МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2014
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2014
Proceedings of the Forty Third Spring Conference
of the Union of Bulgarian Mathematicians
Borovetz, April 2–6, 2014

CYLINDRICAL HELFRICH SURFACES*

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The equilibrium shapes of fluid membranes in the spontaneous curvature model (Helfrich's model) are described by the so called Helfrich equation. Surfaces obtained as solutions of the governing equation are called Helfrich surfaces, respectively. By making use of the conformal coordinates and applying Lie group reduction method we construct group-invariant solutions, expressed in terms of the Weierstrass elliptic $\wp$-function. The explicit analytic formulas for the position vector allow to find out the closed directrices generating the Helfrich cylindrical surfaces and to display some of their graphs.

1. Helfrich equation. Membranous biophysical systems, formed in aqueous solution, such as membranes of cells and lipid configurations in living organisms (e.g. lipid vesicles), are generally called *fluid membranes*. Any membrane itself is regarded as a smooth surface $\mathcal{S}$ in the Euclidean space $\mathbb{R}^3$ with the mean and the Gaussian curvatures H and K, respectively. In the spontaneous curvature model (Helfrich's model), widely accepted now, the equilibrium shapes of fluid membranes are determined by solving the *Helfrich equation* [1]

$$\Delta_{\mathcal{S}} H + 2(H^2 + \mathbb{h} H - K)(H - \mathbb{h}) - \frac{2\lambda H}{k} + \frac{p}{k} = 0 \qquad (1)$$

which is the Euler-Lagrange equation, obtained in a variational approach by minimizing the free elastic energy functional

$$\mathcal{F} = \frac{k}{2}\int\limits_{\mathcal{S}} (2H + \mathbb{h})^2 \mathrm{d}S + \bar{k}\int\limits_{\mathcal{S}} K \mathrm{d}S$$

under the constraints of fixed enclosed volume and surface area of the membrane.

In the aforementioned formulas the bending rigidity k, the Gaussian rigidity $\bar{k}$, the tensile stress λ, the pressure difference p between the outer and the inner media of the surface (osmotic pressure) and the spontaneous mean curvature $\mathbb{h}$ are the physical characteristics of the membrane, and $\Delta_{\mathcal{S}}$ denotes the Laplace-Beltrami operator on $\mathcal{S}$.

We will call the surfaces obtained as solutions of the Helfrich equation *Helfrich surfaces*.

* **2010 Mathematics Subject Classification:** 53A05, 74A50, 74K15.
Key words: symmetries, membranes, variational problems, surfaces.

2. Conformal metric on $\mathcal{S}$. Let on $\mathcal{S}$ be given the *conformal metric*

$$\mathrm{d}s^2 = 4q^2\varphi^2(\mathrm{d}x^2 + \mathrm{d}y^2) \tag{2}$$

and the matrix of the *second fundamental form*

$$b = \begin{pmatrix} \theta & \omega \\ \omega & 8q^2\varphi(1 + \mathbb{h}\varphi) - \theta \end{pmatrix} \tag{3}$$

where $q = q(x, y), \varphi = \varphi(x, y), \theta = \theta(x, y)$ and $\omega = \omega(x, y)$ are unknown functions of the conformal coordinates (x, y). In order to ensure the Helfrich surface to exist, these four functions must satisfy, along with the Helfrich equation (1), three additional differential equations, known as the Gauss-Codazzi-Mainardi compatibility conditions [2]. Consequently, the Helfrich's model is represented by a system of four second order partial differential equations [3, 4, 5]

$$\begin{aligned}
&q^2(\varphi_{xx} + \varphi_{yy}) + 2q\varphi(q_{xx} + q_{yy}) \\
&\qquad -2\varphi(q_x^2 + q_y^2) + q^4(8\varphi + \alpha_2\varphi^2 + \alpha_3\varphi^3 + \alpha_4\varphi^4) = 0 \\
&\theta_y - \omega_x - (8 + \frac{\alpha_2}{3}\varphi)q(\varphi q_y + q\varphi_y) = 0 \\
&\omega_y + \theta_x - \frac{\alpha_2}{3}q\varphi(\varphi q_x + q\varphi_x) - 8q\varphi q_x = 0 \\
&4q^2\varphi(\varphi_{xx} + \varphi_{yy}) + 4q\varphi^2(q_{xx} + q_{yy}) \\
&\qquad -4\varphi^2(q_x^2 + q_y^2) - 4q^2(\varphi_x^2 + \varphi_y^2) - \omega^2 - \theta^2 + (8 + \frac{\alpha_2}{3}\varphi)q^2\varphi\theta = 0
\end{aligned} \tag{4}$$

where $\alpha_2 = 24\mathbb{h}$, $\alpha_3 = 8(2\mathbb{h}^2 - \frac{\lambda}{k})$, $\alpha_4 = \frac{4p}{k} - \frac{8\lambda\mathbb{h}}{k}$ are phenomenological constants, and $\varphi_x = \partial\varphi/\partial x$, etc. Subsequently in this paper we will refer to (4) as the *Helfrich system.*

3. Symmetry Lie algebra. The *symmetry Lie algebra* of a given system of differential equations is found by solving the so called determining system of equations [6]. The solution of the determining system is the most general infinitesimal operator of the one-parameter symmetry Lie group of the system.

For the *symmetry Lie group* of the Helfrich system (4) the infinitesimal operator has the form

$$V = \sum_{i=1}^{2} \xi^i(\vec{x}, \vec{u})\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{4} \eta^\alpha(\vec{x}, \vec{u})\frac{\partial}{\partial u^\alpha}$$

where $\vec{x} = (x^1, x^2)$ and $\vec{u} = (u^1, u^2, u^3, u^4)$ are vectors in the spaces of the independent and dependent variables respectively: $x^1 = x$, $x^2 = y$, $u^1 = q$, $u^2 = \varphi$, $u^3 = \theta$, $u^4 = \omega$. The coefficient functions $\xi^i(\vec{x}, \vec{u})$, $\eta^\alpha(\vec{x}, \vec{u})$, $i = 1, 2$, $\alpha = 1, 2, 3, 4$ are found in [3, 5] by solving the corresponding determining system for two distinguished cases:

Case 1. $|\alpha_2| + |\alpha_3| + |\alpha_4| \neq 0$

$$\begin{aligned}
V^{\mathrm{I}}(\xi^1, \xi^2) \quad = \quad & \xi^1\partial_x + \xi^2\partial_y - q\xi_x^1\partial_q - 2(\theta\xi_x^1 + \omega\xi_x^2)\partial_\theta \\
& -2\left[\omega\xi_x^1 - \left(\theta - 4q^2\varphi - \frac{\alpha_2 q^2\varphi^2}{6}\right)\xi_x^2\right]\partial_\omega
\end{aligned}$$

Case 2. $\alpha_2 = \alpha_3 = \alpha_4 = 0$

$$V^{\mathrm{II}}(\xi^1, \xi^2) = V_1(\xi^1, \xi^2) + cV_2, \qquad c \in \mathbb{R}$$

$$V_1(\xi^1, \xi^2) = \xi^1 \partial_x + \xi^2 \partial_y - q\xi^1_x \partial_q - 2(\theta \xi^1_x + \omega \xi^2_x)\partial_\theta - 2\left[\omega \xi^1_x - \left(\theta - 4q^2\varphi\right)\xi^2_x\right]\partial_\omega$$

$$V_2 = \varphi \partial_\varphi + \theta \partial_\theta + \omega \partial_\omega$$

with $\xi^1 = \xi^1(x, y)$, $\xi^2 = \xi^2(x, y)$ – arbitrary real-valued harmonic functions satisfying, the Cauchy-Riemann conditions $\xi^1_y = -\xi^2_x$, $\xi^1_x = \xi^2_y$ and $\partial_x \equiv \partial/\partial x$, etc.

The full sets of group generators $V^{\mathrm{I}}(\xi^1, \xi^2)$ and $V^{\mathrm{II}}(\xi^1, \xi^2)$ constitute two symmetry Lie algebras L^{I} and L^{II} for each one of the considered cases. The Lie algebras L^{I} and L^{II} are infinite dimensional.

4. Group-invariant solution. A *group-invariant solution* of a given system of partial differential equations is a solution invariant under the group action of the symmetry group. It can be obtained as a solution of a new *reduced system* of differential equations. The main advantage is that the independent variables in the reduced system are less than in the original system [6]. We look for group invariant solutions of the Helfrich system (4), related to the subalgebra of the general symmetry Lie algebra spanned by the group generator $V^{\mathrm{I}}(1, 0) \equiv \partial_x$, i.e., solutions invariant under the translations of the variable x. The symmetry reduction procedure leads to the invariant solution of the form [3]

$$q = q(y), \qquad \varphi = \varphi(y), \qquad \theta = \theta(y), \qquad \omega(y) = \alpha_5 \equiv \text{const}$$

and the reduced system of ordinary differential equations

$$\begin{aligned} q^2\varphi_{yy} + 2q\varphi q_{yy} - 2\varphi q_y^2 + q^4(8\varphi + \alpha_2\varphi^2 + \alpha_3\varphi^3 + \alpha_4\varphi^4) &= 0 \\ \theta_y - q(8 + \frac{\alpha_2}{3}\varphi)(\varphi q_y + q\varphi_y) &= 0 \\ 4q\varphi^2 q_{yy} + 4\varphi q^2\varphi_{yy} - 4\varphi^2 q_y^2 - 4q^2\varphi_y^2 - \alpha_5^2 - \theta^2 + q^2\varphi\theta(8 + \frac{\alpha_2}{3}\varphi) &= 0. \end{aligned} \tag{5}$$

Now, we make the assumption

$$q(y) = \frac{1}{\varphi(y)}$$

and from (5) we readily obtain

$$\theta(y) = 0, \qquad \omega(y) = 0 \tag{6}$$

$$\int (C_1\varphi^4 - 2\alpha_4\varphi^3 - \alpha_3\varphi^2 - \frac{2}{3}\alpha_2\varphi - 4)^{-1/2}\mathrm{d}\varphi = y + C_2 \tag{7}$$

where C_1, C_2 are integration constants. The latter integral can be expressed in a standard way in terms of elliptic functions (or elementary functions) [7]. But first we will introduce new variables, which will return us to the original geometrical considerations.

Following classical differential geometry of surfaces [2], and based on (2), (3) and (6), we calculate the mean curvature

$$H(y) = \frac{1}{\varphi(y)} + \mathbb{h}$$

and in addition to this, the Gaussian curvature $K = 0$, which allows us, from now on, to study the cylindrical surfaces in $\mathbb{R}^3$, whose directrices are plane curves Γ of curvature

$\kappa(s) = 2H(s)$, parametrized by their arclength $s = 2y$. It means also that via (7) $\kappa(s)$ can be expressed by the Weierstrass elliptic function $\wp(s)$ [7]

$$\kappa(s) = a + \frac{f'(a)}{4}\left(\wp\left(\frac{s}{2} + C_2\right) - \frac{f''(a)}{24}\right)^{-1} \tag{8}$$

where a is an arbitrary root of the polynomial

$$f(t) = -t^4 + 2\mu t^2 + 4\sigma t + 8E$$

and the invariants of $\wp(s)$ are

$$g_2 = \frac{\mu^2}{3} - 8E, \qquad g_3 = \sigma^2 - \frac{\mu^3}{27} - \frac{8\mu E}{3}$$

with $\mu = 4(\mathbb{h}^2 + \frac{\lambda}{k})$, $\sigma = -\frac{4p}{k}$, $E = \frac{4\mathbb{h}}{k}(p - \lambda\mathbb{h}) - 2\mathbb{h}^4 + \frac{C_1}{2}$, and $f'(a) \equiv \mathrm{d}f(a)/\mathrm{d}t, \ldots$.

In this way a special type of cylindrical Helfrich surfaces with directrices defined by their intrinsic equation (8) in terms of the Weierstrass $\wp$-function are obtained.

5. Cylindrical Helfrich surfaces. Given the intrinsic equation $\kappa = \kappa(s)$ of a plane curve Γ, it is possible to recover (up to a rigid motion) the position vector of the curve $\mathbf{x}(s) = (\tilde{x}(s), \tilde{z}(s))$ in the plane $\mathbb{R}^2$ in a standard manner by calculating the quadratures [2]

$$\tilde{x}(s) = \int \cos\psi(s)\mathrm{d}s, \qquad \tilde{z}(s) = \int \sin\psi(s)\mathrm{d}s \tag{9}$$

where $\psi(s)$ is the slope angle of Γ, expressed by

$$\psi(s) = \int \kappa(s)\mathrm{d}s. \tag{10}$$

In the following we will confine to the case $\sigma \neq 0$. For the directrices of the Helfrich cylindrical surfaces with intrinsic equation (8) the integration in (9), for $\sigma \neq 0$, can be avoided (cf [8] and references cited therein), which is a specific consequence of the fact that the curvature function (8) satisfies the equation

$$2\frac{\mathrm{d}^2\kappa(s)}{\mathrm{d}s^2} + \kappa^3(s) - \mu\kappa(s) - \sigma = 0.$$

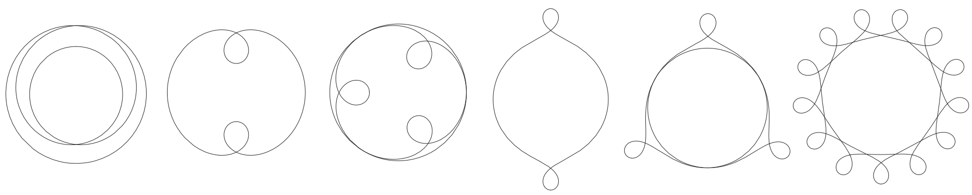

Fig. 1. Closed directrices with self-intersections

As a result the Cartesian coordinates of the position vector take the form [8]

$$\begin{aligned} \tilde{x}(s) &= \frac{2}{\sigma}\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\cos\psi(s) + \frac{\kappa^2(s) - \mu}{\sigma}\sin\psi(s) \\ \tilde{z}(s) &= \frac{2}{\sigma}\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\sin\psi(s) - \frac{\kappa^2(s) - \mu}{\sigma}\cos\psi(s) \end{aligned} \tag{11}$$

where the slope angle $\psi(s)$ is expressed explicitly via the quadrature (10) in terms of the Weierstrass elliptic functions $\wp(s)$, $\sigma(s)$ and $\zeta(s)$ [7]

$$\psi(s) = as + \frac{f'(a)}{2\wp'(\mathring{s})}\left[s\zeta(\mathring{s}) + \ln\frac{\sigma(\frac{s}{2} - \mathring{s})}{\sigma(\frac{s}{2} + \mathring{s})}\right] \tag{12}$$

with $\mathring{s}$ defined by $\wp(\mathring{s}) = \frac{f''(a)}{24}$ $(C_2 = 0)$.

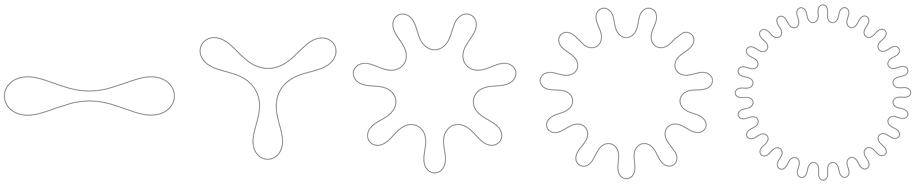

Fig. 2. Closed directrices without self-intersections

Several graphs of closed directrices of the so obtained cylindrical Helfrich surfaces, defined by (11) and (12), for different values of the parameters σ, μ, E and a, are presented in Fig. 1 and Fig. 2.

REFERENCES

[1] W. Helfrich. Elastic Properties of Lipid Bilayers: Theory and Possible Experiments, *Z. Naturforsch C*, **28** (1973), 693–703.

[2] J. Oprea. Differential Geometry and its Applications, Washington D. C., Mathematical Association of America, 2007.

[3] G. de Matteis, G. Manno. Lie Algebra Symmetry Analysis of the Helfrich and Willmore Surface Shape Equations, *Comm. Pure Appl. Anal.*, **13** (2014), 453–481.

[4] B. Konopelchenko. On Solutions of the Shape Equation for Membranes and Strings, *Phys. Lett. B*, **414** (1997), 58–64.

[5] V. Pulov, E. Chacarov, M. Hadzhilazova, I. Mladenov. Symmetry Properties of the Membrane Shape Equation, In: Geometry, Integrability and Quantization XIV, (Eds I. Mladenov, A. Ludu, A. Yoshioka), Avangard Prima, Sofia 2013, pp 152–159.

[6] P. Olver. Applications of Lie Groups to Differential Equations, Berlin, Springer, 1986.

[7] M. Abramowitz, I. Stegun (Eds). Handbook of Mathematical Functions, New York, Dover, 1972.

[8] V. Vassilev, P. Djondjorov, I. Mladenov. Cylindrical Equilibrium Shapes of Fluid Membranes, *J. Phys. A*: *Math. Theor.*, **41** (2008), 435201, 16 pp.

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ЦИЛИНДРИЧНИ ПОВЪРХНИНИ НА HELFRICH

Владимир Пулов, Мариана Хаджилазова, Ивайло М. Младенов

Намерени са транслационно-инвариантни решения на уравнението на Ou-Yang и Helfrich, което в рамките на техния модел, задава клас от цилиндрични повърхнини, представляващи равновесни форми на биологични мембрани. За целта са използвани конформни координати и групов анализ. Представена е явна аналитична параметризация на повърхнините чрез елиптични функции на Weierstrass. Представени са графики на директриси на повърхнини, които се затварят гладко.