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ON SOME NUMERICAL CHARACTERISTICS OF A BIPARTITE GRAPH*

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The paper considers an equivalence relation in the set of vertices of a bipartite graph. Some numerical characteristics showing the cardinality of equivalence classes are introduced. A combinatorial identity that is in relationship to these characteristics of the set of all bipartite graphs of the type $g = \langle R_g \cup C_g, E_g\rangle$ is formulated and proved, where $V = R_g \cup C_g$ is the set of vertices, E_g is the set of edges of the graph g, $|R_g| = m \geq 1$, $|C_g| = n \geq 1$, $|E_g| = k \geq 0$, m, n and k are integers.

1. Introduction. The widespread use of graph theory in different areas of science and technology is well known. For example, graph theory is a good tool for the modelling of computing devices and computational processes. So many of graph algorithms have been developed [7, 9]. One of the latest applications of graph theory is calculating the number of all disjoint pairs of S-permutation matrices [10, 11]. The concept of disjoint S-permutation matrices was introduced by Geir Dahl [3] in relation to the popular Sudoku puzzle. On the other hand, Sudoku matrices are special cases of Latin squares in the class of gerechte designs [2].

Let p be a positive integer. By $[p]$ we denote the set

$$[p] = \{1, 2, \ldots, p\}.$$

A bipartite graph is the ordered triplet

$$g = \langle R_g \cup C_g, E_g\rangle,$$

where R_g and C_g are sets such that $R_g \neq \emptyset$, $C_g \neq \emptyset$, and $R_g \cap C_g = \emptyset$. The elements of the set

$$V_g = R_g \cup C_g$$

will be called *vertices*. The set

$$E_g \subseteq R_g \times C_g = \{\langle r, c\rangle \mid r \in R_g, c \in C_g\}$$

will be called the set of *edges*. Repeated edges are not allowed in our considerations.

Let $g' = \langle R_{g'} \cup C_{g'}, E_{g'}\rangle$ and $g'' = \langle R_{g''} \cup C_{g''}, E_{g''}\rangle$. We will say that the graphs g' and g'' are *isomorphic* and we will write $g' \cong g''$, if $R_{g'} = R_{g''}$, $C_{g'} = C_{g''}$, $|R_{g'}| = |R_{g''}| = m$, $|C_{g'}| = |C_{g''}| = n$ and there are $\rho \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_n$, where $\mathcal{S}_p$ is *the symmetric group*, such that $\langle r, c\rangle \in E_{g'} \Longleftrightarrow \langle \rho(r), \sigma(c)\rangle \in E_{g''}$. The object of this work is bipartite graphs considered up to isomorphism.

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Let m, n and k be integers, $m \geq 4$, $n \geq 1$, and let $0 \leq k \leq mn$. Let us denote by $\mathfrak{G}_{m,n,k}$ the set of all bipartite graphs without repeated edges of the type $g = \langle R_g \cup C_g, E_g \rangle$, considered up to isomorphism, such that $|R_g| = m$, $|C_g| = n$ and $|E_g| = k$.

For more details on graph theory see [4, 6, 7].

In [5] Roberto Fontana proposed an algorithm which randomly obtain a family of $n^2 \times n^2$ mutually disjoint S-permutation matrices, where $n = 2, 3$. In $n = 3$ he ran the algorithm 1000 times and found 105 different families of nine mutually disjoint S-permutation matrices. Then he obtained $9! \cdot 105 = 38\ 102\ 400$ Sudoku matrices. In relation with Fontana's algorithm, it looks useful to calculate the probability of two randomly generated S-permutation matrices being disjoint.

The solution of this problem is given in [11], where a formula is described for calculating all pairs of mutually disjoint S-permutation matrices. The application of this formula when $n = 2$ and $n = 3$ is explained in detail in [10].

To do that, graph theory techniques have been used. It has been shown that to count the number of disjoint pairs of $n^2 \times n^2$ S-permutation matrices, it is sufficient to obtain some numerical characteristics of the set $\mathfrak{G}_{n,n,k}$ of all bipartite graphs of the type $g = \langle R_g \cup C_g, E_g \rangle$, where $V_g = R_g \cup C_g$ is the set of vertices, and E_g is the set of edges of the graph g, $R_g \cap C_g = \emptyset$, $|R_g| = |C_g| = n$, $|E_g| = k$.

The aim of this work is to formulate and prove a combinatorial problem related to some numerical characteristics of the elements of the set $\mathfrak{G}_{n,n,k}$.

For the classification of all non defined concepts and notations as well as for common assertions which have not been proved here see [1, 4, 8].

2. An equivalence relation in a bipartite graph. Let

$$g = \langle R_g, C_g, E_g \rangle \in \mathfrak{G}_{m,n,k}$$

for some natural numbers m, n and k and let $v \in V_g = R_g \cup C_g$.

By $N(v)$ we denote the set of all vertices of V_g, adjacent with v, i.e., $u \in N(v)$ if and only if there is an edge in E_g connecting u and v. In other words if $v \in R_g$, then $N(v) = \{u \in C_g \mid \langle v, u \rangle \in E_g\}$ and if $v \in C_g$, then $N(v) = \{u \in R_g \mid \langle u, v \rangle \in E_g\}$. If v is an isolated vertex (i.e., there is no edge incident with v), then by definition $N(v) = \emptyset$ and degree$(v) = |N(v)| = 0$.

Since in g there are no repeated edges, it is easy to see that

$$\sum_{u \in R_g} |N(u)| = k \quad \& \quad \sum_{v \in C_g} |N(v)| = k \quad \Longrightarrow \quad \sum_{w \in V_g} |N(w)| = 2k.$$

Let $g = \langle R_g, C_g, E_g \rangle \in \mathfrak{G}_{m,n,k}$ and let $u, v \in V_g = R_g \cup C_g$. We will say that u and v are equivalent and we will write $u \sim v$ if $N(u) = N(v)$. If u and v are isolated, then by definition $u \sim v$ if and only if $u, v \in R_g$ or $u, v \in C_g$. Obviously if $u \sim v$, then $u \in R_g \Leftrightarrow v \in R_g$ and $u \in C_g \Leftrightarrow v \in C_g$. It is easy to see that the relation introduced above is an equivalence relation.

By $V_{g_{/\sim}}$ we denote the obtained factor-set (the set of the equivalence classes) with respect to the relation $\sim$ and let

$$V_{g_{/\sim}} = \{\Delta_1, \Delta_2, \ldots, \Delta_s\},$$

where $\Delta_i \subseteq R_g$, or $\Delta_i \subseteq C_g$, $i = 1, 2, \ldots s$, $2 \leq s \leq 2n$. We assume that

$$\delta_i = |\Delta_i|, \quad 1 \leq \delta_i \leq n, \quad i = 1, 2, \ldots, s$$

and for every $g \in \mathfrak{G}_{m,n,k}$ we define the multi-set (set with repetition)

$$[g] = \{\delta_1, \delta_2, \dots \delta_s\},$$

where $\delta_1, \delta_2, \dots, \delta_s$ are natural numbers, obtained as above.

Obviously

$$\sum_{i=1}^{s} \delta_i = m + n.$$

The next assertion is a generalization of Corollary 1 of Lemma 1 from [11].

Theorem 1. *For any positive integers m, n and any nonnegative integer k such that $0 \le k \le mn$ the following equality holds:*

$$\sum_{g \in \mathfrak{G}_{m,n,k}} \frac{1}{\prod_{\delta \in [g]} \delta!} = \frac{(mn)!}{m!n!k!(mn-k)!}$$

Proof. A binary (or boolean, or $(0,1)$)-matrix is a matrix all of whose elements belong to the set $\mathfrak{B} = \{0, 1\}$. With $b(m, n, k)$ we will denote the number of all $m \times n$ binary matrices with exactly k elements equal to 1, $k = 0, 1, \dots, mn$.

It is easy to see that

$$b(m, n, k) = \binom{mn}{k} = \frac{(mn)!}{k!(mn-k)!} \tag{1}$$

We will prove that

$$b(m, n, k) = m!n! \sum_{g \in \mathfrak{G}_{m,n,k}} \frac{1}{\prod_{\delta \in [g]} \delta!} \tag{2}$$

Let $A = [a_{ij}]_{m \times n}$ be a $m \times n$ binary matrix with exactly k 1s. Then we construct a graph $g = \langle R_g \cup C_g, E_g \rangle$, such that the set $R_g = \{r_1, r_2, \dots, r_m\}$ corresponds to the rows of A, and $C_g = \{c_1, c_2, \dots, c_n\}$ corresponds to the columns of A, however there is an edge connecting the vertices r_i and c_j if and only if $a_{ij} = 1$. The graph which has been constructed obviously belongs to $\mathfrak{G}_{m,n,k}$.

Conversely, let $g = \langle R_g \cup C_g, E_g \rangle \in \mathfrak{G}_{m,n,k}$. We number in a random way the vertices of R_g by natural numbers from 1 to m without repeating any of the numbers. This can be done by $m!$ ways. We analogously number the vertices of C_g by natural numbers from 1 to n. This can be done by $n!$ ways. Then we construct the binary $m \times n$ matrix $A = [a_{ij}]_{m \times n}$, such that $a_{ij} = 1$ if and only if there is an edge in E_g connecting the vertex with number i of R_g with the vertex with number j of C_g. Since $g \in \mathfrak{G}_{m,n,k}$, the matrix that has been constructed has exactly k 1s. It is easy to see that when $q, r \in [m]$, the qth and rth rows of A are equal to each other (i.e., the matrix A does not change if we exchange the places of these two rows) if and only if the vertices of R_g corresponding to numbers q and r are equivalent according to relation $\sim$.

The analogous assertion is true about the columns of the matrix A and the edges of the set C_g, which proves formula (2).

From (1) and (2) it follows that

$$m!n!\sum_{g\in\mathfrak{G}_{m,n,k}}\frac{1}{\prod\limits_{\delta\in[g]}\delta!}=\frac{(mn)!}{k!(mn-k)!},$$

which proves the theorem. □

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ВЪРХУ НЯКОИ ЧИСЛОВИ ХАРАКТЕРИСТИКИ НА БИПОЛЯРЕН ГРАФ

Красимир Янков Йорджев

В статията се разглежда една релация на еквивалентност в множеството от върхове на произволен биполярен граф. Въвеждат се някои числови характеристики показващи мощността на отделните класове на еквивалентност. Получено и доказано е едно комбинаторно тъждество свързано с тези характеристики в множеството от всички биполярни графи от вида $g=\langle R_g\cup C_g, E_g\rangle$, където $V=R_g\cup C_g$ е множеството от върхове, а E_g е множеството от ребра на графа g, $|R_g|=m\geq 1$, $|C_g|=n\geq 1$, $|E_g|=k\geq 0$, m,n и k са цели числа.