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ON THE VOTING PARADOX OF LUXEMBOURG AND DECISION POWER INDICES*

Zdravko D. Slavov, Christina S. Evans

In this paper we consider the properties of weighted voting games having a dummy player and two basic decision power indices. In principle, a player can be assigned weight zero, but in practice this player would be a dummy. We will discuss the games in which a player has a positive weight and can also be a dummy. The most famous example of this somewhat paradoxical phenomenon is offered by Luxembourg in the Council of Ministers of the EU between 1958 and 1973. Luxembourg held weight one but was a dummy and we call this case the voting paradox of Luxembourg.

1. Introduction. The notion of a simple game was introduced by John von Neumann and Oscar Morgenstern in their monumental book *Theory of Games and Economic Behavior* in 1944 [5]. This game is a conflict in which the only objective is winning and the only rule is an algorithm to decide which coalitions are winning. When the number of players is small in a voting game it can occur that one of the players has no influence on the result of this game.

In order to study the voting paradox of Luxembourg mathematically we must first define a few terms.

Definition 1. *A simple game is a pair (N, v) where $N = \{1, 2, ..., n\}$ is a set of players, $n = |N| \geq 2$, and $v : 2^N \rightarrow \{0, 1\}$ is the characteristic function which satisfies the following three conditions:*

(1) $v(\emptyset) = 0$.

(2) $v(N) = 1$.

(3) *if S and T are coalitions of players in N such that $S \subset T$, then $v(S) \leq v(T)$.*

The characteristic function v for a coalition S indicates the value of S. For each coalition $S \subset N$ there is either $v(S) = 0$ or $v(S) = 1$.

Definition 2. *For a coalition $S \subset N$, we define that coalition S is winning if and only if $v(S) = 1$, coalition S is losing if and only if $v(S) = 0$, and the losing coalition S is blocking if and only if $N \backslash S$ is a losing coalition too. The collection of all winning coalitions is denoted by W.*

By definition $N \in W$; therefore, W is nonempty.

Definition 3. *Let us label the players by $1, 2, \ldots, n$ and let $S \in W$, S is called a minimal winning coalition if and only if $S \backslash \{i\}$ is a losing coalition for all $i \in S$. The collection of all minimal winning coalitions is denoted by MW.*

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It is easy to prove that $MW \subset W$ is nonempty.

Definition 4. *A player who does not belong to any minimal winning coalition is called a dummy, i.e., player $i \in N$ is a dummy if and only if $i \notin S$ for all $S \in MW$. A player who belongs to all minimal coalitions is called a veto player, i.e., a player $i \in N$ has capacity to veto if and only if $i \in S$ for all $S \in MW$. A player $i \in N$ is a dictator if and only if $\{i\}$ is a winning coalition, i.e. $\{i\} \in MW$.*

Of course, if $i \in N$ and $v(S \cup \{i\}) - v(S) = 0$ for all $S \subset N$, then player i is a dummy.

2. Weighted voting games. We will consider a special class of simple games called weighted voting games with dichotomous voting rule − acceptance or rejection. The basic formal framework of this study is as follows. The symbol $[q; w_1, w_2, \ldots, w_n]$ will be used, where q and $w_1, w_2, \ldots, w_n$ are nonnegative integer numbers such that $w_i \leq q \leq \sum_{k=1}^{n} w_k = \tau$ for all $i \in N$. By convention, we take $w_i \geq w_j$ if $i < j$. Here we have the following properties:

(1) $1 \leq q \leq \tau$.

(2) $n = |N| \geq 2$ is the number of players.

(3) $w_i \geq 0$ is the number of votes of player $i \in N$ and $w_1 \geq 1$.

(4) q is the needed quota so that a coalition can win.

The symbol $[q; w_1, w_2, \ldots, w_n]$ represents the simple game (N, v) defined by

$$v(S) = \begin{cases} 1, & \sum\limits_{k \in S} w_k \geq q \\ 0, & \sum\limits_{k \in S} w_k < q \end{cases},$$

where $S \subset N$.

Of course, if $i \in N$ and $w_i = 0$, then player i is a dummy.

Remark 1. A simple game (N, v) is called proper if and only if $v(S) + v(N \backslash S) \leq 1$ for all $S \subset N$. In a proper game S and $N \backslash S$ cannot both be winning coalition. It is easy to show that if $2q > \tau$, then the weighted voting game (N, v) is proper. In an improper game there will be at least one pair of non-intersection winning coalitions. If $v(S) + v(N \backslash S) = 1$ for all $S \subset N$, then the game is said to be decisive.

Example 1. The voting method of the Security Council of the United Nations, formed by 5 permanent and 10 temporary members, is the game in which each one of the permanent members has 7 votes and each one of the temporary members has only one vote, the established quota being 39 votes, all votes are 45. We observe that any coalition which does not include all of the 5 permanent members has at most $4 \times 7 + 10 = 38$ votes, which is a number inferior to the fixed quota. As a result this coalition will not be winning. Hence, each one of the permanent members has capacity to veto any proposal. For more information see [2] and [4].

Example 2. The Bulgarian Parliament with 240 seats uses two rules: a simple majority by quota 121 (more than 50%) and a qualified majority by quota 161 (more than $\frac{2}{3}$). The Finish Parliament with 200 seats uses three rules: a simple majority by quota 101 (more than 50%), a qualified majority by quota 134 (more than $\frac{2}{3}$), and in some special case by quota 167 (more than $\frac{5}{6}$) [2].

Example 3. The U. S. Congress has a nonvoting delegate who represents the District of Columbia; therefore, this delegate is a dummy.

Example 4. Note that in principle a player can be assigned weight zero, but in practice this player would be silly, because it would be a dummy. However, a player having positive weight can also be a dummy. It is known that Luxembourg as a member of the European Union Council of Ministers in the period 1958–1973 had weight one but was a dummy. Now, we call this situation the voting paradox of Luxembourg.

3. Decision power indices of the players. The weighted voting games employ mathematical models which are used to analyze the distribution of decision power of the players. This analysis of power is central in political science. In general, it is difficult to define the idea of power, but for the special case of voting power mathematical power indices have been used. There are two most widely used measures of voting power in the weighted voting games – the Shapley–Shubik power index and the Banzhaf power index. The Shapley–Shubik power index in a voting situation depends on the number of orderings in which each player is pivotal. The Banzhaf power index depends on the number of ways in which each player can affect a negative swing.

3.1. The Shapley–Shubik power index. The Shapley–Shubik power index was introduced by the mathematician Lloyd Shapley and the economist Martin Shubik in 1954 [3]. For player $i \in N$ this index is defined by

$$\phi_i = \sum_{S \notin W, S \cup \{i\} \in W} \frac{s!(n-s-1)!}{n!},$$

where $s = |S|$. If we assume that all $n!$ orderings are equiprobable, then ϕ_i is the probability of player i being pivotal in a winning coalition, that is, $S \cup (\{i\}$ is a winning and S is a losing coalition.

Remark 2. In the classical theory a negative swing for player $i \in N$ is a pair of coalitions $(S \cup \{i\}, S)$ such that $S \cup \{i\}$ is winning and S is losing, i.e., $i \notin S$, $v(S \cup \{i\}) = 1$ and $v(S) = 0$. It is easy to show that $v(S) - v(S \backslash \{i\})$ is always either zero or one for all $S \subset N$ and all $i \in N$. If $S \subset N$ and $i \notin S$, then $v(S) - v(S \backslash \{i\}) = 0$. If $S \subset N$ and $i \in S$, then $v(S) - v(S \backslash \{i\}) = 1$ (when S is winning and $S \backslash \{i\}$ is losing) or $v(S) - v(S \backslash \{i\}) = 0$ (when S and $S \backslash \{i\}$ are winning or S and $S \backslash \{i\}$ are losing).

From Remark 2 it follows that

$$\phi_i = \sum_{S \subset N} \frac{(s-1)!(n-s)!}{n!} \left(v(S) - v(S \backslash \{i\})\right)$$
$$= \sum_{S \subset N : i \in S} \frac{(s-1)!(n-s)!}{n!} \left(v(S) - v(S \backslash \{i\})\right).$$

For each player $i \in N$ we also obtain

$$\phi_i = \sum_{j=0}^{n-1} \frac{j!(n-j-1)!}{n!} d_j^i,$$

where each d_j^i is the number of negative swings of player i in a coalition of size j.

The Shapley-Shubik index is the vector $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ and it has the normalization property, i.e., $\sum_{i=1}^{n} \phi_i = 1$.

3.2. The Banzhaf power index. The Banzhaf power index was introduced by the American jurist and law professor John Banzhaf III in 1965 [1]. The absolute Banzhaf index concerns the number of times each player $i \in N$ could change a coalition from losing to winning and it requires that we know the number of negative swings for each player i. For each player $i \in N$, the absolute Banzhaf index is denoted by η_i and it equals the number of negative swings for this player.

The normalized Banzhaf power index is the vector $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, given by

$$\beta_i = \frac{\eta_i}{\sum_{k=1}^{n} \eta_k} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Remark 3. It is easy to show that

$$\eta_i = \sum_{S \subset N} (v(S) - v(S \backslash \{i\})) = \sum_{S \subset N: i \in S} (v(S) - v(S \backslash \{i\}))$$

for all $i = 1, 2, \ldots, n$. See also Remark 2. The Banzhaf index is similar of the Penrose-Banzhaf (or Banzhaf-Coleman) index which is defined by

$$b_i = \sum_{S \subset N} \frac{v(S) - v(S \backslash \{i\})}{2^{n-1}} = \frac{\eta_i}{2^{n-1}} \text{ for } i = 1, 2, \ldots, n.$$

Remark 4. Let $S \subset N$ be a minimal winning coalition and $i \in N$. It is easy to show that $v(S) - v(S \backslash \{i\}) = 1$ for all $i \in S$ and $v(S) - v(S \backslash \{i\}) = 0$ for all $i \notin S$.

4. Analysis of the voting paradox of Luxembourg. Consider the Council of Ministers in the period 1958–1973. The decision rule is a weighted voting game $[12; 4, 4, 4, 2, 2, 1]$. In this game the players are: Germany, France, Italy, Belgium, Netherlands and Luxembourg. Note that the total sum of the weights is 17 and the quota is 12, i.e., $\tau = 17$ and $q = 12$. The number of winning coalitions is 14 and they are: {1, 2, 3, 4, 5, 6}, {1, 2, 3, 4, 5}, {1, 2, 3, 4, 6}, {1, 2, 3, 5, 6}, {1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 3, 6}, {1, 2, 4, 5, 6}, {1, 3, 4, 5, 6}, {2, 3, 4, 5, 6}, {1, 2, 3}, {1, 2, 4, 5}, {1, 3, 4, 5} and {2, 3, 4, 5}. The number of minimal winning coalitions is 4 and they are: {1,2,3}, {1, 2, 4, 5}, {1, 3, 4, 5} and {2, 3, 4, 5}. As a result we see that Luxembourg (player 6) is a dummy player. This game has no veto player and no dictator. We also get that this game has at least one blocking coalition, for example $\{1, 4\}$. Note that coalition $\{4, 5, 6\}$ is losing but not blocking.

These notes allow us to discuss the case when a player has positive weight and is a dummy.

Let us consider a proper game $[q; w_1, w_2, \ldots, w_{n-1}, w_n]$ when $n \geq 3$ and $w_n > 0$. Then, $w_i > 0$ for all $i \in N$.

Theorem 1. *Let $i, j \in N$ and $i \neq j$. The following statements are true.*

(a) *If player i is not a dummy and player j is a dummy, then $w_i > w_j$.*

(b) *If player i is a dummy and $w_i \geq w_j$, then player j is also a dummy.*

Proof. (a) Let us denote $MW_k = \{S \in MW : k \in S\}$ for $k \in N$. It is easy to show that MW_i is not empty and MW_j is empty.

If $S \in MW_i \subset MW$, then $\sum_{k \in S} w_k \geq q$. For $T = S \backslash \{i\}$ it follows that $\sum_{k \in T} w_k + w_i \geq q$ and $\sum_{k \in T} w_k < q$, i.e., T is a losing coalition. It is known that player j is a dummy; therefore, $j \notin S$ and $j \notin T$. Let us now consider the coalition $P = T \cup \{j\}$. There are two cases:

Case 1. Let P be a losing coalition. In this case we have $\sum_{k\in T} w_k + w_j < q$.

Case 2. Let P be a winning coalition. From the condition that j is a dummy it follows that $T = P\backslash\{j\}$ is a winning coalition too. This leads to a contradiction.

Finally, we obtain $\sum_{k\in T} w_k + w_j < q$.

From the inequalities $\sum_{k\in T} w_k + w_i \geq q$ and $\sum_{k\in T} w_k + w_j < q$ we get $w_i > w_j$.

(b) Let us assume that player j is not a dummy. From player i being a dummy and part (a) it follows that $w_i < w_j$. This leads to a contradiction with the condition $w_i \geq w_j$; therefore, player j is a dummy.

The theorem is proven. □

Theorem 2. *Let $i \in N$. The following statements are equivalent:*

(a) *Player i is a dummy.*

(b) $\phi_i = 0$.

(c) $\eta_i = 0$.

Proof. First step: $(a) \Rightarrow (b)$. Let player i be a dummy.

For $S \subset N$ and $i \in S$ there are two cases:

Case 1. If $S \notin W$, then $v(S) = 0$. It is clear that also $v(S\backslash\{i\}) = 0$. As a result we get $v(S) - v(S\backslash\{i\}) = 0$.

Case 2. If $S \in W$, then $v(S) = 1$. It is easy to show that $S \notin MW$; therefore, $v(S\backslash\{i\}) = 1$. As a result we also get $v(S) - v(S\backslash\{i\}) = 0$.

Finally, we obtain $v(S) - v(S\backslash\{i\}) = 0$ for all $S \subset N$. Thus, for the Shapley-Shubik power index we have

$$\phi_i = \sum_{S\subset N: i\in S} \frac{(s-1)!(n-s)!}{n!}\,(v(S) - v(S\backslash\{i\})) = \sum_{S\subset N: i\in S} \frac{(s-1)!(n-s)!}{n!} 0 = 0.$$

Second step: $(b) \Rightarrow (c)$. In this case, let $\phi_i = 0$.

From $\phi_i = \sum_{S\subset N: i\in S} \frac{(s-1)!(n-s)!}{n!}\,(v(S) - v(S\backslash\{i\})) = 0$ it follows that $v(S) - v(S\backslash\{i\}) = 0$ for all $S \subset N$ when $i \in S$. Hence, for the Banzhaf power index we have $\eta_i = \sum_{S\subset N: i\in S} (v(S) - v(S\backslash\{i\})) = 0$. As a result we obtain $\beta_i = \eta_i = 0$.

Third step: $(c) \Rightarrow (a)$. Finally, let $\beta_i = 0$.

From $\eta_i = \sum_{S\subset N: i\in S} (v(S) - v(S\backslash\{i\})) = 0$ it follows that $v(S) - v(S\backslash\{i\}) = 0$ for all $S \subset N$ when $i \in S$. There are two cases:

Case 1. If $v(S) = 1$ and $v(S\backslash\{i\}) = 1$, then $S \in W$ and $S \notin MW$.

Case 2. If $v(S) = 0$ and $v(S\backslash\{i\}) = 0$, then $S \notin W$.

Finally, we obtain $S \notin MW$ for all $S \subset N$ when $i \in S$, i.e., player i is a dummy.

The theorem is proven. □

Remark 5. Note that the collection of all winning coalitions, the collection of all minimal winning coalitions and the power indices of players in the weighted voting game $[q; w_1, w_2, \ldots, w_n]$ are the same as the collection of all winning coalitions, the collection of all minimal winning coalitions and the power indices in the weighted voting game $[\lambda q; \lambda w_1, \lambda w_2, \ldots, \lambda w_n]$ for every positive integer number λ, respectively. As a result we obtain that the game $[12; 4, 4, 4, 2, 2, 1]$ can be equivalently represented as $[12; 4, 4, 4, 2, 2, 0]$ and $[6; 2, 2, 2, 1, 1, 0]$.

Remark 6. From Theorem 2 it follows that $\phi_6 = 0$ and $\eta_6 = 0$ in the weighted voting game $[12; 4, 4, 4, 2, 2, 1]$, i.e., the Shapley-Shubik and the normalized Banzhaf power indices for Luxembourg are zero. It is easy to calculate that the Shapley-Shubik power index equals to $(0.233, 0.233, 0.233, 0.150, 0.150, 0)$ and the normalized Banzhaf power index equals to $(0.238, 0.238, 0.238, 0.143, 0.143, 0)$ [2].

Theorem 3. *If $w_{n-1} \geq 2$, $0 < w_n < w_{n-1}$, $q \equiv 0$ modulo w_{n-1} and $w_i \equiv 0$ modulo w_{n-1} for all $i = 1, 2, \ldots n-2$, then player n is a dummy.*

Proof. Let p be a positive integer number between q and τ, and denote $W(p) = \{S \in W : \sum_{i \in S} w_i = p\}$ and $MW(p) = MW \cap W(p)$. It is easy to show that $W = \bigcup_{p=q}^{\tau} W(p)$ and $MW = \bigcup_{p=q}^{\tau} MW(p)$. Since W is nonempty it follows that there exists $p \in [q, \tau]$ such that $W(p)$ is nonempty, too.

Of course, if $W(p)$ is nonempty, then $p \equiv 0$ modulo w_{n-1} or $p \equiv w_n$ modulo w_{n-1}.

Let $W(p)$ be nonempty. Here there are two cases.

Case 1. Let $p \equiv 0$ modulo w_{n-1}.

In this case there exists $S \in W(p)$ such that $\sum_{i \in S} w_i = p$. From $\sum_{i \in S} w_i \equiv 0$ modulo w_{n-1} it follows that $n \notin S$. As a result we obtain that $S \in W(p)$ implies $n \notin S$. Let us denote $A = \{p \in [q, \tau] : n \notin S \in W(p)\}$. Obviously, $n \notin S$ for all $S \in \bigcup_{p \in A} W(p)$.

Case 2. Let $p \equiv w_n$ modulo w_{n-1}.

In this case we have $p > q$. Let us assume that there exists $S \in W(p)$, i.e. $\sum_{i \in S} w_i = p$ and $p \geq q$. From $\sum_{i \in S} w_i \equiv w_n$ modulo w_{n-1} it follows that $n \in S$ and $p \geq q + w_n$. Consider a coalition $T = S \backslash \{n\}$. It is easy to see that $T \in W(p - w_n)$; therefore, $S \notin MW$. As a result we get $S \notin MW(p)$, i.e. $MW(p)$ is empty.

Finally, we have that $n \notin S$ for all $S \in \bigcup_{p \in A} W(p)$ and $MW = \bigcup_{p=q}^{\tau} MW(p) = \bigcup_{p \in A} MW(p) \subset \bigcup_{p \in A} W(p)$. As a result we obtain $n \notin S$ for all $S \in \bigcup_{p \in A} MW(p)$ and $MW \subset \bigcup_{p \in A} W(p)$. It is easy to show that $n \notin S$ for all $S \in MW$, i.e., player n is a dummy.

The theorem is proved. □

Remark 7. From Theorem 3 it follows that player 6 is a dummy in the weighted voting game $[12; 4, 4, 4, 2, 2, 1]$, i.e., Luxembourg is a dummy. Note that player 6 also is a dummy in the weighted voting games $[15; 9, 9, 6, 3, 3, 1]$ and $[15; 9, 9, 6, 3, 3, 2]$.

Remark 8. We showed that player 6 is a dummy in the weighted voting game $[12; 4, 4, 4, 2, 2, 1]$. Now consider an extension by a small new player, who receives the same weight as player 6. We have two cases on the quota in the new game.

Case 1. Let $q = 12$, i.e., consider the weighted voting game $[12; 4, 4, 4, 2, 2, 1, 1]$. It is easy to show that $\{1, 2, 4, 6, 7\}$ is a minimal winning coalition. As a result in this game players 6 and 7 are not dummies.

Case 2. Let $q = 13$, i.e., consider the weighted voting game $[13; 4, 4, 4, 2, 2, 1, 1]$. It is easy to show that $\{1, 2, 3, 6\}$ and $\{1, 2, 3, 7\}$ are two minimal winning coalitions. As a result in this game players 6 and 7 are not dummies.

Finally, the two new games have no dummy player.

Remark 9. Consider Case 1 of Remark 8, i.e., the weighted voting game [12; 4, 4, 4, 2, 2, 1, 1]. Let us assume that players 4 and 5 be hard partners, i.e., for each $S \in W$, $4 \in S$ if and only if $5 \in S$. In this case the minimal winning coalitions are {1,2,3}, {1,2,4,5}, {1,3,4,5} and {2,3,4,5}. As a result we find that players 6 and 7 are dummies. See also Theorem 3.

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Zdravko D. Slavov
Varna Free University
Chaika Resort
9007 Varna, Bulgaria
e-mail: slavovibz@yahoo.com

Christina S. Evans
The George Washington University
Washington DC, USA
e-mail: evans.christina.s@gmail.com

ВЪРХУ ПАРАДОКСА С ГЛАСУВАНЕТО НА ЛЮКСЕМБУРГ И ИНДЕКСИ НА ВЛИЯНИЕ ВЪРХУ РЕШЕНИЕТО

Здравко Д. Славов, Христина С. Еванс

В тази статия ние разглеждаме свойствата на тегловни игри с гласуване, имащи фиктивен играч и два основни индекса на влияние върху решението. По принцип на един играч може да има присъдено тегло нула, но на практика този играч ще бъде фиктивен. Ние разглеждаме игри, в които един играч има положително тегло и също може да бъде фиктивен. Най-известният пример за това донякъде парадоксално явление се вижда при Люксембург в Съвета на министрите на ЕС между 1958 и 1973. Люксембург притежава тегло едно, но беше фиктивен и ние наричаме този случай парадокса с гласуването на Люксембург.

Ключови думи: тегловни игри с гласуване, фиктивен играч, парадокс, индекс на влияние, печеливша коалиция.