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BULGARIAN 2013 PARLIAMENTARY ELECTIONS: COMPARISON OF THREE BI-PROPORTIONAL METHODS*

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The paper deals with the Bulgarian 2013 parliamentary elections. The results are analyzed and a comparison of three bi-proportional methods (the methods of the Central Electoral Commission (CEC) of 2009 and 2013 and a Double Hare-Niemeyer method) is presented. Alabama paradoxes of higher order are considered as well as the effects from the introduction of a new electoral region for the votes cast abroad.

Bulgarian parliamentary elections. In the period 1991–2013 there were seven parliamentary elections in Bulgaria for Ordinary National Assembly of 240 members of parliament (MP). Several variants of proportional electoral systems have been used in these elections (in 2009 the system was mechanically mixed with 31 MP elected in single constituencies). The voting is carried out in 31 regions in the country and abroad. A party takes part in the distribution of seats if it has obtained no less than 4 per cent of the valid votes. Independent candidates can also participate in the elections but none have been elected in this period.

The electoral system functions in two steps. First the total number of seats (or mandates) of each party is determined at nationwide level. This has been done by the D'Hondt method (1991, 1994, 1997, 2001, 2005) and the Hare-Niemeyer method (HNM) (2009, 2013).

Next the mandates of each party are personified by 31 regional candidate lists. In each region the number of mandates is determined proportionally to its population. This is done by a bi-proportional algorithm. Such algorithms have been studied, e.g., in [3, 7]. The mathematical aspects of the Bulgarian parliamentary elections are considered in [4, 5, 6, 7].

We shall use the notations: floor(x) – the entire part of the array (scalar, vector, or matrix) x; sum(x) – the sum of elements of x; $\preceq$ – the component-wise partial order relation for arrays of same size. If $S = [s_{ij}]$ is a matrix with elements s_{ij} then $s_{i\bullet}$ and $s_{\bullet j}$ are the ith row and the jth column of S respectively. If λ is a scalar and x is an array then $\lambda + x$ is the array with elements equal to those of x increased by λ. When $x = [x_1, x_2, \ldots, x_n]$ is a row array then its transpose is written as $x^\top = [x_1; x_2; \ldots; x_n]$ (some of these notations are inspired by MATLAB, a trademark of MathWorks).

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Plain proportional methods (PPM). The PPM allocates M seats (or mandates) among n parties $\Pi_1, \Pi_2, \dots, \Pi_n$ with number of votes $v_1, v_2, \dots, v_n$. Let s_j be the number of seats assigned to party Π_j. Setting

$$v = [v_1, v_2, \dots, v_n],\ v^* = v/\mathrm{sum}(v), \quad s = [s_1, s_2, \dots, s_n],\ s^* = s/M$$

we have $\mathrm{sum}(v^*) = 1$, $\mathrm{sum}(s) = M$ and $\mathrm{sum}(s^*) = 1$. Here the numbers s_j are approximately proportional to v_j. Denoting by

$$\sigma = Mv^* = [\sigma_1, \sigma_2, \dots, \sigma_n]$$

the vector of fractional seats we have $\mathrm{sum}(\sigma) = M$.

The method is inplemented by an algorithm $\mathbf{F} = [\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n]$, $\mathbf{F}_j = \mathbf{F}_j(M, v)$ such that $s = \mathbf{F}(M, v)$. It includes a random choice mechanism (when e.g. a mandate must be distributed to two or more parties with equal votes). Note that here v and s are row vectors. We shall use a similar notation for column vectors setting

$$\mathbf{F}(M; v^\top) = (\mathbf{F}(M, v))^\top$$

More generally, let $V = [v_{ij}]$ be an $m \times n$ matrix of votes, $s = [s_j]$ be the $1 \times n$ vector of party seats at nationwide level and $r = [r_i]$ be the $m \times 1$ vector of seats allocated to the electoral regions with $\mathrm{sum}(s) = \mathrm{sum}(r) = M$. Then $\mathbf{F}(s; V)$ is the $m \times n$ matrix of seats with elements $\mathbf{F}_{ij}(s; V)$ such that $\mathbf{F}_{\bullet j}(s; V) = \mathbf{F}(s_j; v_{\bullet j})$ while $\mathbf{F}(r, V)$ is an $m \times n$ matrix with elements $\mathbf{F}_{ij}(r, V)$ such that $\mathbf{F}_{i\bullet}(r, V) = \mathbf{F}(r_i, v_{i\bullet})$, where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Desirable properties of PPM. The next three conditions for PPM are desirable.

C1 If a given party has more votes than another party then it shall not have fewer seats: $(v_i - v_j)(s_i - s_j) \geq 0$.

C2 If more seats are allocated then no party shall lose a seat: $\mathbf{F}(v, M) \preceq \mathbf{F}(v, M + k)$, $k \geq 1$ (the violation of this condition is known as *the Alabama paradox*).

C3 The *fair share* condition $q \preceq s \preceq 1 + q$ is satisfied, where $q = \mathrm{floor}(\sigma)$.

An important result [1] says that *there is no PPM that satisfies* **C1**–**C3**.

The plain HNM. This method acts as follows: 1. the vectors σ and $q = [q_j] = \mathrm{floor}(\sigma)$ are calculated; 2. each party Π_j obtains q_j initial seats and if $q = \sigma$ then all M seats are allocated; 3. if $q \neq \sigma$ then there are $\mathrm{sum}(\sigma - q)$ seats for additional allocation; 4. the party Π_k with maximum remainder $\sigma_k - q_k = \max_j\{\sigma_j - q_j\}$ gets one additional seat first; 5. the second seat goes to the party with the next largest remainder, etc. If this procedure cannot be continued because of equal remainders then additional criteria or ties are applied.

HNM satisfies conditions **C1** and **C3** but may violate condition **C2**, i.e., it suffers from the Alabama paradox. The result produced by this method is denoted as

$$s = \mathbf{HN}(M, v)$$

More on Alabama paradoxes. The concept of Alabama paradox [2] may be generalized as follows (see also [5]).

The pair (M, v) is said to be an *Alabama pair of order* k if the inequalities $\mathbf{F}(M, v) \preceq \mathbf{F}(M + j, v)$ are *violated* for $j = 1, 2, \dots, k$ but $\mathbf{F}(M, v) \preceq \mathbf{F}(M + k + 1, v)$.

Thus the standard Alabama paradox corresponds to an Alabama pair of order 1. We also stress that a necessary condition for existence of the paradox is $k \leq \min\{n, M\} - 2$. In particular $n, M \geq 3$ must be fulfilled, see Example 4 below.

Example 1. Let $v = [1, 3, 3]$. Then $\mathbf{HN}(3, v) = [1, 1, 1]$, $\mathbf{HN}(4, v) = [0, 2, 2]$ and hence $(3, v)$ is an Alabama pair of order 1.

Example 2. Let $v = [1, 4, 4, 4]$. Then $\mathbf{HN}(4, v) = [1, 1, 1, 1]$, $\mathbf{HN}(5, v) = [0, 2, 2, 1]$, $\mathbf{HN}(6, v) = [0, 2, 2, 2]$ and $\mathbf{HN}(7, v) = [1, 2, 2, 2]$. Thus $(4, v)$ is Alabama pair of order 2.

Example 3. Let $M = n$, $v = [v_1, v_2, \ldots, v_n]$, $v_1 = 1$, $v_2 = v_3 = \cdots = v_n = n$ and let the parties $\Pi_2, \ldots \Pi_n$ be ordered by ties. Then $\mathbf{HN}(n, v) = [1, 1, \ldots, 1]$,

$$\mathbf{HN}(n + i, v) = [0, 2, \ldots, 1], \; i = 1, 2, \ldots, n - 2$$

and $\mathbf{HN}(2n - 1, v) = [1, 2, \ldots, 2]$. Thus (n, v) is an Alabama pair of order $n - 2$.

Example 4. Generalizing the above examples, let $M = n \geq 3$, $v_1 = 1$, $v_2 = v_3 = \cdots = v_n$ and $v_2 \in (1 + n/2, n + 1)$. Then $\mathbf{HN}(n, v) = [1, 1, \ldots, 1]$,

$$\mathbf{HN}(n + j, v) = [0, 2, \ldots, 1], \; j = 1, 2, \ldots, n - 2$$

and $\mathbf{HN}(2n - 1, v) = [1, 2, \ldots, 2]$. Thus (n, v) is an Alabama pair of order $n - 2$. Here this phenomenon disappears if $v_2 < 1 + n/2$ or $v_2 > n + 1$.

The Alabama paradox is important for the Bulgarian electoral system (described below) since it may occur when a regional list contains fewer names than there are seats allocated to this list. Such a theoretical possibility existed also in the 2007 elections for members of the European parliament from Bulgaria. However, the purpose of the analysis of Alabama-type paradoxes is not to avoid them but rather to inform the political parties and the public about this phenomenon.

The Bulgarian electoral system (BES). The BES functions as follows. First the country is divided into $m = 31$ electoral regions $R_1, R_2, \ldots, R_m$. In each region R_i a number of seats r_i is preassigned proportionally to its population by HNM. Denote $r = [r_1; r_2; \ldots; r_m]$ (recently there is an additional requirement $r_i \geq 4$). There is also voting abroad, the corresponding results being accounted for in different ways.

Let parties $\Pi_1, \Pi_2, \ldots, \Pi_n$ have votes $v_1, v_2, \ldots, v_n$ so that $v_j/V_0 \geq 0.04$ (the 4-percent barrier!), where V_0 is the sum of all valid votes. We have $v_j = c_j + a_j$, or $v = c + a$, where c_j and a_j are the votes for party Π_j in the country and abroad respectively. The party seats at national level are determined (when there are no independent candidates elected) from $s = [s_1, s_2, \ldots, s_n] = \mathbf{HN}(240, v)$.

The data for distribution of seats among party lists Π_{ij} is the $m \times n$ matrix $V = [v_{ij}]$ and the vector $a = [a_1, a_2, \ldots, a_n]$, where v_{ij} is the number of votes for the party list Π_{ij} of party Π_j in region R_i. The data is augmented as an $(m + 1) \times n$ matrix $W = [V; a]$ of votes in country regions and abroad.

The result is the $m \times n$ non-negative integer matrix $S = [s_{ij}]$, where s_{ij} is the number of seats assigned to Π_{ij}. The matrix S satisfies $\mathrm{sum}(s_{i\bullet}) = r_i$ and $\mathrm{sum}(s_{\bullet j}) = s_j$.

Region for votes abroad (RVA). In the above scheme the vote vector a is accounted for only at national level and this leads to a certain distortsion in the regional allocation of seats. A possible way to overcome this problem is to introduce RVA. This is motivated also by constitutional reasons: nowadays voters abroad cannot vote for particular party lists as well as for independent candidates, which is a violation of their constitutional rights.

A problem here is how to determine the number of seats in this region. This can hardly be done *a priori*. Indeed, there is no reliable data on the Bulgarian population abroad. At the same time the number of voters abroad is approximately one million but this data also cannot be used since the voting rate abroad is very low.

A way to overcome the problem is to determine the number of seats in RVA *a posteriori* on the basis of votes cast therein. However, the same rule has to be applied to the electoral regions in the country. Thus the number of mandates allocated to electoral regions is determined proportionally to the number of votes cast (including invalid ones). A similar rule is applied for federal elections in Swiss. The *a posteriori* determination of regional seats has even a certain advantage: the voter knows beforehand that the number of seats in his (her) region depends on the vote turnout and is thus additionally motivated to vote.

When this latter approach is applied, denote by $u = [u_1, u_2, \ldots, u_{32}]$ the vector of votes cast (RVA is the 32nd electoral region). Then the 32-vector of seats allocated to the electoral regions is $\mathbf{HN}(240 - N, u)$, where N is the total number of independent candidates elected (if any). More comments on this issue are given below after Table 3.

CEC-2009 method. The method used by CEC in 2009 is a particular case of the (α, β, γ)-algorithm [7] for $\alpha = \gamma = 0$ and $\beta = 1$. The method starts with allocation of seats among the lists of each party by HNM. At each step the algorithm computes a current $m \times n$ matrix $T = [t_{ij}]$ of seats. As a result lists Π_{ij} and regions R_i obtain a current number of seats t_{ij} and $\text{sum}(t_{i\bullet})$ respectively, while each party Π_j has a constant number $s_j = \text{sum}(t_{\bullet j})$ of seats. The region R_i is said to be with *balance*, *surplus* or *shortage* of seats if its current number of seats is equal to, more than or less than r_i respectively. A brief description of the method is as follows.

1. The s_j seats of each party Π_j are distributed among its lists $\Pi_{1j}, \Pi_{2j}, \ldots, \Pi_{mj}$ by HNM. As a result the current seat matrix is set to $T = [t_{ij}] = \mathbf{F}(s; V)$. The allocation of seats in regions with balance is final and these regions are excluded.
2. If there are regions with surplus then there are regions with shortage of seats. A seat is taken from the list Π_{ij} with $t_{ij} \geq 1$ in regions with surplus and with least price v_{ij}/t_{ij} and is given to the list Π_{kj} in regions with shortage and with greatest future price $v_{kj}/(t_{kj} + 1)$.

CEC-2013 method. This method, contrary to the previous one, starts with allocation of seats in the regions. At each step the algorithm computes a current $m \times n$ matrix $T = [t_{ij}]$ of seats. Initially the method ensures that the number of seats (obtained as the integer parts of fractional mandates) is such that the column and row sums of T do not exceed the corresponding elements of s and r. The lists Π_{ij} and parties Π_j obtain a current number of seats t_{ij} and $\text{sum}(t_{\bullet j})$ respectively, while each region R_i has a constant number $r_i = \text{sum}(t_{i\bullet})$ of seats. The party Π_j is said to be with *balance*, *surplus*, or *shortage* of seats if its current number of seats is equal to, more than or less than s_j respectively.

A brief description of a modification of the method follows since its published version has been inconsistent (for the correct version see e.g. `apa.bg`).

1. The r_i seats in each region R_i are distributed among its lists $\Pi_{i1}, \Pi_{i2}, \ldots, \Pi_{in}$ by HNM. The current seat matrix is set to $T = [t_{ij}] = \text{floor}(\mathbf{F}(r, V))$. The allocation of seats for parties (regions) with balance is final and these parties (regions) are excluded. The matrix V and the vector s (r) are updated deleting the corresponding columns (rows).
2. If there are parties with surplus then a seat is taken from the list Π_{ij} with $t_{ij} \geq 1$ for

parties with surplus and with least price v_{ij}/t_{ij}. New prices of lists are computed and the procedure is repeated until each party is either excluded or with shortage. The vector r is updated reducing the old values of r_i with the sum of seats of excluded parties in region R_i.

3. The remaining seats are distributed by HNM according to $T = \mathbf{F}(r, V)$, where (r, V) is the updated data. If $t_{ij} = 1 + \text{floor}(\sigma_{ij})$, where $\sigma_{ij} = r_i v_{ij}/\text{sum}(v_{i\bullet})$ the list Π_{ij} is marked. If a party is with balance it is excluded.
4. If there are parties with surplus than a seat is taken from the marked list Π_{ij} among such parties with least remainder $\sigma_{ij} - \text{floor}(\sigma_{ij})$ and is given to the unmarked list Π_{ik} with greatest remainder. The list Π_{ij} is unmarked and the list Π_{ik} is marked.
5. If the procedure from step 3 cannot be repeated because there are no appropriate lists and there are seats for distribution then a seat is given to the list of party with shortage and with the greatest future price. The data is updated.

The Double Hare-Niemeyer method (DHNM). This bi-proportional method has been proposed in [7] and acts as follows (see also `apa.bg`).

1. A matrix T_0 with elements $t_{ij} = Mv_{ij}/\text{sum}(V)$ is constructed. The current distribution of seats is set to $T = \text{floor}(T_0)$. Regions and parties with balance are excluded and the data are updated.
2. If $\text{sum}(t_{\bullet j}) > s_j$ for some j a seat is taken from the list Π_{ij} with least price of one seat. Parties with balance are excluded and the data is updated.
3. If there are additional seats for allocation after steps 1–2 then the list Π_{ij} with the greatest remainder $t_{ij} - \text{floor}(t_{ij})$ among the remaining lists takes the first additional seat. Parties with balance are excluded and the data are updated.
4. If there are additional seats for allocation after step 3 then the list with the dreatest future price takes the first additional seat.

Measures for estimating seat distributions. There are many possible measures to estimate the quality of a seat distribution S corresponding to a given matrix W.

One of the most unpleasant effects in a seat distribution is connected to the so-called *discordances.*

The apportionment S is called *party-wise monotone*, *region-wise monotone* and *completely monotone* if $(v_{ij} - v_{kj})(s_{ij} - s_{kj}) \geq 0$, $(v_{ij} - v_{il})(s_{ij} - s_{il}) \geq 0$ and $(v_{ij} - v_{kl})(s_{ij} - s_{kl}) \geq 0$ for $i, k = 1, 2, \ldots, m$ and $j, l = 1, 2, \ldots, n$, respectively. The violation of a monotonicity condition is called a *discordance.*

The number of party and region discordances are denoted as Δ_P and Δ_R while the number of discordances among arbitrary lists is Δ (note that usually $\Delta_R + \Delta_P < \Delta$).

A desirable property of a distribution is that the numbers of discordances are small. As we shall see, this can only be achieved for regional discordances. Which, however, are the most important ones from a psychological point of view.

Two other measures of the quality of a distribution are the number $v_{\min}$ of minimum votes among the lists with one mandate and the number $v_{\max}$ of maximum votes among the lists with no mandates.

Another set of measures is connected with the price $C_{ij} = v_{ij}/s_{ij}$ of one mandate for a list among the lists with $s_{ij} \geq 1$.

Denote by $\overline{C_{i\bullet}}$ and $\underline{C_{i\bullet}}$ the maximum and minimum prices among lists in region R_i and by $\overline{C_{\bullet j}}$ and $\underline{C_{\bullet j}}$ the maximum and minimum prices among lists of party Π_j. Let also

Table 1. Votes cast in the country and abroad in 2013 elections

Reg	BSP	Ataka	CEDB	MRF	Reg	BSP	Ataka	CEDB	MRF
1	32060	11301	47664	19005	18	9721	2034	10325	28149
2	45566	7797	66533	22681	19	27581	9486	32545	10827
3	48500	19908	73171	14219	20	14052	2614	16127	17411
4	38381	9989	32966	7393	21	23874	6610	25791	5764
5	17662	3370	12335	474	22	17638	1403	15652	10148
6	28772	7669	23044	7975	23	62101	13036	78176	1156
7	15678	5046	21871	2983	24	46484	12367	63952	1128
8	23049	6016	25199	10465	25	54650	14951	70035	1553
9	8902	1325	12729	46407	26	35917	9921	37815	3278
10	22038	7312	19610	445	27	50103	12565	48090	13097
11	20218	4763	19431	4672	28	13306	2751	10783	21593
12	23062	8197	19721	2525	29	33219	8408	34038	17637
13	33395	13359	34994	17257	30	19478	5304	21603	25291
14	20687	7166	20046	455	31	21585	4364	19152	1770
15	41856	11448	34210	9270	Abr	4907	3018	23090	54353
16	41247	13252	62003	6200	Tot	942541	258481	1081605	400466
17	46852	11731	48904	14885	Man	84	23	97	36

$\overline{C}$ and $\underline{C}$ be the maximum and minimum prices among all lists. Set

$$\rho_R = \max\{\overline{C_{i\bullet}}/\underline{C_{i\bullet}} : i = 1, 2, \ldots, m\},\ \rho_P = \max\{\overline{C_{\bullet j}}/\underline{C_{\bullet j}} : j = 1, 2, \ldots, n\},\ \rho = \overline{C}/\underline{C}$$

where $m = 31$ without RVA and $m = 32$ with RVA.

Votes cast in the elections. The votes cast are given in Table 1, where the parties are denoted as BSP (Bulgarian Socialist Party), Ataka, CEDB (from Citizens for European Development of Bulgaria) and MRF (Movement for Rights and Freedoms).

Seat allocation. Six seat allocations are computed by the three methods (DHNM, CEC-2009 and CEC-2013) without and with RVA as follows (Table 2).

The two rows named “Man” in Table 3 clearly show the difference in the regional number of seats with and without RVA.

The total number of seats in RVA is 8. Nine regions in the country (no. 1, 2, 3, 5, 13, 17, 19, 23, 25) “lose” a seat while region no. 10 “wins” a seat (this looks like an Alabama paradox but in fact it is not) when RVA is introduced. If we denote the 31-vectors of seats in the country regions with and without RVA by r and r_{RVA}, respectively, then

$$\frac{\mathrm{sum}(\mathrm{abs}\,(r - r_{\mathrm{RVA}}))}{\mathrm{sum}(r)} = \frac{9}{240} = 0.0375$$

i.e. the relative difference is less than 4%.

Comparison of results. The results obtained are compared according to Table 4, where the methods get 6, 5, ..., 1 points for first, second, ..., sixth place.

The ranking is 1. CEC-13, 2. CEC-09, 3. DHNM for both cases without and with RVA. We see that the introduction of RVA relaxes the problem and reduces the values of the quality costs.

MATLAB codes for simulation of the above (and many other) methods have been developed and widely used.

Table 2. Distribution of seats without RVA by three methods:
Double Hare-Niemeyer, Modified CEC-2009, Modified CEC-2013

Reg	BSP	Ataka	CEDB	MRF	Man	Reg	BSP	Ataka	CEDB	MRF	Man
1	3,3,3	1,1,1	4,4,5	3,3,2	11	17	4,4,4	1,1,1	5,5,4	1,1,2	11
2	4,4,4	1,1,1	6,6,7	3,3,2	14	18	1,1,1	0,0,0	1,1,1	2,2,2	4
3	4,4,5	2,2,2	7,7,7	2,2,1	15	19	3,2,3	1,1,1	3,3,3	1,2,1	8
4	3,3,3	1,1,1	3,3,3	1,1,1	8	20	1,1,1	0,0,0	1,1,1	2,2,2	4
5	2,2,2	0,0,0	1,1,1	1,1,1	4	21	2,2,2	1,1,1	2,2,2	1,1,1	6
6	2,3,2	1,0,1	2,2,2	1,1,1	6	22	2,2,1	0,0,1	1,1,1	1,1,1	4
7	1,1,1	1,1,0	2,2,2	0,0,1	4	23	6,6,6	1,3,1	7,7,8	2,0,1	16
8	2,2,2	1,1,1	2,2,2	1,1,1	6	24	4,4,4	1,2,1	6,6,6	1,0,1	12
9	0,0,1	0,0,0	1,1,1	4,4,3	5	25	5,5,5	2,2,1	7,7,7	0,0,1	14
10	2,2,2	0,0,1	2,2,1	0,0,0	4	26	3,3,3	1,1,1	4,4,3	0,0,1	8
11	2,2,2	1,0,1	2,2,2	0,1,0	5	27	5,5,4	1,1,1	4,4,4	1,1,2	11
12	2,2,2	1,1,1	2,2,2	0,0,0	5	28	1,1,1	0,0,0	1,1,1	2,2,2	4
13	3,3,3	1,1,1	3,3,3	2,2,2	9	29	3,3,3	1,0,1	3,3,3	1,2,1	8
14	2,2,2	0,0,0	2,2,2	0.0,0	4	30	2,2,2	0,0,0	2,2,2	2,2,2	6
15	4,4,4	1,1,1	3,3,3	1,1,1	9	31	2,2,2	0,0,0	2,2,2	0,0,0	4
16	4,4,4	1,1,1	6,6,6	0,0,0	11	Tot	84	23	97	36	240

Table 3. Distribution of seats with RVA by three methods:
Double Hare-Niemeyer, Modified CEC-2009, Modified CEC-2013

Reg	BSP	Ataka	CEDB	MRF	Man	Reg	BSP	Ataka	CEDB	MRF	Man
1	3,3,3	1,1,1	4,4,4	2,2,2	10(–1)	17	4,4,4	1,1,1	4,4,4	1,1,1	10(–1)
2	4,4,4	1,1,1	6,6,6	2,2,2	13(–1)	18	1,1,1	0,0,0	1,1,1	2,2,2	4
3	4,4,4	2,2,2	7,7,7	1,1,1	14(–1)	19	2,2,2	1,1,1	3,3,3	1,1,1	7(–1)
4	3,3,3	1,1,1	3,3,3	1,1,1	8	20	1,1,1	0,0,0	1,1,1	2,2,2	4
5	2,2,2	0,0,0	1,1,1	0,0,0	3(–1)	21	2,2,2	1,1,1	2,2,2	1,1,1	6
6	2,3,2	1,0,1	2,2,2	1,1,1	6	22	2,2,2	0,0,0	1,1,1	1,1,1	4
7	1,1,1	1,1,1	2,2,2	0,0,0	4	23	6,6,6	1,2,1	7,7,8	1,0,0	15(–1)
8	2,2,2	1,1,1	2,2,2	1,1,1	6	24	4,4,5	1,2,1	6,6,6	1,0,0	12
9	0,0,1	0,0,0	1,1,1	4,4,3	5	25	5,5,5	1,1,1	7,7,6	0,0,1	13(–1)
10	2,2,2	1,1,1	2,2,2	0,0,0	5(+1)	26	3,3,3	1,2,1	4,3,4	0,0,0	8
11	2,2,2	1,0,1	2,2,2	0,1,0	5	27	5,5,4	1,1,1	4,4,4	1,1,2	11
12	2,2,2	1,1,1	2,2,2	0,0,0	5	28	1,1,1	0,0,0	1,1,1	2,2,2	4
13	3,3,3	1,1,1	3,3,3	1,1,1	8(–1)	29	3,3,3	1,0,1	3,3,3	1,2,1	8
14	2,2,2	0,0,0	2,2,2	0,0,0	4	30	2,2,2	0,0,0	2,2,2	2,2,2	6
15	4,4,4	1,1,1	3,3,3	1,1,1	9	31	2,2,2	0,0,0	2,2,2	0,0,0	4
16	4,4,4	1,1,1	5,6,6	1,0,0	11	RVA	1,0,0	0,0,0	2,2,2	5,6,6	8(+8)
						Tot	84	23	97	36	240

Table 4. Comparison of results

Met/RVA	Δ_R	Δ_P	Δ	$v_{\min}$	$v_{\max}$	C_R	C_P	C	Score
CEC-09/No	2	40	168	474	8902	26.02	31.40	34.02	18.5
CEC-09/Yes	1	31	91	4672	8902	3.11	3.69	3.69	38.5
CEC-13/No	2	45	126	474	7166	26.02	37.21	41.37	19.5
CEC-13/Yes	0	22	56	1553	7166	9.63	11.36	11.36	42.0
DHNM/No	2	52	166	474	8902	26.02	37.21	37.21	14.5
DHNM/Yes	0	20	60	1128	8902	11.28	15.64	15.64	35.0

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БЪЛГАРСКИТЕ ПАРЛАМЕНТАРНИ ИЗБОРИ 2013: СРАВНЕНИЕ НА ТРИ ДВОЙНО-ПРОПОРЦИОНАЛНИ МЕТОДА

Михаил М. Константинов, Галина Б. Пелова

Работата е посветена на резултатите от парламентарните избори в България на 12 май 2013. Направено е сравнение на действието на три двойно-пропорционални метода (методите на ЦИК от 2009 и 2013 и двойния метод на Хеър-Нимайер). Разгледани са парадокси на Алабама от по-висок ред, както и ефектите от въвеждане на нов изборен район за гласовете от чужбина.