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ONE SUFFICIENT CONDITION FOR THINNESS OF SEQUENCES*

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It is known that every interpolating sequence of type 1 for H^∞ is a thin sequence, which satisfies an additional topological condition by R. Mortini. In this paper, we present another proof of this fact. More precisely, we prove that if there exists a thin radial sequence in the open unit disk such that the interpolation problem admits a solution with norm 1, then this condition is fulfilled.

1. Introduction. Let H^∞ be the Banach algebra of all bounded analytic functions in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ with the supremum norm. Its spectrum, or maximal ideal space, is the space $M(H^\infty)$ of all nonzero multiplicative linear functionals on H^∞ endowed with the weak*-topology. Then $M(H^\infty)$ is a compact Hausdorff space and Carleson's corona theorem says that D is dense in $M(H^\infty)$. As usual we identify a function $f \in H^\infty$ with its Gelfand transform $\hat{f}$, defined by $\hat{f}(x) = x(f)$ for $x \in M(H^\infty)$. Taking the boundary values of the functions on $\partial D = \{z \in \mathbb{C} : |z| = 1\}$, we can consider $H^\infty = H^\infty(\partial D)$ as an essentially supremum-norm closed subalgebra of $L^\infty = L^\infty(\partial D)$. The maximal ideal space $M(L^\infty)$ of L^∞ can be identified with the Shilov boundary of H^∞.

To proceed, we need to present a few definitions. For points x and y in $M(H^\infty)$ the pseudohyperbolic distance is defined by

$$\rho(x,y) = \sup\{|h(x)| : h \in ball(H^\infty), h(y) = 0\}.$$

where $ball(H^\infty)$ stands for the closed unit ball of H^∞. By Schwarz-Pick's lemma $\rho(z,w) = |z-w|/|1-\overline{z}w|$ for z and w in D. It is well known that the relation defined on $M(H^\infty)$ by $x \sim y \Longleftrightarrow \rho(x,y) < 1$ defines an equivalence relation on $M(H^\infty)$. The equivalence class containing a point $x \in M(H^\infty)$ is called the Gleason part of x and is denoted by $P(x)$. If $P(x)$ consists of a single point, we call the part (or point) trivial. If the part consists of more than one point, the part (or point) is called nontrivial ([1]) .

A sequence $\{x_n\}_n$ in $M(H^\infty)$ is called interpolating if for every bounded sequence $\{a_n\}_n$ of complex numbers there is a function $f \in H^\infty$ such that $f(x_n) = a_n$ for all n. A sequence $\{x_n\}_n$ in $M(H^\infty)$ is said to be discrete if there exists a sequence of open sets $\{U_n\}_n$ with $x_n \in U_n$ for every n, whose closures are pairwise disjoint. Every interpolating sequence is discrete.

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An interpolating sequence $\{z_n\}_n$ in D is characterized by Carleson ([2]) as follows:

$$\inf_j \prod_{n:n\neq j}^{\infty} \left| \frac{z_j - z_n}{1 - \overline{z}_n z_j} \right| > 0,$$

For a sequence $\{z_n\}_n$ in D with $\sum\limits_{n=1}^{\infty} (1 - |z_n|) < \infty$, the function:

$$B(z) = \prod_{n=1}^{\infty} \frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z}, z \in D,$$

is called the Blaschke product with zeroes $\{z_n\}_n$. If $\{z_n\}_n$ is an interpolating sequence, then $B(z)$ is also called interpolating.

The study of interpolating sequences is useful in many areas of function theory and operator theory. In [1] K. Hoffman proved that $P(x) \neq \{x\}$ if and only if x belongs to the closure of some interpolating sequence. Most of the results for interpolating sequences in $M(H^\infty)$ have been obtained by P. Gorkin, H.-M. Lingenberg, R. Mortini [3] , S. Axler and P. Gorkin [4] and K. Izuchi [5].

An interpolating sequence $\{x_n\}_n \subset M(H^\infty)$ is said to be thin if it satisfies the following condition:

(T) $$\lim_{j\to\infty} \prod_{n:n\neq j}^{\infty} \rho(x_j, x_n) = 1.$$

A Blaschke product is called thin Blaschke product if its zero set $\{z_n\}_n \subset D$ is a thin sequence. Some results for thin sequences in different subsets of $M(H^\infty)$ can be found in [6],[7],[8].

In [9] R. Mortini introduces another, more topological sufficient condition for a sequence of nontrivial points to be interpolating.

Theorem 1.1 ([9])**.** *Let $\{x_n\}_n$ be a discrete sequence of nontrivial points in $M(H^\infty)$. Suppose that there exist pairwise disjoint open sets $\{U_n\}_n$ in $M(H^\infty)$, $x_n \in U_n$ such that*

$$\delta = \inf_j \prod_{n:n\neq j}^{\infty} \rho(U_j, U_n) > 0,$$

Then $\{x_n\}_n$ is an interpolating sequence.

Here we shall consider the following condition:

(M) There exist pairwise disjoint open sets $\{U_n\}_n$ in $M(H^\infty)$, $x_n \in U_n$ such that $\lim\limits_{j\to\infty} \prod\limits_{n:n\neq j}^{\infty} \rho(U_j, U_n) = 1.$

From the inequality $\prod\limits_{n:n\neq j}^{\infty} \rho(U_j, U_n) \leqslant \prod\limits_{n:n\neq j}^{\infty} \rho(x_j, x_n)$ we obtain that condition (M) implies condition (T) for a sequence $\{x_n\}_n \subset M(H^\infty)$.

A sequence $\{x_n\}_n$ in $M(H^\infty)$ is called an interpolating sequence of type 1 (or an isometric interpolating sequence) if for every sequence $\{a_n\}_n \subset l^\infty$, $\sup\limits_n |a_n| \leq 1$, there exists a function $f \in H^\infty$, $\|f\| = 1$, such that $f(x_n) = a_n$ for all n. The maximum principle for holomorphic functions shows that any interpolating sequence of type 1 is necessarily contained in the corona $M(H^\infty) \backslash D$ of H^∞. In [8] it is shown that every interpolating sequence of type 1 for H^∞ satisfies condition (M).

In this paper, we present another proof of this fact. More precisely, we prove that if there exist a thin radial sequence $\{a_n\}_n \subset D$ and a function $f \in H^\infty$, $\|f\| \le 1$, such that $f(x_n) = a_n$ for all n, then $\{x_n\}_n \subset M(H^\infty)$ satisfies condition (M).

2. Sequences satisfying the condition (M). First, let us show that there are sequences $\{x_n\}_n \subset M(H^\infty)$ with the following property: there exist a thin radial sequence $\{a_n\}_n \subset D$ and a function $f \in H^\infty$, $\|f\| \le 1$, such that $f(x_n) = a_n$ for all n. This follows from the following well-known facts:

1. In [6] P. Gorkin and R. Mortini proved that every discrete sequence in the Shilov boundary of H^∞ is an interpolating sequence of type 1.

2. In [7] R. Mortini proved the following theorem:

Theorem 2.1 ([7]). *Let $E = \{x_n : n \in \mathbb{N}\}$ be a discrete sequence of points in $M(H^\infty)$. Then, for every sequence $\{a_n\}_n \subset l^\infty$ with $\sup\limits_n |a_n| \le 1$ the interpolation problem $f(x_n) = a_n$, $\|f\| \le 1$, admits a thin Blaschke product as a solution if and only if E is contained in the zero set of a thin Blaschke product b on the corona of H^∞.*

Lemma 2.2 ([10]). *Let $\{z_n\}_n$ be a thin sequence in D. Then there exist sequences $\{\tau_n\}_n \subset (0,1)$ and $\{\gamma_n\}_n \subset (0,1)$ with $\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \gamma_n = 1$ such that whenever $\{\xi_n\}_n$ is a sequence in D satisfying $\rho(z_n, \xi_n) \le \tau_n$, $n \in \mathbb{N}$, it follows that $\prod\limits_{k:k\neq j}^{\infty} \rho(\xi_j, \xi_k) \ge \gamma_j$, $j \in \mathbb{N}$. In particular, $\{\xi_n\}_n$ is a thin sequence again.*

Theorem 2.3. *Let $\{x_n\}_n$ be a sequence in $M(H^\infty)$. Let there exist a thin radial sequence $\{a_n\}_n \subset D$, $|a_n| < |a_k|$ for $n < k$, and a function $f \in H^\infty$, $\|f\| \le 1$, such that $f(x_n) = a_n$ for all n. Then $\{x_n\}_n$ satisfies condition (M).*

Proof. We need some properties of the pseudohyperbolic distance in the open unit disk D. It is well known ([2]) that $\rho(z,w) = \dfrac{1}{2}\tanh \psi(z,w)$, whenever $\psi(z,w)$ is the hyperbolic distance in D from z to w, i.e., the length of the arc on the circle via the points z and w, which is orthogonal to the unit circle $\partial D = \{z \in \mathbb{C} : |z| = 1\}$. If z and w are on a diameter of D, then the hyperbolic distance $\psi(z,w)$ coincides with the Euclidian distance between z and w. As is well known the pseudohyperbolic disk $K(z_0, r)$ is an Euclidean disk with center $c = \dfrac{z_0(1-r^2)}{(1-r^2|z_0|^2)}$ and radius $R = \dfrac{r(1-|z_0|^2)}{(1-r^2|z_0|^2)}$. We note that if z_0 is a real number, then c is a real number and $|c| < 1$.

Since $\{a_n\}_n \subset D$ is a radial sequence, i.e., we have $\arg a_n = \arg a_k$ for $n \neq k$, and $|a_n| < |a_k|$ for $n < k$, there exists $\theta \in [0, 2\pi)$ such that if $g(z) = ze^{-i\theta}$, then:

$$\upsilon_n = g(a_n) \subset [0,1) \text{ for every } n \text{ and } \upsilon_n < \upsilon_k \text{ for } n < k.$$

But the distance $\rho(z,w)$ for z and w in D is invariant under Möbius transformations ([2]). Therefore,

$$\rho(a_n, a_k) = \rho(g(a_n), g(a_k)) = \rho(\upsilon_n, \upsilon_k)$$

for all $n \in \mathbb{N}$, $k \in \mathbb{N}$, and we obtain that the sequence $\{\upsilon_n\}_n$ is interpolating and thin. Moreover, the function $F = g \circ f$ belongs to H^∞, $\|F\| \le 1$ and $f(x_n) = \upsilon_n$ for all n.

Since $\{\upsilon_n\}_n$ is an interpolating sequence, then $\{\upsilon_n\}_n$ is discrete, i.e., there exists is a sequence $\{V_n\}_n$ of open sets in the open unit disk D, such that $\upsilon_n \in V_n$ and $\overline{V}_n \cap \overline{V}_k = \varnothing$ for $n \neq k$. By Lemma 2.2 there exist two sequences $\{\tau_n\}_n \subset (0,1)$ and $\{\gamma_n\}_n \subset (0,1)$

with $\lim_{n\to\infty}\tau_n = \lim_{n\to\infty}\gamma_n = 1$ such that whenever $\{\xi_n\}_n$ is a sequence in the open unit disk D satisfying $\rho(\upsilon_n,\xi_n) \le \tau_n$ for every n, it follows that $\prod\limits_{k:k\neq j}^{\infty} \rho(\xi_j,\xi_k) \ge \gamma_j$, for every j. Let W_n, $n\in\mathbb{N}$, be the open pseudohyperbolic disk with centre υ_n and radius τ_n. Then W_n is an open Euclidean disk with centre $[0,1)$. We obtain that $U_n = V_n\cap W_n$ is an open set (intersection of two open Euclidean disks), $\upsilon_n\in U_n$ for every n and $\overline{U}_n\cap\overline{U}_k = \varnothing$ for $n\neq k$. If $[\eta_n,t_n] = \overline{U}_n\cap \operatorname{Re} z$ then $\eta_n < t_n$ for every n, and the increasing sequences $\{\eta_n\}_n$ and $\{t_n\}_n$ tend to 1, because the sequence $\{\upsilon_n\}_n$ is an increasing and $\lim_{n\to\infty}\upsilon_n = 1$.

Now fix $j\in N$, $j>1$.

Let $A_n\in\overline{U}_n$, $A_j\in\overline{U}_j$ for $n<j$ are arbitrary points. Write $\alpha_{n,j} = \psi\left(A_n,A_j\right)$ and $p_{n,j} = |A_n - A_j|$. Since A_n and A_j are points of Euclidean disks, then $\alpha_{n,j}\ge p_{n,j}\ge \eta_j - t_n$. For the pseudohyperbolic distance between $\overline{U}_n$ and $\overline{U}_j$ we obtain:

$$\begin{aligned}\rho(\overline{U}_n,\overline{U}_j) &= \frac{1}{2}\tanh\psi(\overline{U}_n,\overline{U}_j) = \frac{1}{2}\tanh(\inf\alpha_{n,j})\\ &= \frac{1}{2}\tanh(\eta_j - t_n) = \frac{1}{2}\tanh\psi(t_n,\eta_j) = \rho(t_n,\eta_j).\end{aligned}$$

because t_n and η_j lie on a diameter of D.

Let $A_n\in\overline{U}_n$, $A_j\in\overline{U}_j$ for $n>j$ are an arbitrary points. Write $\alpha_{n,j} = \psi\left(A_n,A_j\right)$ and $p_{n,j} = |A_n - A_j|$. Analogously, since A_n and A_j are points of Euclidean disks, then $\alpha_{n,j}\ge p_{n,j}\ge \eta_n - t_j$. For the pseudohyperbolic distance between $\overline{U}_n$ and $\overline{U}_j$ we obtain:

$$\begin{aligned}\rho(\overline{U}_n,\overline{U}_j) &= \frac{1}{2}\tanh\psi(\overline{U}_n,\overline{U}_j) = \frac{1}{2}\tanh(\inf\alpha_{n,j})\\ &= \frac{1}{2}\tanh(\eta_n - t_j) = \frac{1}{2}\tanh\psi(t_j,\eta_n) = \rho(t_j,\eta_n).\end{aligned}$$

because t_j and η_n lie on a diameter of D.

Put $\xi'_n = t_n$ for every $n\in\mathbb{N}$, $n\neq j$, and $\xi'_j = \eta_j$. We get $\xi'_n\in\overline{U}_n = \overline{V}_n\cap\overline{W}_n$, i.e. $\rho(\upsilon_n,\xi'_n)\le\tau_n$ for every n. By the choice of $\{\tau_n\}_n$ and $\{\gamma_n\}_n$ we have:

$$\prod_{n:n<j}^{\infty}\rho(t_n,\eta_j) = \prod_{n:n<j}^{\infty}\rho(\xi'_n,\xi'_j) \ge \prod_{n:n\neq j}^{\infty}\rho(\xi'_n,\xi'_j)\ge\gamma_j. \tag{1}$$

Put $\xi''_n = \eta_n$ for every $n\in\mathbb{N}$, $n\neq j$, and $\xi''_j = t_j$. We get $\xi''_n\in\overline{U}_n$, for every n and analogously:

$$\prod_{n:n>j}^{\infty}\rho(\eta_n,t_j) = \prod_{n:n>j}^{\infty}\rho(\xi''_n,\xi''_j) \ge \prod_{n:n\neq j}^{\infty}\rho(\xi''_n,\xi''_j)\ge\gamma_j. \tag{2}$$

Now we apply (1) and (2):

$$\prod_{n:n\neq j}^{\infty}\rho(\overline{U}_n,\overline{U}_j) = \prod_{n:n<j}^{\infty}\rho(t_n,\eta_j).\prod_{n:n>j}^{\infty}\rho(\eta_n,t_j)\ge\gamma_j^2.$$

Since the function $F = g\circ f$ belongs to H^∞, $\|F\|\le 1$, and $f\left(x_n\right) = \upsilon_n$ for all n, then the set $O_n = F^{-1}\left(U_n\right)$ is open, $x_n\in O_n$ for every $n\in\mathbb{N}$, and $O_n\cap O_k = \varnothing$ for $n\neq k$. By the properties of the pseudohyperbolic distance ([2]):

$$\rho(\varphi_n,\varphi_j)\ge\rho(F\left(\varphi_n\right),F\left(\varphi_j\right))\ge\rho(U_n,U_j),$$

where $\varphi_n \in O_n$ and $\varphi_j \in O_j$ are arbitrary points. Hence

$$\prod_{n:n\neq j}^{\infty} \rho(O_n, O_j) \geq \prod_{n:n\neq j}^{\infty} \rho(\overline{U}_n, \overline{U}_j) \geq \gamma_j^2$$

and we obtain $\lim\limits_{j\to\infty} \prod\limits_{n:n\neq j}^{\infty} \rho(O_n, O_j) = 1$, since $\lim\limits_{j\to\infty} \gamma_j = 1$. The theorem is proved. □

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ЕДНО ДОСТАТЪЧНО УСЛОВИЕ ЗА ТЪНКОСТ НА РЕДИЦИ

Димчо Костов Станков

Известно е, че всяка интерполационна редица от тип 1 за е тънка редица, която удовлетворява едно допълнително условие на Р. Мортини. В тази работа предлагаме друго доказателство на този факт. По-точно доказваме, че ако съществува тънка радиална редица в отворения единичен кръг, за която интерполационната задача допуска решение с норма 1, то това условие е изпълнено.