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APPROXIMATION OF SOME IMPULSE FUNCTIONS – IMPLEMENTATION IN PROGRAMMING ENVIRONMENT MATHEMATICA*

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The basic problem in antenna synthesis is, given a radiation pattern, to find such a current distribution that implements it. In this paper we consider Fourier Transform and Hausdorff approximation approaches for approximate solution of the problem. These two methods are compared numerically and their implementation in the programming environment MATHEMATICA is considered. The MATHEMATICA codes provided facilitate the research in the field of antenna-feeder technics, analysis and synthesis of antenna patterns and filters, noise minimization by suitable approximation of impulse functions.

1. Fourier transform approach. In the case of a line source the normalized antenna pattern $F(\omega)$ and its current distribution $i(s)$ are connected as follows [5]

$$F(\omega) = \int_{-\frac{L}{2\lambda}}^{\frac{L}{2\lambda}} i(s)e^{j2\pi\omega s}\, ds, \tag{1}$$

where L is the antenna apperture length, λ is the length of the wave, ω is the frequency. If we suppose that $i(s) = 0$ for $s \notin [-\dfrac{L}{2\lambda}, \dfrac{L}{2\lambda}]$ then equality (1) takes the form

$$F(\omega) = \int_{-\infty}^{\infty} i(s)e^{j2\pi\omega s}\, ds, \tag{2}$$

which is the well known Fourier transform of the function $i(s)$ and the inverse Fourier transform gives

$$i(s) = \int_{-\infty}^{\infty} F(\omega)e^{-j2\pi\omega s}\, d\omega. \tag{3}$$

The function $i(s)$ represents the current distribution in the pattern and is computed from (3). In antenna-feeder technics the most fraquent signals F are of rectangular type as is shown in Fig. 1. This is the reason the notion of spectral density of rectangular signal is used.

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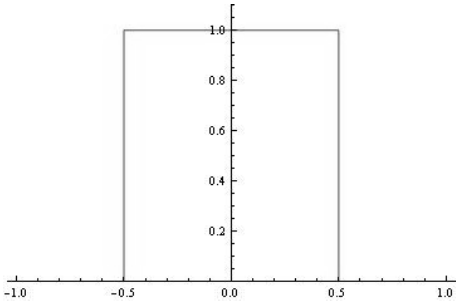


Fig. 1

Examples ([5])**.** If

$$F(\omega) = \begin{cases} 1, & |\omega| \le c, \\ 0, & |\omega| > c. \end{cases} \tag{4}$$

we get from (3) that $i(s)$ is

$$i(s) = 2c\frac{\sin(2\pi cs)}{2\pi cs}.$$

If we truncate the function $i(s)$ and define it as

$$i(s) := \begin{cases} i(s), & |s| \le \dfrac{L}{2\lambda}, \\ 0, & |s| > \dfrac{L}{2\lambda} \end{cases}$$

then the expression (2) returns an approximation $\tilde{F}(\omega)$ to the function $F(\omega)$. It follows from (2) that F can be written approximately as

$$F(\omega) \approx \frac{1}{\pi}\left(Si\left(\frac{L}{\lambda}\pi(\omega + c)\right) - Si\left(\frac{L}{\lambda}\pi(\omega - c)\right)\right).$$

In the programming environment MATHEMATICA the following operator can be used for the above calculations [1]: "SinIntegral[z] gives the sine integral function Si(z)". Let us fix $c = 0.5$, and $\frac{L}{\lambda} = 10$. The following is a simple MATHEMATICA code which makes the calculations and provides (see Fig. 2) the graph of the function F and the graph of the emitting chart.

```
R[x_] = (SinIntegral[10 ∗ π ∗ (x + 0.5)] − SinIntegral[10 ∗ π ∗ (x − 0.5)])/π

Plot[R[x], {x, −1, 1}, AxesLabel → {Style[x, Large, Bold, Red],
Style[F[x], Large, Bold, Blue]}, LabelStyle → Directive[Orange, Bold],
PlotLabel → R[x]]
```

Let us note that the Fourier transform is closely connected with Gibb's phenomena. This is evident in Fig. 2. One way to avoid this unpleasant effect is the approximation of function $F(\omega)$ in Hausdorff metric.

In Fig. 2 the variable x is used instead of ω. Another simple MATHEMATICA code for drawing the graph of the function i is

```
R1[x_] = 2 ∗ 0.5 ∗ Sin[2 ∗ π ∗ 0.5 ∗ x]/(2 ∗ π ∗ 0.5 ∗ x)
```

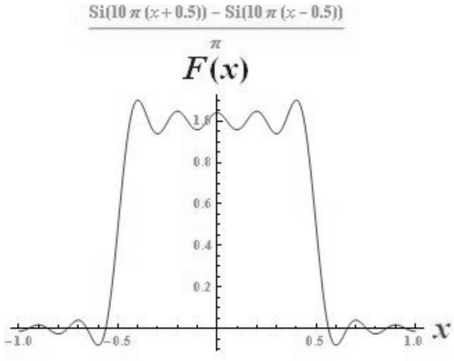


Fig. 2

Plot[R1[x], {x, −5, 5}, PlotRange → Full, AxesLabel → {Style[x, Large, Bold, Red],
Style[i[x], Large, Bold, Blue]}, LabelStyle → Directive[Orange, Bold], PlotLabel → R1[x]]

In Fig. 3 the graph of i is given where x is used instead of s.

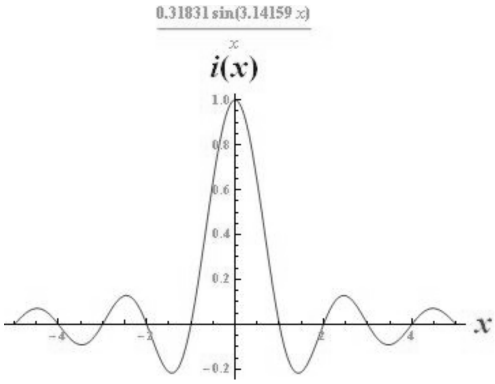


Fig. 3

Note. In the programming environment MATHEMATICA there exist various specialized operators such as

FourierTransform[...]; FourierSinTransform[...]; InverseFourierSinTransform[...],

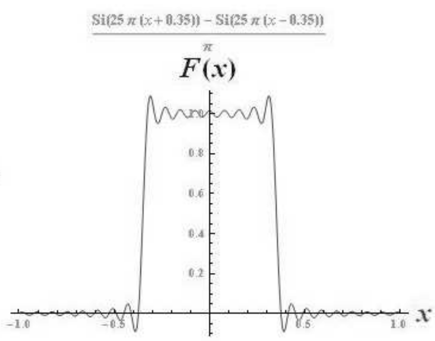
etc., which can be used for Fourier approximation of non-periodical signals [1].

2. Hausdorff approximation approach. Let us consider the function $sgn\, x$ whose graph is given in Fig. 5. This function plays an important role in the theory of impulse technics.

One consequence from Haar's theorem is the assertion that for any natural number n and any number $0 < \lambda < 1$ there exists a unique polynomial $Q_{2n+1} = \sum_{k=0}^{n} q_k x^{2k+1}$ of the best uniform approximation of the constant 1 in the interval $[\lambda, 1]$. The polynomials Q_{2n+1} take part in some technical problems such as antenna synthesis and electrical

```
Print["F_d(x)={ 1,  |x| ≤ c,
                0,  |x| > c,"];
c = Input["Give the value of the parameter c "];
Print["The value of the parameter: c = ", c];
Print["The antenna design techniques using Fourier analysis"];
Print["F(x)=1/π (Si(L/λ π(x+c))-Si(L/λ π(x-c)))"];
ρ = Input["Give the value of the parameter ρ=L/λ"];
Print["The value of the parameter: ρ = ", ρ];
Print["Graph of the function F(x): "];
R[x_] = (SinIntegral[ρ*Pi*(x+c)] - SinIntegral[ρ*Pi*(x-c)]) / Pi;
Plot[R[x], {x, -1, 1}, AxesLabel → {Style[x, Large, Bold, Red],
   Style[F[x], Large, Bold, Blue]}, LabelStyle → Directive[Orange, Bold], PlotLabel → R[x]]
```

```
F_d(x)={ 1,  |x| ≤ c,
         0,  |x| > c,
The value of the parameter: c = 0.35
The antenna design techniques using Fourier analysis
F(x)=1/π (Si(L/λ π(x+c))-Si(L/λ π(x-c)))
The value of the parameter: ρ = 25
Graph of the function F(x):
```



Manipulate[Plot[(SinIntegralρ] * Pi * (x + c)] − SinIntegral[ρ] * Pi * (x − c)])/Pi,
{x, −1, 1}, PlotRange → Full], {c, 0.1, 0.9, Appearance → "Open"}, {ρ], 10, 100,
Appearance → "Open"}]

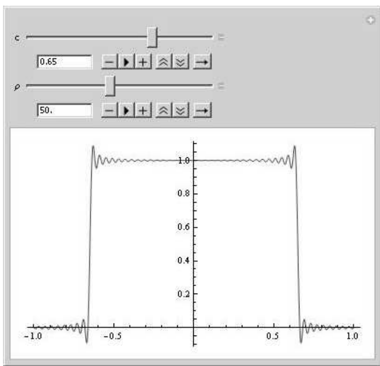


Fig. 4

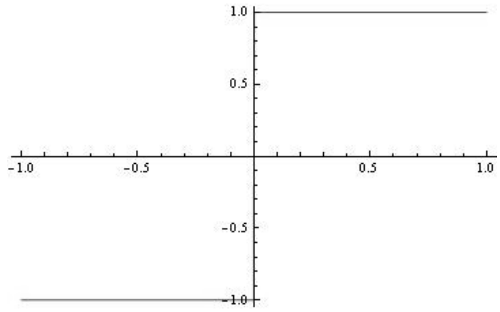


Fig. 5

schemes. Therefore their explicit finding excites a certain interest. It follows from the generalized Chebyshev theorem that:

1. if $\sigma_1, \sigma_2, \dots, \sigma_n$ are internal points of maximum deviation of Q_{2n+1} in the interval $[\lambda, 1]$, then $Q'_{2n+1}(\pm\sigma_i) = 0$ for $i = 1, \dots, n$ which leads to

$$Q'_{2n+1}(x) = \frac{dQ_{2n+1}(x)}{dx} = \prod_{i=1}^{n}(x^2 - \sigma_i^2). \tag{5}$$

The solution of (5) is

$$Q_{2n+1}(x) = C(n)I_n(x), \tag{6}$$

where

$$I_n(x) = \int_0^x \prod_{i=1}^{n}(t^2 - \sigma_i^2)\, dt \tag{7}$$

and $C(n)$ is a constant;

2. if $\sigma_0 = \lambda$ and $\sigma_{n+1} = 1$, then $Q_{2n+1}(\sigma_0) = Q_{2n+1}(\sigma_2) = \dots = Q_{2n+1}(\sigma_s)$; $Q_{2n+1}(\sigma_1) = Q_{2n+1}(\sigma_3) = \dots = Q_{2n+1}(\sigma_l)$, where $s = 2\left[\frac{n+1}{2}\right]$; $l = 2\left[\frac{n}{2}\right] + 1$. The condition 2 shows that [4]

$$U(\sigma_1, \dots, \sigma_n) = [I_n(\sigma_2) - I_n(\sigma_0)]^2 + [I_n(\sigma_3) - I_n(\sigma_1)]^2 + \dots + [I_n(1) - I_n(\sigma_{n-1})]^2 = 0. \tag{8}$$

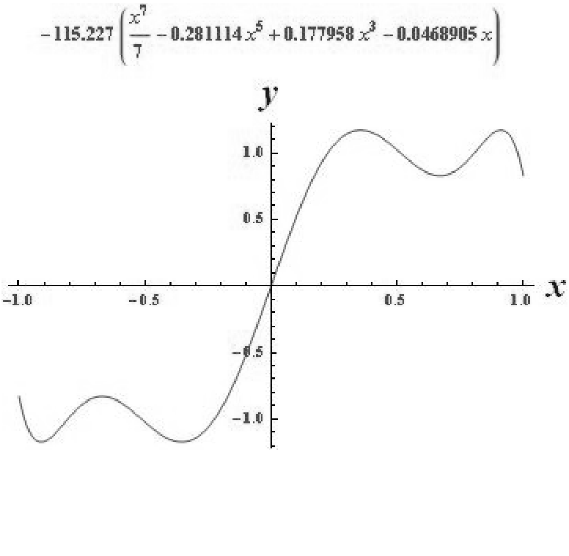
The function U depends on the variables $\sigma_1, \sigma_2, \dots, \sigma_n$ and from (7) it follows

$$I_n(x) = \frac{1}{2n+1}x^{2n+1} - \frac{1}{2n-1}\sum_{i=1}^{n}\sigma_i^2 x^{2n-1} + \dots + (-1)^n \sigma_1^2 \sigma_2^2 \dots \sigma_n^2 .x.$$

The constants $\sigma_1, \sigma_2, \dots, \sigma_n$ can be uniquely determined from (8) whereupon the polynomial Q_{2n+1} can be explicitly represented using (6) since $C(n) = \frac{2}{I_n(\sigma_0) + I_n(\sigma_1)}$. In [4] a numerical method is proposed for finding of alternation points $\sigma_1, \sigma_2, \dots, \sigma_n$. The authors suggest a modification of the difference-gradient method for minimization of the function $U(\sigma_1, \sigma_2, \dots, \sigma_n)$. The numerical experiments show that the solution of (8) is a hard problem and the question of computing of the polynomials (6) is open. We propose another approach based on the MATHEMATICA options.

Remark. Let the polynomial $P_{2n+1}(x) = \sum_{k=0}^{n} p_k x^{2k+1}$ satisfy: $P_{2n+1}(\lambda_n) = 1 - \lambda_n$, $\sigma_0 = \lambda_n$; $P_{2n+1}(\sigma_0) = P_{2n+1}(\sigma_2) = \cdots = P_{2n+1}(\sigma_s)$; $P_{2n+1}(\sigma_1) = P_{2n+1}(\sigma_3) = \cdots = P_{2n+1}(\sigma_l)$, where $s = 2\left[\frac{n+1}{2}\right]$; $l = 2\left[\frac{n}{2}\right] + 1$. It is proved in [4] that the polynomial P_{2n+1} is the polynomial of the best Hausdorff approximation of the function $sgn\,(x)$ in $[-1, 1]$ by algebraic polynomials of degree $2n+1$ and λ_n is the best Hausdorff approximation. The conditions above determine completely the polynomial P_{2n+1} and for finding λ_n, σ_1, σ_2, $\ldots$, σ_n, the following operator must be used from the programming environment MATHEMATICA: $\mathtt{FindMinimum[f, \{\{x, x_0\}, \{y, y_0\}, \ldots\}]}$ which "searches for a local minimum in a function of several variables". In what follows we illustrate the possibilities of MATHEMATICA for finding the polynomial of the best Hausdorff approximation of the function $\operatorname{sgn} x$.

Example. Let $n = 3$. We are looking for the polynomial $P_7(x)$. Let us start with $\lambda_7 \approx 0.171622$ and accuracy $\varepsilon = 10^{-6}$. The test provided in our control example follows. Let $\mathtt{f[x\_]} = \mathtt{1/7*x^7 - 1/5*(\theta 1^2 + \theta 2^2 + \theta 3^2)*x^5 + 1/3*(\theta 2^2\theta 3^2 + \theta 1^2\theta 3^2 + \theta 1^2\theta 2^2)*x^3 - (\theta 3^2 * \theta 1^2 * \theta 2^2)*x}$. Using the operators $\mathtt{FindMinimum[(f[\theta 2] - f[t])^2 + (f[\theta 3] - f[\theta 1])^2 + (f[1] - f[\theta 2])^2, \{\{\theta 1, 0.3\}, \{\theta 2, 0.6\}, \{\theta 3, 0.9\}\}]}$; $\mathtt{Print[TableForm[\%]]}$ we find $2.7747 * 10^{-20}$; $\theta 1 = 0.354572720549729$, $\theta 2 = 0.6699168186173807$, $\theta 3 = 0.9116252786309404$. Now the following operators must be applied: $\mathtt{c = 2/(f[t] + f[0.3134934291772114]);}$: $\mathtt{P[x\_] = c*(1/7*x^7 - 1/5*(\theta 1^2 + \theta 2^2 + \theta 3^2)*x^5 + 1/3*(\theta 2^2\theta 3^2 + \theta 1^2\theta 3^2 + \theta 1^2\theta 2^2)*x^3 - (\theta 3^2 * \theta 1^2 * \theta 2^2)*x)}$; $\mathtt{Abs[P[t] - 1]}$; $\mathtt{Abs[t - Abs[P[t] - 1]]}$; $\mathtt{Plot[P[x], \{x, -1, 1\}}$, $PlotRange \to Full$, $AxesLabel \to \{x, y\}$, $PlotLabel \to P[x]]$. They give -115.227; 0.171623; $6.39626 * 10^{-7}$. The graph of the polynomial P_7 of the best approximation is shown below. This polynomial satisfies the accuracy condition $|\lambda_7 - |P_7(\lambda_7) - 1|| \leq \epsilon$, where $\epsilon = 6.39626 * 10^{-7}$.



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АПРОКСИМИРАНЕ НА НЯКОИ ИМПУЛСНИ ФУНКЦИИ – ИМПЛЕМЕНТАЦИЯ В ПРОГРАМНАТА СРЕДА MATHEMATICA

Андрей С. Андреев, Николай В. Кюркчиев

В тази статия се разглежда задачата за апроксимиране на някои импулсни функции, които играят важна роля в областта на антенно-фидерната техника при синтез и анализ на диаграми на излъчване и цифрови филтри. Разработени са модули в програмната среда MATHEMATICA за облекчаване на инженерните разчети.