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COVARIANT VECTOR DECOMPOSITION OF THREE-DIMENSIONAL ROTATIONS*

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The main purpose of this paper is to provide an alternative representation for the generalized *Euler* decomposition (with respect to arbitrary axes) obtained in [1, 2] by means of vector parametrization of the *Lie* group $\mathsf{SO}(3)$. The scalar (angular) parameters of the decomposition are explicitly written here as functions depending only on the contravariant components of the compound vector-parameter in the basis, determined by the three axes. We also consider the case of coplanar axes, in which the basis needs to be completed by a third vector and in particular, two-axes decompositions.

1. Vector-parameters in the *Euler* decomposition. Vector-parameters, also known as *Rodrigues'* or *Gibbs'* vectors, are naturally introduced via stereographic projection. For the rotation group in $\mathbb{R}^3$ we consider the spin cover $\mathsf{SU}(2) \cong \mathbb{S}^3 \longrightarrow \mathsf{SO}(3) \cong \mathbb{RP}^3$ and identify $\mathbb{S}^3$ with the set of the unit quaternions (cf. [4])

$$\zeta = (\zeta_0, \boldsymbol{\zeta}) = \zeta_0 + \zeta_1 \mathbf{i} + \zeta_2 \mathbf{j} + \zeta_3 \mathbf{k}, \qquad |\zeta|^2 = \zeta\bar{\zeta} = 1, \qquad \bar{\zeta} = (\zeta_0, -\boldsymbol{\zeta}), \qquad \zeta_\alpha \in \mathbb{R}.$$

The corresponding group morphism is given by the adjoint action of $\mathbb{S}^3$ in its *Lie* algebra of skew-*Hermitian* matrices, in which we expand vectors $\mathbf{x} \in \mathbb{R}^3 \to x_1\mathbf{i}+x_2\mathbf{j}+x_3\mathbf{k} \in \mathfrak{su}(2)$. The resulting $\mathsf{SO}(3)$ matrix transforming the *Cartesian* coordinates of $\mathbf{x}$ has the form

$$\mathcal{R}(\zeta) = (\zeta_0^2 - \boldsymbol{\zeta}^2)\mathcal{I} + 2\,\boldsymbol{\zeta}\otimes\boldsymbol{\zeta}^t + 2\,\zeta_0\boldsymbol{\zeta}^\times, \tag{1}$$

where $\mathcal{I}$ and $\boldsymbol{\zeta}\otimes\boldsymbol{\zeta}^t$ denote the identity and the tensor (dyadic) product in $\mathbb{R}^3$ respectively, whereas $\boldsymbol{\zeta}^\times$ is the skew-symmetric matrix, associated with the vector $\boldsymbol{\zeta}$ via *Hodge* duality. The famous *Rodrigues'* rotation formula then follows directly with the substitution

$$\zeta_0 = \cos\frac{\varphi}{2}, \qquad \boldsymbol{\zeta} = \sin\frac{\varphi}{2}\,\mathbf{n}, \qquad (\mathbf{n},\mathbf{n}) = 1.$$

On the other hand, we may choose to get rid of the unnecessary fourth coordinate by projecting $\zeta \to \mathbf{c} = \dfrac{\boldsymbol{\zeta}}{\zeta_0} = \tan\left(\dfrac{\varphi}{2}\right)\mathbf{n}$ and thus obtain the entries of the rotation matrix (1) expressed as rational functions of the *vector-parameter* $\mathbf{c}$ in the form

$$\mathcal{R}(\mathbf{c}) = \frac{(1-\mathbf{c}^2)\,\mathcal{I} + 2\,\mathbf{c}\otimes\mathbf{c}^t + 2\,\mathbf{c}^\times}{1+\mathbf{c}^2}. \tag{2}$$

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Quaternion multiplication then gives the composition law of vector-parameters as

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1}{1 - (\mathbf{c}_2, \mathbf{c}_1)}, \qquad \mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_1) = \mathcal{R}(\langle \mathbf{c}_2, \mathbf{c}_1 \rangle) \tag{3}$$

and in the case of three rotations $\mathbf{c} = \langle \mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1 \rangle$ we have

$$\mathbf{c} = \frac{\mathbf{c}_3 + \mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_3 \times \mathbf{c}_2 + \mathbf{c}_3 \times \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1 + (\mathbf{c}_3 \times \mathbf{c}_2) \times \mathbf{c}_1 - (\mathbf{c}_3, \mathbf{c}_2)\, \mathbf{c}_1}{1 - (\mathbf{c}_3, \mathbf{c}_2) - (\mathbf{c}_3, \mathbf{c}_1) - (\mathbf{c}_2, \mathbf{c}_1) + (\mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1)} \cdot \tag{4}$$

It is not difficult to see that the operation is associative and constitutes a representation of $\mathsf{SO}(3)$, since the identity and inverse elements are also well-defined by $\langle \mathbf{c},\, 0 \rangle = \langle 0,\, \mathbf{c} \rangle = \mathbf{c}$, $\langle \mathbf{c},\, -\mathbf{c} \rangle = 0$. Among the advantages of this representation are more economical calculations, rational expressions for the matrix entries of $\mathcal{R}(\mathbf{c})$ and a correct description of the topology of $\mathsf{SO}(3) \cong \mathbb{RP}^3$. For applications in rigid body mechanics we refer to [3, 5].

As for the generalized *Euler* decompositions, we start with the much simpler two axes setting $\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}_2)\, \mathcal{R}(\mathbf{c}_1)$, where $\mathbf{c}_k = \tau_k \hat{\mathbf{c}}_k$ and $\mathbf{c} = \tau \mathbf{n}$ $(\hat{\mathbf{c}}_k^2 = \mathbf{n}^2 = 1)$ are the corresponding vector-parameters. We also denote $(\hat{\mathbf{c}}_j, \mathcal{R}(\mathbf{c})\, \hat{\mathbf{c}}_k) = r_{jk}$ and $(\hat{\mathbf{c}}_j,\, \hat{\mathbf{c}}_k) = g_{jk}$. Taking an appropriate scalar product provides the necessary and sufficient condition for the existence of the above decomposition in the form $r_{21} = g_{21}$. Next, multiplying $\mathbf{c} = \langle \mathbf{c}_2, \mathbf{c}_1 \rangle$ on the left by $\mathbf{n}^{\times}$ and projecting along $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_2$ respectively, we obtain

$$\tau_1 = \frac{\tilde{\upsilon}_3}{g_{12}\upsilon_1 - \upsilon_2}, \qquad \tau_2 = \frac{\tilde{\upsilon}_3}{g_{12}\upsilon_2 - \upsilon_1}, \tag{5}$$

where we make use of the notations

$$\upsilon_k = (\hat{\mathbf{c}}_k, \mathbf{n}), \qquad \tilde{\upsilon}_1 = (\hat{\mathbf{c}}_2 \times \hat{\mathbf{c}}_3, \mathbf{n}), \qquad \tilde{\upsilon}_2 = (\hat{\mathbf{c}}_3 \times \hat{\mathbf{c}}_1, \mathbf{n}), \qquad \tilde{\upsilon}_3 = (\hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2, \mathbf{n}) \cdot$$

Note that vanishing denominators in the above expressions are related to *half-turns*, i.e., rotations by a straight angle. In particular, if $\mathbf{n} \perp \hat{\mathbf{c}}_{1,2}$ $(\upsilon_1 = \upsilon_2 = 0)$, we have a decomposition into a pair of reflections, which is a well-known result in elementary geometry.

In the case of three axes $\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}_3)\mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_1)$, such that $\hat{\mathbf{c}}_2$ cannot be parallel to $\hat{\mathbf{c}}_1$ or $\hat{\mathbf{c}}_3$, we use the scalar product $(\hat{\mathbf{c}}_3, \mathcal{R}(\mathbf{c})\, \hat{\mathbf{c}}_1) = (\hat{\mathbf{c}}_3, \mathcal{R}(\tau_2 \hat{\mathbf{c}}_2)\, \hat{\mathbf{c}}_1)$ to obtain

$$(r_{31} + g_{31} - 2g_{12}g_{23})\, \tau_2^2 + 2\omega\, \tau_2 + r_{31} - g_{31} = 0, \qquad \omega = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2 \times \hat{\mathbf{c}}_3).$$

The above quadratic equation has real roots given by

$$\tau_2^{\pm} = \frac{-\omega \pm \sqrt{\Delta}}{r_{31} + g_{31} - 2g_{12}g_{23}} \tag{6}$$

as long as its discriminant is non-negative

$$\Delta = \begin{vmatrix} 1 & g_{12} & r_{31} \\ g_{21} & 1 & g_{23} \\ r_{31} & g_{32} & 1 \end{vmatrix} \geq 0 \tag{7}$$

which plays the role of a necessary and sufficient condition for the existence of the decomposition. In order to find the remaining two scalar parameters, we use the composition

$$\mathbf{c}_1 = \langle -\mathbf{c}_2, -\mathbf{c}_3, \mathbf{c} \rangle, \qquad \mathbf{c}_2 = \langle -\mathbf{c}_3, \mathbf{c}, -\mathbf{c}_1 \rangle, \qquad \mathbf{c}_3 = \langle \mathbf{c}, -\mathbf{c}_1, -\mathbf{c}_2 \rangle.$$

Namely, multiplying by $\hat{\mathbf{c}}_k^\times$ on the left and projecting over $\mathbf{n}$, we obtain the linear-fractional relations between τ_k, which yield the solutions for the generic case in the form

$$(8)\qquad \begin{aligned} \tau_1^\pm &= \frac{g_{32} - r_{32}}{(g_{32} + r_{32})\tau\upsilon_1 - (g_{31} + r_{31})\tau\upsilon_2 + (r_{31} - g_{31})/\tau_2^\pm} \\ \tau_3^\pm &= \frac{g_{21} - r_{21}}{(g_{21} + r_{21})\tau\upsilon_3 - (g_{31} + r_{31})\tau\upsilon_2 + (r_{31} - g_{31})/\tau_2^\pm} \end{aligned}$$

while in the symmetric one we consider the limit $\tau \to \infty$ and thus obtain

$$(9)\qquad \tau_1^\pm = \frac{g_{23} - \upsilon_2\upsilon_3}{\upsilon_1\tilde{\upsilon}_1 + \upsilon_2\tilde{\upsilon}_2 + (\upsilon_1\upsilon_3 - g_{13})/\tau_2^\pm}, \qquad \tau_3^\pm = \frac{g_{12} - \upsilon_1\upsilon_2}{\upsilon_2\tilde{\upsilon}_2 + \upsilon_3\tilde{\upsilon}_3 + (\upsilon_1\upsilon_3 - g_{13})/\tau_2^\pm}.$$

In the three axes setting we may also have degenerate solutions, related to a singularity of the map $\mathbb{RP}^3 \to \mathbb{T}^3$, known as *gimbal lock*, which is given by the condition

$$(10)\qquad \hat{\mathbf{c}}_3 = \pm\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1.$$

In that case the parameters τ_1 and τ_3 cannot be determined independently. Instead, we have the effective two-axes decomposition $\mathcal{R}(\mathbf{c}) = \mathcal{R}(\tau_2\hat{\mathbf{c}}_2)\mathcal{R}(\tilde{\tau}_1\hat{\mathbf{c}}_1)$, where the solutions

$$(11)\qquad \tilde{\tau}_1 = \frac{\tau_1 \pm \tau_3}{1 \mp \tau_1\tau_3} = \frac{\tilde{\upsilon}_3}{g_{12}\upsilon_1 - \upsilon_2}, \qquad \tau_2 = \frac{\tilde{\upsilon}_3}{g_{12}\upsilon_2 - \upsilon_1}$$

form an one-parameter set, expressed in terms of the generalized *Euler* angles as

$$\varphi_1 \pm \varphi_3 = 2\arctan\left(\frac{\tilde{\upsilon}_3}{g_{12}\upsilon_1 - \upsilon_2}\right), \qquad \varphi_2 = 2\arctan\left(\frac{\tilde{\upsilon}_3}{g_{12}\upsilon_2 - \upsilon_1}\right).$$

2. Covariant form of the solutions. First, we consider the simpler case of two axes $\mathbf{c} = \langle\tau_2\hat{\mathbf{c}}_2, \tau_1\hat{\mathbf{c}}_1\rangle$, in which it is necessary to complete the basis with a third vector

$$(12)\qquad \mathbf{c} = \xi_1\,\hat{\mathbf{c}}_1 + \xi_2\,\hat{\mathbf{c}}_2 + \xi_3\,\hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2.$$

If we denote the adjoint matrix of g with γ, we have $|\,\hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2\,|^2 = 1 - g_{12}^2 = \gamma^{33}$. Note that in formula (5) we use the covariant components of $\mathbf{c}$ in the same basis

$$(13)\qquad \tau\upsilon_1 = \xi_1 + g_{12}\xi_2, \qquad \tau\upsilon_2 = \xi_2 + g_{12}\xi_1, \qquad \tau\tilde{\upsilon}_3 = \gamma^{33}\xi_3.$$

Thus, by direct substitution, we obtain the decomposability condition $r_{21} = g_{21}$ as

$$(14)\qquad \xi_1\xi_2 + (1 - g_{12}\xi_3)\xi_3 = 0$$

and the solutions themselves are given by the expressions

$$(15)\qquad \tau_1 = -\xi_3/\xi_2, \qquad \tau_2 = -\xi_3/\xi_1.$$

One peculiar symmetry becomes apparent from the above formula, namely $\tau_1\xi_2 - \tau_2\xi_1 = 0$.

In the three axes setting we first consider the case, in which $\{\hat{\mathbf{c}}_k\}$ constitutes a basis

$$\mathbf{c} = \xi_1\,\hat{\mathbf{c}}_1 + \xi_2\,\hat{\mathbf{c}}_2 + \xi_3\,\hat{\mathbf{c}}_3 = \langle\tau_3\hat{\mathbf{c}}_3, \tau_2\hat{\mathbf{c}}_2, \tau_1\hat{\mathbf{c}}_1\rangle.$$

We substitute the matrix entries r_{ij}, calculated according to (2) in the solutions (6), (8) and use the inverse metric tensor $g^{-1} = \omega^{-2}\gamma$ for lifting the indices of $\mathbf{c}$. Thus, we obtain

$$\tau_2^{\pm} = \frac{-\omega \pm \sqrt{\omega^2 - \sigma^2 + 2\gamma^{13}\sigma}}{\sigma - 2\gamma^{13}}, \qquad \sigma = 2\frac{\gamma^{13}\xi_2^2 - \gamma^{23}\xi_1\xi_2 - \gamma^{12}\xi_2\xi_3 + \gamma^{22}\xi_1\xi_3 - \omega\xi_2}{\xi_1^2 + \xi_2^2 + \xi_3^2 + 2g_{12}\xi_1\xi_2 + 2g_{23}\xi_2\xi_3 + 2g_{13}\xi_1\xi_3}$$

for the middle parameter and for the other two

$$\begin{aligned} \tau_1^{\pm} &= \frac{\gamma^{13}\xi_1\xi_2 + \gamma^{12}\xi_1\xi_3 - \gamma^{11}\xi_2\xi_3 - \gamma^{23}\xi_1^2 - \omega\xi_1}{\omega\left(\xi_1^2 + \xi_2^2 + 2g_{12}\xi_1\xi_2 + g_{13}\xi_1\xi_3 + g_{23}\xi_2\xi_3\right) - \gamma^{23}\xi_1 + \gamma^{13}\xi_2 + \kappa_2/\tau_2^{\pm}} \\ \tau_3^{\pm} &= \frac{\gamma^{13}\xi_2\xi_3 + \gamma^{23}\xi_1\xi_3 - \gamma^{33}\xi_1\xi_2 - \gamma^{12}\xi_3^2 - \omega\xi_3}{\omega\left(\xi_2^2 + \xi_3^2 + g_{12}\xi_1\xi_2 + g_{13}\xi_1\xi_3 + 2g_{23}\xi_2\xi_3\right) + \gamma^{13}\xi_2 - \gamma^{12}\xi_3 + \kappa_2/\tau_2^{\pm}} \end{aligned} \tag{16}$$

respectively, in which we use the notation $\kappa_2 = \gamma^{13}\xi_2^2 - \gamma^{23}\xi_1\xi_2 - \gamma^{12}\xi_2\xi_3 + \gamma^{22}\xi_1\xi_3 - \omega\xi_2$.

In the case $\omega = 0$ we use expansion in the basis (12) and the explicit relations (13) between the covariant and contravariant components of $\mathbf{c}$ in order to obtain

$$\tau_2^{\pm} = \pm\sqrt{\frac{\mathring{\sigma}}{2\gamma^{13} - \mathring{\sigma}}}, \qquad \mathring{\sigma} = 2\frac{\gamma^{13}\xi_2^2 - \gamma^{23}(\xi_1\xi_2 + \xi_3) - g_{13}\gamma^{33}\xi_3^2}{1 + \xi_1^2 + \xi_2^2 + \gamma^{33}\xi_3^2 + 2g_{12}\xi_1\xi_2}. \tag{17}$$

Denoting $\mathring{\kappa}_2 = \gamma^{13}\xi_2^2 - \gamma^{23}(\xi_1\xi_2 + \xi_3) - g_{13}\gamma^{33}\xi_3^2$, we have for $\tau_{1,3}$ the expressions

$$\begin{aligned} \tau_1^{\pm} &= \frac{\gamma^{13}(\xi_1\xi_2 - \xi_3) - \gamma^{23}\xi_1^2 + g_{23}\gamma^{33}\xi_3^2}{(\gamma^{13} + g_{12}\gamma^{23})\xi_1\xi_3 + (\gamma^{23} + g_{12}\gamma^{13})\xi_2\xi_3 + \gamma^{13}\xi_2 - \gamma^{23}\xi_1 + \mathring{\kappa}_2/\tau_2^{\pm}} \\ \tau_3^{\pm} &= \frac{g_{12}\xi_3^2 - \gamma^{33}(\xi_1\xi_2 + \xi_3)}{(g_{12}\gamma^{23} + g_{13}\gamma^{33})\xi_1\xi_3 + (\gamma^{23} + g_{23}\gamma^{33})\xi_2\xi_3 + \gamma^{13}\xi_2 + \mathring{\kappa}_2/\tau_2^{\pm}}. \end{aligned} \tag{18}$$

If the compound rotation is symmetric, i.e., $\varphi = \pi$ and $\mathcal{R}(\mathbf{c}) = \mathcal{O}(\mathbf{n}) = 2\mathbf{n} \otimes \mathbf{n}^t - \mathcal{I}$, considering the limit $\tau \to \infty$ in the solutions we substitute the coordinates ξ_k with the contravariant components η_k in the expansion of the unit vector $\mathbf{n}$ ($\xi_k = \tau\eta_k$) dropping all linear and constant terms in the expressions. For example, in the case $\omega = 0$ we have

$$\begin{aligned} \tau_1^{\pm} &= \frac{\gamma^{13}\eta_1\eta_2 - \gamma^{23}\eta_1^2 + g_{23}\gamma^{33}\eta_3^2}{(\gamma^{13} + g_{12}\gamma^{23})\eta_1\eta_3 + (\gamma^{23} + g_{12}\gamma^{13})\eta_2\eta_3 + (\gamma^{13}\eta_2^2 - \gamma^{23}\eta_1\eta_2 - g_{13}\gamma^{33}\eta_3^2)/\tau_2^{\pm}} \\ \tau_3^{\pm} &= \frac{g_{12}\eta_3^2 - \gamma^{33}\eta_1\eta_2}{(g_{12}\gamma^{23} + g_{13}\gamma^{33})\eta_1\eta_3 + (\gamma^{23} + g_{23}\gamma^{33})\eta_2\eta_3 + (\gamma^{13}\eta_2^2 - \gamma^{23}\eta_1\eta_2 - g_{13}\gamma^{33}\eta_3^2)/\tau_2^{\pm}} \end{aligned} \tag{19}$$

where

$$\tau_2^{\pm} = \pm\sqrt{\frac{\mathring{\sigma}}{2\gamma^{13} - \mathring{\sigma}}}, \qquad \mathring{\sigma} = 2\frac{\gamma^{13}\eta_2^2 - \gamma^{23}\eta_1\eta_2 - g_{13}\gamma^{33}\eta_3^2}{\eta_1^2 + \eta_2^2 + \gamma^{33}\eta_3^2 + 2g_{12}\eta_1\eta_2}. \tag{20}$$

The case $\omega \neq 0$ is treated similarly and so is the decomposition with respect to two axes.

As for the degenerate case (10), if $\omega = 0$ we may use the result obtained in the two axes setting combined with (11) in order to express

$$\tilde{\tau}_1 = \frac{\tau_1 \pm \tau_3}{1 \mp \tau_1\tau_3} = -\frac{\xi_3}{\xi_2}, \qquad \tau_2 = -\frac{\xi_3}{\xi_1}. \tag{21}$$

If $\omega \neq 0$ on the other hand, the solutions are given by

$$\tilde{\tau}_1 = \frac{\tau_1 \pm \tau_3}{1 \mp \tau_1\tau_3} = \frac{\omega\,\xi_3}{\gamma^{23}\xi_3 - \gamma^{33}\xi_1}, \qquad \tau_2 = \frac{\omega\,\xi_3}{\gamma^{13}\xi_3 - \gamma^{33}\xi_1}\,. \tag{22}$$

In both cases we may use η_k instead of ξ_k so that the expressions are valid when $\tau \to \infty$.

If we need to express ξ_k on the other hand, it is straightforward to use the composition law (4) and then take the correct scalar products. Thus, in the case $\omega \neq 0$ we obtain

$$\begin{aligned}
\xi_1 &= \frac{(1 - g_{23}\tau_2\tau_3)\tau_1 + \omega^{-1}(\gamma^{12}\tau_1\tau_3 - \gamma^{13}\tau_1\tau_2 - \gamma^{11}\tau_2\tau_3)}{1 - g_{12}\tau_1\tau_2 - g_{13}\tau_1\tau_3 - g_{23}\tau_2\tau_3 + \omega\tau_1\tau_2\tau_3} \\
\xi_2 &= \frac{(1 + g_{13}\tau_1\tau_3)\tau_2 + \omega^{-1}(\gamma^{22}\tau_1\tau_3 - \gamma^{12}\tau_2\tau_3 - \gamma^{23}\tau_1\tau_2)}{1 - g_{12}\tau_1\tau_2 - g_{13}\tau_1\tau_3 - g_{23}\tau_2\tau_3 + \omega\tau_1\tau_2\tau_3} \\
\xi_3 &= \frac{(1 - g_{12}\tau_1\tau_2)\tau_3 + \omega^{-1}(\gamma^{23}\tau_1\tau_3 - \gamma^{13}\tau_2\tau_3 - \gamma^{33}\tau_1\tau_2)}{1 - g_{12}\tau_1\tau_2 - g_{13}\tau_1\tau_3 - g_{23}\tau_2\tau_3 + \omega\tau_1\tau_2\tau_3}\,.
\end{aligned} \tag{23}$$

For $\omega = 0$ the corresponding result is

$$\mathring{\xi}_1 = \frac{(1 - g_{23}\tau_2\tau_3)\tau_1 - \gamma^{13}(1 - g_{12}\tau_1\tau_2)\tau_3/\gamma^{33}}{1 - g_{12}\tau_1\tau_2 - g_{13}\tau_1\tau_3 - g_{23}\tau_2\tau_3}$$

$$\mathring{\xi}_2 = \frac{(1 + g_{13}\tau_1\tau_3)\tau_2 - \gamma^{23}(1 - g_{12}\tau_1\tau_2)\tau_3/\gamma^{33}}{1 - g_{12}\tau_1\tau_2 - g_{13}\tau_1\tau_3 - g_{23}\tau_2\tau_3}, \quad \mathring{\xi}_3 = \frac{\gamma^{23}\tau_1\tau_3/\gamma^{33} - \gamma^{13}\tau_2\tau_3/\gamma^{33} - \tau_1\tau_2}{1 - g_{12}\tau_1\tau_2 - g_{13}\tau_1\tau_3 - g_{23}\tau_2\tau_3}.$$

Likewise, in the case of two axes we have a linear system for $\xi_{1,2}$ with solutions, given by

$$\xi_1 = \frac{\tau_1}{1 - g_{12}\tau_1\tau_2}, \qquad \xi_2 = \frac{\tau_2}{1 - g_{12}\tau_1\tau_2}\,. \tag{24}$$

Since the above expressions are rational in terms of the parameters τ_j, if any of these diverges, i.e., $\varphi_k = \pi$, we can still obtain the correct formulae, applying *l'Hôpital's* rule.

Similar expressions hold for the relations between the scalar parameters and the covariant components of $\mathbf{c}$ in the corresponding basis. However, these are almost straightforward to write considering the results obtained in [1, 2]. Another possible generalization involves the hyperbolic case, i.e., the three-dimensional *Lorentz* group $\mathsf{SO}(2,1)$, which can be treated in an analogous way. Some of the advantages of this new representation for the numerous applications of the generalized Euler decomposition (cf. [3, 4, 5]) are quite obvious. The explicit dependence only on the contravariant components allows, apart from its purely geometric merits, for straightforward differentiation, as well as for obtaining the decomposition in a rotated frame from one that has been given.

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КОВАРИАНТНО РАЗЛАГАНЕ НА ТРИМЕРНИ РОТАЦИИ

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В тази статия предлагаме алтернативно представяне на решенията, получени преди това в [1, 2] за обобщеното разлагане на *Euler* (около три произволни оси) чрез векторна параметризация на групата $\mathsf{SO}(3)$. Скаларните (ъглови) параметри в разлагането са представени като явни функции, зависещи само от контравариантните компоненти на вектор-параметъра на композитната ротация в базиса, определен от трите оси в разлагането. Отделно сме разгледали случаите, в които осите са компланарни и базисът следва да бъде допълнен, и в частност разлагането на две въртения.