0\, \dots\, 0 \,\wr$ is the zero element, so, there are nilpotent endomorphisms. Now, from Proposition 2.1, it follows that $\mathcal{N}_n^{[0]} = \mathcal{N}_n$ is a subsemiring of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$.

Let $\alpha, \beta \in \mathcal{N}_n^{[k]}$, where n and k are fixed and $k \in \{1, \dots, n-2\}$.

For $r = 0, \dots, k-1$ from Lemma 3.1 it follows that $\alpha(r) > r$ and $\beta(r) > r$. Then $(\alpha + \beta)(r) = \alpha(r) \vee \beta(r) > r \vee r = r$ and $(\alpha \cdot \beta)(r) = \beta(\alpha(r)) \geq \beta(r) > r$.

From Lemma 3.1 it follows that $\alpha(k) = k$ and $\beta(k) = k$. Then $(\alpha + \beta)(k) = \alpha(k) \vee \beta(k) = k \vee k = k$ and $(\alpha \cdot \beta)(k) = \beta(\alpha(k)) = \beta(k) = k$.

For $s = k+1, \dots, n-1$ from Lemma 3.1 it follows that $\alpha(s) < s$ and $\beta(s) < s$. Then $(\alpha + \beta)(s) = \alpha(s) \vee \beta(s) < s \vee s = s$ and $(\alpha \cdot \beta)(s) = \beta(\alpha(s)) \leq \beta(s) < s$.

Hence, $\mathcal{N}_n^{[k]}$ is a subsemiring of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$, where $k \in \{1, \dots, n-2\}$.

Let $k = n-1$ and $\alpha, \beta \in \mathcal{N}_n^{[n-1]}$. Then for $i = 0, \dots, n-2$ from Lemma 3.2 it follows that $\alpha(i) > i$ and $\beta(i) > i$. Then $(\alpha + \beta)(i) = \alpha(i) \vee \beta(i) > i \vee i = i$ and $(\alpha \cdot \beta)(i) = \beta(\alpha(i)) \geq \beta(i) > i$. So $\mathcal{N}_n^{[n-1]}$ is a subsemiring of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$.

Thus we prove that for any $k \in \{0, 1, \dots, n-2, n-1\}$ the set of k-nilpotent endomorphisms $\mathcal{N}_n^{[k]}$ is a subsemiring of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$.

From this proof and Lemma 3.1 it follows that endomorphism $\wr\, 1, \dots, 1, k, k, \dots k \,\wr$ is the neutral element of the semigroup $\left(\mathcal{N}_n^{[k]}, +\right)$ for $k \in \{1, \dots, n-2\}$. It is easy to calculate that $\wr\, 1, \dots, 1, k, k, \dots k \,\wr \cdot \wr\, k, \dots, k \,\wr = \wr\, k, \dots, k \,\wr$ which means that there is not a zero element in the semiring $\mathcal{N}_n^{[k]}$, where $k \in \{1, \dots, n-2\}$.

From this proof and Lemma 3.2 it follows that $\wr\, 1, \dots, 1, n-1, n-1 \,\wr$ is the neutral element of the semigroup $\left(\mathcal{N}_n^{[n-1]}, +\right)$. From $\wr\, 1, \dots, 1, n-1, n-1 \,\wr \cdot \wr\, n-1, \dots, n-1 \,\wr = \wr\, n-1, \dots, n-1 \,\wr$ it follows that in semiring $\mathcal{N}_n^{[n-1]}$ there is not a zero element.

Let us consider arbitrary endomorphism $\alpha = \wr\, i_0, \dots, i_{k-1}, k, i_{k+1}, \dots, i_{n-1} \,\wr$ of semiring $\mathcal{N}_n^{[k]}$ and let $(k, i_{k+1}, \dots, i_{n-1})$ be the ordered $n-k$-tuple corresponding to the last part of α. Since $i_s < s$ for any $s = k+1, \dots, n-1$ it follows that $k+1 > i_{k+1} \geq i_k = k$, i.e. $i_{k+1} = k$. The number of these $n-k$-tuples $(k, i_{k+1}, \dots, i_{n-1})$ is equal to the number of $n-k$-tuples $(0, i_{k+1}-k, \dots, i_{n-1}-k)$, which satisfies the conditions of Proposition 2.1. Hence, their number is C_{n-k-1}, which is the $(n-k-1)$-th Catalan number.

Let $(i_0, \dots, i_{k-1}, k)$ be the ordered $k+1$-tuple corresponding to the first part of α. Since $i_r > r$ for any $r = 0, \dots, k-1$ it follows that $k = i_k \geq i_{k-1} > k-1$, i.e. $i_{k-1} = k$. Instead of $k+1$-tuple $(i_0, \dots, i_{k-1}, k)$ we consider $k+1$-tuple $(j_0, \dots, j_k)$, where $j_0 = 0$, $j_{m+1} = k - i_{k-m-1}$, $m = 0, \dots k-1$. Then $j_m < m$ and $j_m \leq j_{m+1}$. The number of all $k+1$-tuples of this kind is C_k.

Since every $k+1$-tuple $(i_0, \dots, i_{k-1}, k)$ can be combined with every $n-k$-tuple $(k, i_{k+1}, \dots, i_{n-1})$, then the order of semiring $\mathcal{N}_n^{[k]}$ is equal to $\left|\mathcal{N}_n^{[k]}\right| = C_k.C_{n-k-1}$.

Corollary 3.4 *The semirings $\mathcal{N}_n^{[k]}$ are disjoint.*

Proof. Let $\alpha \in \mathcal{N}_n^{[k]}$ and $\alpha \in \mathcal{N}_n^{[\ell]}$. Then $\alpha(k) = k$ and $\alpha(\ell) = \ell$. If $\ell < k$, then $\alpha(\ell) > \ell$, and if $\ell > k$, then $\alpha(\ell) < \ell$. So $\ell = k$, i.e. for $k \neq \ell$ it follows that $\mathcal{N}_n^{[k]} \cap \mathcal{N}_n^{[\ell]} = \varnothing$. □

Let us consider for any natural n, $n \geq 2$, and $k \in \{1, \ldots, n-2\}$ the endomorphisms of the type

$$\wr\, i_0, \ldots, i_{k-1}, k, i_{k+1}, \ldots, i_{n-1} \,\wr,$$

where $i_r \geq r$ for $r = 0, \ldots, k-1$ and $i_s \leq s$ for $s = k+1, \ldots, n-1$. These maps are called *near k-nilpotent endomorphisms*. The subset of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$ consisting of all near k-nilpotent endomorphisms is denoted by $\mathcal{NN}_n^{[k]}$.

For $k = 0$ we consider endomorphisms $\wr\, 0, i_1, \ldots, i_{n-1} \,\wr$ such that $i_s \leq s$ for $s = 1, \ldots, n-1$ and they are the near 0-nilpotent endomorphisms. These endomorphisms are just all the over nilpotent endomorphisms, so, the set of near 0-nilpotent endomorphisms is the semiring $\mathcal{ON}_n = \mathcal{NN}_n^{[0]}$, see Proposition 2.2.

The following reasonings are similar to ones from the proof of the last theorem.

Let $\alpha, \beta \in \mathcal{NN}_n^{[k]}$, where n and k are fixed and $k \in \{1, \ldots, n-2\}$

For $r = 0, \ldots, k-1$ it follows that $(\alpha + \beta)(r) = \alpha(r) \vee \beta(r) \geq r \vee r = r$ and $(\alpha \cdot \beta)(r) = \beta(\alpha(r)) \geq \beta(r) \geq r$.

For $s = k$ we have $(\alpha + \beta)(k) = \alpha(k) \vee \beta(k) = k \vee k = k$ and $(\alpha \cdot \beta)(k) = \beta(\alpha(k)) = \beta(k) = k$.

For $s = k+1, \ldots, n-1$ it follows that $(\alpha + \beta)(s) = \alpha(s) \vee \beta(s) \leq s \vee s = s$ and $(\alpha \cdot \beta)(s) = \beta(\alpha(s)) \leq \beta(s) \leq s$.

Hence, $\mathcal{NN}_n^{[k]}$ is a subsemiring of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$, where $k \in \{1, \ldots, n-2\}$.

For $k = n-1$ we consider endomorphisms $\wr\, i_0, i_1, \ldots, i_{n-2}, n-1 \,\wr$ such that $i_r \geq r$ for $r = 0, \ldots, n-2$ and they are the near $n-1$-nilpotent endomorphisms. Note that the near $n-1$-nilpotent endomorphisms are similar to the under maxpotent endomorphisms (see [6], where these endomorphisms have the restriction $i_0 = 0$). Let the set of all near $n-1$-potent endomorphisms be denoted by $\mathcal{NN}_n^{[n-1]}$. For $\alpha, \beta \in \mathcal{NN}_n^{[n-1]}$ and $i = 0, \ldots, n-2$ it follows that $\alpha(i) \geq i$ and $\beta(i) \geq i$. Then $(\alpha + \beta)(i) = \alpha(i) \vee \beta(i) \geq i \vee i = i$ and $(\alpha \cdot \beta)(i) = \beta(\alpha(i)) \geq \beta(i) \geq i$. So $\mathcal{NN}_n^{[n-1]}$ is a subsemiring of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$.

Thus we prove

Proposition 3.5. *For any natural n, $n \geq 2$, and $k \in \{0, 1, \ldots, n-2, n-1\}$ the set of near k-nilpotent endomorphisms $\mathcal{NN}_n^{[k]}$ is a subsemiring of $\widehat{\mathcal{E}}_{\mathcal{C}_n}$.*

Theorem 3.6. *For any natural n, $n \geq 2$, and $k \in \{0, 1, \ldots, n-2, n-1\}$ the semiring $\mathcal{N}_n^{[k]}$ is an ideal of $\mathcal{NN}_n^{([k])}$.*

Proof. The case $k = 0$ is just Theorem 3.9 of [9].

Let $k \in \{1, \ldots, n-2\}$ and let us choose arbitrary $\alpha \in \mathcal{N}_n^{[k]}$ and $\beta \in \mathcal{NN}_n^{[k]}$. Then for any $r = 0, \ldots, k-1$ it follows that $\alpha(r) > r$ and $\beta(r) \geq r$. Now we calculate

$$(\alpha \cdot \beta)(r) = \beta(\alpha(r)) \geq \alpha(r) > r, \; (\beta \cdot \alpha)(r) = \alpha(\beta(r)) \geq \alpha(r) > r.$$

For $r = k$ we find

$$(\alpha \cdot \beta)(k) = \beta(\alpha(k)) = \alpha(k) = k, \; (\beta \cdot \alpha)(k) = \alpha(\beta(k)) = \alpha(k) = k.$$

For any $s = k+1, \dots, n-1$ it follows that $\alpha(s) < s$ and $\beta(s) \leq s$, so, we have

$$(\alpha \cdot \beta)(s) = \beta(\alpha(s)) \leq \alpha(s) < s,\ (\beta \cdot \alpha)(s) = \alpha(\beta(s)) \leq \alpha(s) < s.$$

Hence for $k \in \{1, \dots, n-2\}$ the semiring $\mathcal{N}_n^{[k]}$ is an ideal of $\mathcal{NN}_n^{[k]}$.

Let $k = n-1$. Then for any $i = 0, \dots, n-2$ it follows that $\alpha(i) > i$ and $\beta(i) \geq i$ and we find

$$(\alpha \cdot \beta)(i) = \beta(\alpha(i)) \geq \alpha(i) > i,\ (\beta \cdot \alpha)(i) = \alpha(\beta(i)) \geq \alpha(i) > i$$

and this completes the proof. □

Proposition 3.7. *The order of semiring $\mathcal{NN}_n^{[k]}$ is* $\left|\mathcal{NN}_n^{[k]}\right| = C_{k+1}C_{n-k}$.

Proof. It follows immediately from the proof of Theorem 3.3 and Proposition 2.2. □

We denote by $\mathcal{N}_n^{[k]}(k_1, \dots, k_s)$ the subset of $\mathcal{NN}_n^{[k]}$ consisting of endomorphisms with *just* $s+1$ different fixed points k and $k_1, \dots, k_s$.

Proposition 3.8. ' *For any $k_1, \dots, k_s \in \mathcal{C}_n$, where $k_i \neq k$, the set $\mathcal{N}_n^{[k]}(k_1, \dots, k_s)$ is a subsemiring of $\mathcal{NN}_n^{[k]}$.*

Proof. Let $\alpha, \beta \in \mathcal{N}_n^{[k]}(k_1, \dots, k_s)$. Then $(\alpha+\beta)(k_i) = k_i$ and $(\alpha \cdot \beta)(k_i) = k_i$ where $i = 1, \dots, s$. Let $\ell < k$ and $\ell \neq k_i$, $i = 1, \dots, s$. So, $\alpha(\ell) < \ell$, $\beta(\ell) < \ell$ and then $(\alpha+\beta)(\ell) < \ell$ and $(\alpha \cdot \beta)(\ell) < \ell$. Analogously, if $\ell > k$, we receive $(\alpha+\beta)(\ell) > \ell$ and $(\alpha \cdot \beta)(\ell) > \ell$. Hence, $\mathcal{N}_n^{[k]}(k_1, \dots, k_s)$ is a semiring. □

Proposition 3.9. *Let, in semiring $\mathcal{N}_n^{[k]}(k_1, \dots, k_s)$, all the fixed points by renumbering be $\ell_1 < \ell_2 < \cdots \ell_{s+1}$. Then the order of semiring $\mathcal{N}_n^{[k]}(k_1, \dots, k_s)$ is equal to* $C_{\ell_1} \prod_{i=1}^{s} C_{\ell_{i+1}-\ell_i-1} C_{n-\ell_{s+1}-1}$.

Proof. Let $\ell_i < \ell_{i+1} \leq k$ and let there be no other fixed point between ℓ_i and ℓ_{i+1}. Let $\alpha \in \mathcal{N}_n^{[k]}(k_1, \dots, k_s)$. Using the reasonings from the proof of Theorem 3.3 it follows that, for the part between ℓ_i and ℓ_{i+1} of the n-tuple, representing α, there are $C_{\ell_{i+1}-\ell_i-1}$ possibilities. By the same way we find number $C_{\ell_{i+1}-\ell_i-1}$ in the case when $k \leq \ell_i < \ell_{i+1}$. For the first part of the n-tuple, representing α, we consider ℓ_1+1-tuples $(i_0, \dots, i_{\ell_1-1}, \ell_1)$, where $\ell_1 \geq i_m > m$ for $m = 0, \dots, \ell_1 - 1$. Hence, using the proof of Theorem 3.3, the number of these ℓ_1+1-tuples is C_{ℓ_1}. Similarly, we find that the number of possibilities for the last part of the n-tuple, representing α, are $C_{n-\ell_{s+1}-1}$. So, the product of all these numbers, equal to $C_{\ell_1} \prod_{i=1}^{s} C_{\ell_{i+1}-\ell_i-1} C_{n-\ell_{s+1}-1}$, is the order of the semiring.

We denote by $\left(\mathcal{N}_n^{[k]},\ k_1, \dots, k_s\right)$ the subset of $\mathcal{NN}_n^{[k]}$ consisting of endomorphisms with *at most* $s+1$ different fixed points k and $k_1, \dots, k_s$. □

Theorem 3.10. *For any $k_1, \dots, k_s \in \mathcal{C}_n$, where $k_i \neq k$, the set $\left(\mathcal{N}_n^{[k]},\ k_1, \dots, k_s\right)$ is an ideal of semiring $\mathcal{NN}_n^{[k]}$.*

Proof. Obviously $\left(\mathcal{N}_n^{[k]},\ k_1, \dots, k_s\right)$ contains any of semirings $\mathcal{N}_n^{[k]}(k'_1, \dots, k'_m)$, where $\{k'_1, \dots, k'_m\} \subseteq \{k_1, \dots, k_s\}$ and also contains the ideal $\mathcal{N}_n^{[k]}$. Consequently, we choose the endomorphisms from different semirings. Let $\alpha \in \mathcal{N}_n^{[k]}(A)$, where $A = \{k'_1, \ldots, k'_m\}$, $\beta \in \mathcal{N}_n^{[k]}(B)$, where $B = \{k''_1, \ldots, k''_r\}$ and $A, B \subseteq \{k_1, \ldots, k_s\}$. Consider $\bar{k} \in A$ and $\bar{k} \notin B$. Let $\bar{k} > k$. Then $\alpha(\bar{k}) = \bar{k}$ and $\beta(\bar{k}) < \bar{k}$ implies $(\alpha + \beta)(\bar{k}) = \bar{k}$ and $(\alpha \cdot \beta)(\bar{k}) < \bar{k}$. Hence, it follows that $\alpha + \beta$, $\alpha \cdot \beta \in \left(\mathcal{N}_n^{[k]},\ A \cup B\right)$. Let $\bar{k} < k$. Then $\alpha(\bar{k}) = \bar{k}$ and $\beta(\bar{k}) > \bar{k}$ implies $(\alpha + \beta)(\bar{k}) > \bar{k}$ and $(\alpha \cdot \beta)(\bar{k}) > \bar{k}$. Hence, it follows that $\alpha + \beta, \alpha \cdot \beta \in \mathcal{N}_n^{[k]}$.

Let $\alpha \in \mathcal{N}_n^{[k]}(A)$, where $A = \{k'_1, \ldots, k'_m\}$ and $\beta \in \mathcal{N}_n^{[k]}$. Assume that $\bar{k} \in A$ and $\bar{k} > k$. Then $\alpha(\bar{k}) = \bar{k}$ and $\beta(\bar{k}) < \bar{k}$ implies $(\alpha + \beta)(\bar{k}) = \bar{k}$ and $(\alpha \cdot \beta)(\bar{k}) < \bar{k}$. Hence, it follows that $\alpha + \beta$, $\alpha \cdot \beta \in \left(\mathcal{N}_n^{[k]},\ A\right)$. Assume that $\bar{k} \in A$ and $\bar{k} < k$. Then $\alpha(\bar{k}) = \bar{k}$ and $\beta(\bar{k}) > \bar{k}$ implies $(\alpha + \beta)(\bar{k}) > \bar{k}$ and $(\alpha \cdot \beta)(\bar{k}) > \bar{k}$. So, it follows that $\alpha + \beta, \alpha \cdot \beta \in \mathcal{N}_n^{[k]}$. Thus we show that $\left(\mathcal{N}_n^{[k]},\ k_1, \ldots, k_s\right)$ is a semiring.

Let $\alpha \in \left(\mathcal{N}_n^{[k]},\ k_1, \ldots, k_s\right)$ and $\gamma \notin \left(\mathcal{N}_n^{[k]},\ k_1, \ldots, k_s\right)$. Hence, there is $\bar{k} \in \mathcal{C}_n$ such that $\bar{k} \neq k$, $\bar{k} \neq k_i$, where $i = 1, \ldots, s$, and $\gamma(\bar{k}) = \bar{k}$. Let $\bar{k} < k$ ($\bar{k} > k$). Then it follows that $\alpha(\bar{k}) > \bar{k}$ ($\alpha(\bar{k}) < \bar{k}$) which implies $(\alpha \cdot \gamma)(\bar{k}) > \bar{k}$ and $(\gamma \cdot \alpha)(\bar{k}) > \bar{k}$ ($(\alpha \cdot \gamma)(\bar{k}) < \bar{k}$ and $(\gamma \cdot \alpha)(\bar{k}) < \bar{k}$). Thus, we prove that $\left(\mathcal{N}_n^{[k]},\ k_1, \ldots, k_s\right)$ is an ideal of $\mathcal{NN}_n^{[k]}$. $\square$

As a consequence, we obtain the following result:

Corollary 3.11. *For any $k_1, \ldots, k_s \in \mathcal{C}_n$, where $k_i \neq k$, in semiring $\mathcal{NN}_n^{[k]}$ there is a chain of ideals $\mathcal{N}_n^{[k]} \subseteq \left(\mathcal{N}_n^{[k]},\ k_1\right) \subseteq \cdots \subseteq \left(\mathcal{N}_n^{[k]},\ k_1, \ldots, k_s\right) \subseteq \cdots \subseteq \mathcal{NN}_n^{[k]}$.*

REFERENCES

[1] R. Brualdi. Introductory combinatorics, Fifth edition, Prentice Hall, 2010.

[2] O. Ganyushkin, V. Mazorchuk. On the structure of $\mathcal{IO}_n$. *Semigroup Forum*, **66**, *3* (2003), 455–483.

[3] O. Ganyushkin, V. Mazorchuk. Combinatorics of nilpotents in symmetric inverse semigroups. *Ann. Combin.*, **8**, *2* (2004), 161–175.

[4] J. Golan. Semirings and Their Applications, Kluwer, Dordrecht, 1999.

[5] J. Jeẑek, T. Kepka, M. Maròti. The endomorphism semiring of a semilattice. *Semigroup Forum*, **78** (2009), 21–26.

[6] A. Laradji, A. Umar. On the number of nilpotents in the partial symmetric semigroup. *Comm. Alg.*, **32**, *8* (2004), 3017–3023.

[7] J. Szigeti. Linear algebra in lattices and nilpotent endomorphisms of semisimple modules. *J. Algebra*, **319** (2008), 296–308.

[8] I. Trendafilov, D. Vladeva. The endomorphism semiring of a finite chain. *Proc. Techn. Univ.-Sofia*, **61**, *1* (2011), 9–18.

[9] I. Trendafilov, D. Vladeva. Subsemirings of the endomorphism semiring of a finite chain. *Proc. Techn. Univ.-Sofia*, **61**, *1* (2011), 19–28.

[10] I. Trendafilov, D. Vladeva. Endomorphism semirings without zero of a finite chain. *Proc. Techn. Univ.-Sofia*, **61**, *2* (2011) 9–18.

[11] I. Trendafilov, D. Vladeva. On some semigroups of the partial transformation semigroup. Appl. Math. in Eng. and Econ. – 38th Int. Conf. AIP Conf. Proc., 2012, (to appear).

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НИЛПОТЕНТНИ ЕЛЕМЕНТИ НА ПОЛУПРЪСТЕНА ОТ ЕНДОМОРФИЗМИ НА КРАЙНА ВЕРИГА И ЧИСЛА НА КАТАЛАН

Иван Д. Трендафилов, Димитринка И. Владева

Целта на статията е да се опишат обобщения на нилпотентните ендоморфизми на крайна верига и да се изследва ролята на числата на Каталан за полупръстените от такива ендоморфизми.