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NEW LEADERS IN MATHEMATICS. FIELDS MEDALISTS FOR 2014 YEAR*

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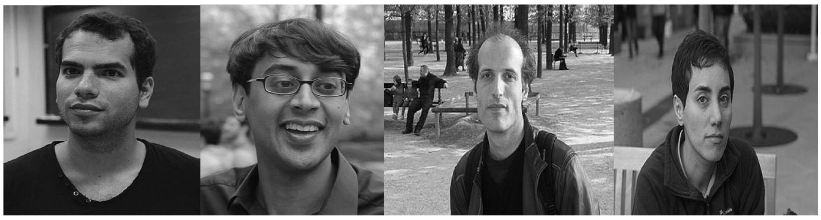
In this talk Fields Medalists for 2014 year will be presented. In the first part of the talk introduction to the main achievements of the new leaders will be discussed. Second part of the talk will deal with the works of Artur Avila and Martin Hairer. Some recent results in the dynamical systems and stochastic partial differential equations will be provided.

1. Introduction. The International Congress of Mathematicians (ICM 2014) held from 13-21 August 2014 in Seoul. The main focus of this congress was the growing role of women in the discipline and encouragement of a stronger mathematics community in the developing countries.

The congress began with the award of the 2014 Fields Medals. One of the four recipients was the first woman to win a medal since awards began in 1936. She is Maryam Mirzkhani of Standford University. The first female president of Korea – Mrs. Geunhye Park who presented the medals to the winners during the opening ceremony said: "In particular, I highly honor and admire the great spitir of challenge and passion of Dr. Mirzakhani, the first female to be awarded the Fields Medal in its history".

The other three Fields Medalist are Artur Avila of the Instituto Nacional de Mathemática Pura e Aplocada in Rio de Janeiro; Manjul Bhargava of Princeton University and



A. Avila M. Bhargava M. Hairer M. Mirzakhani

Fig. 1. Fields Medalists for 2014

*2010 Mathematics Subject Classification: 37E05, 37F45, 30D05, 35Q53,37K10.
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Martin Hairer of the University of Warwick. All medalists were introduced by Ingrid Daubechies, the first female president of the International Mathematical Union (IMU).

In the next section the main achievements of the Fields Medalists will be presented. Main works of Artur Avila in dynamical systems and Martin Hairer in stochastic partial differential equations will be discussed in Section 3 and 4.

2. Achievements of the Fields Medalists. Artur Avila, 35 years old, leads and shapes the field of dynamical systems. With his collaborators he has made essential progress in many areas, including real and complex one-dimensional dynamics, spectral theory of the one-frequency Schrödinger operator, billiards and partially hyperbolic dynamics. With tremendous analytical power Avila made outstanding contributions to dynamical systems. In his papers [2, 3, 4, 5] a wide class of dynamical systems is considered, namely, those arising from maps with a parabolic shape, known as unimodal maps and it is proved that, if one chooses such a map at random, the map will be either regular or stochastic. His work provides a unified, comprehensive picture of the behavior of these systems. This result will be presented in Section 3. Another important results were proofs of long-standing conjectures, that almost every interval exchange transformation is weakly mixing. This work is connected with the weak mixing property of regular polygonal billiards.

Manjul Bhargava, 40 years old, has developed powerful new methods in the geometry of numbers and applied them to count rings of small rank and to bound the average rank of elliptic curves. His thesis provided a reformulation of Gauss's law or the composition of two binary quadratic forms. He showed that the orbits of the group $SL(2 : \mathbf{Z})^3$ on the tensor product of three copies of the standard integral representation correspond to quadratic rings (rings of rank 2 over $\mathbf{Z}$) together with three ideal classes whose product is trivial. This recovers Gauss's composition law in an original and computationally effective manner. He then studied orbits in more complicated integral representations, which correspond to cubic, quartic, and quintic rings, and counted the number of such rings with bounded descriminant.

Martin Hairer, 39 years old, has made outstanding contributions to the theory of stochastic partial differential equations. Hairer's work addresses two central aspects of the theory. Together with Mattingly he employed the Malliavin calculus along with new methods to establish the ergodicity of two-dimensional stochastic Navier-Stokes equation. Building on the rough-path approach of Lyons for stochastic ordinary differential equations, Hairer then created an abstract theory of regularity structures for stochastic partial differential equations (SPDE). The new theory allowed him to construct systematically solutions to singular nonlinear SPDEs as fixed points of renormalization procedure. Hairer was thus able to give, for the first time, a rigorous intrinsic meaning to many SPDEs arising in physics. One such problem concerning dynamical sine-Gordon equation [13] will be presented in Section 4 of this talk.

Maryam Mirzakhani, 37 years old, has made striking and highly original contributions to geometry and dynamical systems. Her work on Riemann surfaces and their moduli spaces bridges several mathematical disciplines and influences them all in return. Mirzakhani's early work concerns closed geodesics on a hyperbolic surface. A now-classic theorem, the so-called prime number theorem for geodesics, says that the number of closed geodesics whose length is bounded by L, grows exponentially with L; specifically, it is asymptotic to e^L/L for large L. She looked at what happens to this theorem when one considers only the simple closed geodesics, meaning that they do not intersect themselves. The behavior is very different in this case: the growth of the number of simple geodesics of length at most L is of the order of L^{6g6} where g is the genus of the surface. Mirzakhani showed that in fact the number is asymptotic to $c.L^{6g6}$ for large L (going to infinity), where the constant c depends only on the hyperbolic structure. Most recently, Mirzakhani, together with Alex Eskin, in part, Amir Mohamadi, made a major breakthrough in understanding another dynamical system on moduli space that is related to the behavior of geodesics in moduli space. Non-closed geodesics in moduli space are very erratic and even pathological, and it is hard to understand how they change when perturbed slightly. However, Mirzakhani et al. have proved that complex geodesics and their closures in moduli space are in fact surprisingly regular, rather than irregular or fractal.

3. Regular or stochastic dynamics in real analytic families of unimodal maps. In the works [2, 3, 4, 5] Avila considers the dynamics of unimodal maps of an interval I, i.e., smooth endomorphisms of I with unique critical point that will be assumed to be quadratic. The simplest and most famous example of this kind is given by the real quadratic family:

$$Q_\lambda : [0,1] \to [0,1], Q_\lambda = \lambda x(1-x),$$

where λ is a real parameter between 1 and 4. In 1976 the article of R. May [16] had a big impact on the scientific community by demonstrating that this simple mathematical model exhibits a very interesting and complex dynamical behavior.

Avila et al. describe a picture of an appropriate space of real analytical unimodal maps, which gives a justification for the special role of the quadratic family. It is used in order to transfer some important dynamical properties from the quadratic family to any non-trivial real analytic family of quasiquadratic maps (a class which includes maps with negative Schwarzian derivative).

A unimodal map is called regular if its critical point belongs to the basin of a hyperbolic periodic attractor and all its periodic orbits are hyperbolic. It is called stochastic if it has an invariant measure absolutely continuous with respect to the Lebesgue measure. Given a smooth one-parameter family $\{f_t\}$ of unimodal maps, parameter t is regular or stochastic if the corresponding map f_t is such. The set of regular parameter values is always open. The set of stochastic parameter values has positive Lebesgue measure for an open set of families containing the quadratic family. In fact, in the case of the quadratic family $\{Q_\lambda\}$ much more is known: 1. the set of regular parameter values λ is open and dense in the quadratic family; 2). the set of stochastic parameter values λ has full Lebesgue measure in the complement of the regular parameters.

The main results of Avila's work [2,3,4,5] can be formulated as follows.

Fix some $a > 0$. The affine space $\mathcal{A}_a$ has a natural involution around its real subspace $\mathcal{A}_a^{\mathbf{R}}$. A subset in $\mathcal{A}_a$ is called $\mathbf{R}$-symmetric if it is invariant under this involution.

Theorem 1. *Every real hybrid class $\mathcal{H}_f^{\mathbf{R}}$, $f \in \mathcal{U}_a$, is an embedded codimension-one real analytic Banach submanifold of $\mathcal{U}_a$. Furthermore, the hybrid classes laminate a neighborhood of any non-parabolic map $f \in \mathcal{U}_a$. More precisely, any non-parabolic map $f \in \mathcal{U}_a$ has an $\mathbf{R}$-symmetric neighborhood ν in the complex affine space $\mathcal{A}_a$ endowed with a codimension-one $\mathbf{R}$-symmetric holomorphic lamination such that for any $g \in \nu \cap \mathcal{U}_a^{\mathbf{R}}$, the intersection of the leaf through g with $\mathcal{A}_a^{\mathbf{R}}$ coincides with $\mathcal{H}_g^{\mathbf{R}} \cap \nu$.*

A real analytic one-parameter family $\{f_\lambda\}_{\lambda\in\Lambda}$ of quasiquadratic maps is called non-trivial if it is not contained in a single real hybrid case.

Theorem 2. *Let $\{f_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{U}_a$ be a non-trivial one-parameter real analytic family of quasiquadratic maps. Then for almost all parameter values $\lambda \in \Lambda$, the map f_λ is either regular or stochastic.*

This theorem extends the following result by Kozlovski [14]: For an open and dense set of parameter values t in a non-trivial real analytic family $\{f_t\}$, the map f_t has a finite number of periodic attractors whose basin has full Lebesgue measure. Theorem 2 fits nicely to the general program of studying attractors in finite parameter families of dynamical systems (in all dimensions) formulated by Palis [17, 18].

Theorem 3. *Under the circumstances of the previous theorem, there is an open and dense set $\Lambda_0 \subset \Lambda$ of parameter values with countable complement such that the straightening map $\chi(f_t)$ is quasisymmetric on any compact interval contained in Λ_0.*

Though these results are concerned with real unimodal maps, they are mostly based on the complex methods. The complex tools which are particularly important for the results are the theory of holomorphic motions and the Pullback Argument, especially, its infinitesimal version. It allows to carry out an infinitesimal analysis of topological classes of unimodal maps which yields the following problem: Is it true that codimension-one topological classes form a lamination in the space of quasiquadratic maps (near parabolic maps as well)?

The main results in [2, 3, 4, 5] are still valid without the negative Schwarzian derivative assumption. The regular or stochastic dichotomy in families of unimodal maps has been recently refined, giving a better statistical description of the dynamics of typical parameters: they satisfy the Collet-Eckmann condition, among other nice properties. Those results, first obtained in the context of the quadratic family [3], is generalized in [4], [2] to non-trivial analytic families using the above stated results and then to generic smooth families.

4. Dynamical sine-Gordon equation. The aim of this work is to provide a solution theory for the stochastic PDE:

$$\partial_t u = \frac{1}{2}\Delta u + c\sin(\beta u + \Theta) + \xi, \tag{1}$$

where c, β, Θ are real valued constants, ξ denotes space-time white noise, and the spatial dimension is fixed to be 2.

The model (1) is interesting for a number of reasons. First and foremost, it is of purely mathematical interest as a very nice testbed for renormalisation techniques. Indeed, even though the authors work with fixed spatial dimension 2, this model exhibits many features comparable to those of various models arising in constructive quantum field theory (QFT) and/or statistical mechanics, but with the dimension d of those models being a function of the parameter β.

More precisely, at least at a heuristic level, (1) is comparable to Φ^3_d Euclidean QFT with $d = 2 + \frac{\beta^2}{2\pi}$, Φ^4_d Euclidean QFT with $d = 2 + \frac{\beta^2}{4\pi}$, or the KPZ equation in dimension $d = \frac{\beta^2}{4\pi}$. In particular, one encounters divergencies when trying to define solutions to (1) or any of the models just mentioned as soon as β is nonzero. (In the case of the KPZ equation recall that, via the Cole-Hopf transform it is equivalent to the stochastic heat equation. In dimension 0, this reduces to the SDE $du = udW$ which is ill-posed if W is a Wiener process but is well-posed as soon as it is replaced by something more regular, say fractional Brownian motion with Hurst parameter greater than 1/2.)

These divergencies can however be cured in all of these models by Wick renormalisation as long as $\beta^2 < 4\pi$. At $\beta^2 = 4\pi$ (corresponding to Φ^3_4, Φ^4_3, and KPZ in dimension 1), Wick renormalisation breaks down and higher order renormalisation schemes need to be introduced. One still expects the theory to be renormalisable until $\beta^2 = 8\pi$, which corresponds to Φ^3_6, Φ^4_4 and KPZ in dimension 2, at which point renormalisability breaks down. This suggests that the value $\beta^2 = 8\pi$ is critical for (1) and that there is no hope to give it any non-trivial meaning beyond that, see for example [8, 10] and, in a slightly different context, [15]. This heuristic (including the fact that Wick renormalisability breaks down at $\beta^2 = 4\pi$) is well-known and has been demonstrated in [6, 8, 11] at the level of the behaviour of the partition function for the corresponding lattice model.

From a more physical perspective, an interesting feature of (1) is that it is closely related to models of a globally neutral gas of interacting charges. With this interpretation, the gas forms a plasma at high temperature (low β) and the various threshold values for β could be interpreted as threshold of formation of macroscopic fractions of dipoles/quadrupoles/etc. The critical value $\beta^2 = 8\pi$ can be interpreted as the critical inverse temperature at which total collapse takes place, giving rise to a Kosterlitz-Thouless phase transition. Finally, the model (1) has also been proposed as a model for the dynamic of crystalvapour interfaces at the roughening transition [7] and as a model of crystal surface fluctuations in equilibrium.

In order to give a non-trivial meaning to (1), the authors first replace ξ by a smoothened version ξ_ε which has a correlation length of order $\varepsilon > 0$ and then study the limit $\varepsilon \to 0$. Since the authors are working in two space dimensions, one expects the solution to become singular (distribution-valued) as $\varepsilon \to 0$. As a consequence, there will be some "averaging effect" so that one expects to have $\sin(\beta u_\varepsilon) \to 0$ in some weak sense as $\varepsilon \to 0$. It therefore seems intuitively clear that if the authors wish to obtain a nontrivial limit, they should simultaneously send the constant c to $+\infty$. This is indeed the case, see Theorem 1 below. The authors also study a class of 2 + 1-dimensional equilibrium interface fluctuation models with more general periodic nonlinearities:

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + c_\varepsilon F_\beta(u_\varepsilon) + \xi_\varepsilon, \tag{2}$$

where F_β is a trigonometric polynomial of the form:

$$F_\beta = \sum_{k=1}^{Z} \zeta_k \sin(k\beta u + \Theta_k), Z \in \mathbf{N}, \tag{3}$$

where ζ_k, β and Θ_k are real valued constants. One may expect that the "averaging effect" of $\sin(k\beta u_\varepsilon + \Theta_k)$ is stronger for larger values of k. This is indeed the case and, as a consequence of this, that provided the constant $c_\varepsilon \to \infty$ at a proper rate, the limiting process obtained as $\varepsilon \to 0$ only depends on F_β via the values β, Θ_1 and ζ_1. In this sense, the equation (1) also arises as the limit of the models (2).

The main result can be formulated as follows.

Theorem 4. *Let* $0 < \beta^2 < \frac{16\pi}{3}$ *and* $\eta \in (-\frac{1}{3}, 0)$. *For* $u^{(0)} \in \mathcal{C}^\eta(\mathbf{T}^2)$ *fixed, consider* *the solution* u_ε *to*

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + C_\rho \varepsilon^{-\beta^2/4\pi} F_\beta(u_\varepsilon) + \xi_\varepsilon, u(0,.) = u^{(0)},$$

where F_β *is defined in* (3), $\xi_\varepsilon = \rho_\varepsilon * \xi$ *with* $\rho_\varepsilon(t,x) = \varepsilon^{-4}\rho(\varepsilon^{-2}t, \varepsilon^{-1}x)$ *for some smooth and compactly supported function* ρ *integrating to 1. Then there exists a constant* C_ρ *(depending only on* β *and the mollifier* ρ*) such that the sequence* u_ε *converges in probability to a limiting distributional process* u *which is independent of* ρ.

More precisely, there exist random variables $\tau > 0$ *and* $u \in \mathcal{D}'(\mathbf{R}_+ \times \mathbf{T}^2)$ *such that, for every* $T' > T > 0$, *the natural restriction of* u *to* $\mathcal{D}'((0,T) \times \mathbf{T}^2)$ *belongs to* $\chi_{T,\eta} = \mathcal{C}([0,T], \mathcal{C}^\eta(\mathbf{T}))$ *on the set* $\tau \geq T'$. *Furthermore, on the same set, one has* $u_\varepsilon \to u$ *in probability in the topology of* $\chi_{T,\eta}$.

Finally, one has $\lim_{t\to\tau} ||u(t,.)||_{\mathcal{C}^\eta(\mathbf{T}^2)} = \infty$ *on the set* $\tau < \infty$. *The limiting process* u *depends on the numerical values* β, ζ_1, *and* Θ_1, *but it depends neither on the choice of mollifier* ρ, *nor on the numerical values* ζ_k *and* Θ_k *for* $k \geq 2$.

As already mentioned, one expects the boundary $\beta^2 = \frac{16\pi}{3}$ to be artificial and a similar result is expected to hold for any $\beta^2 \in (0, 8\pi)$. In fact, 8π is the natural boundary for the method of proof developed in [13] and employed here. However, as $\beta^2 \to 8\pi$, the theory requires proofs of convergence of more and more auxiliary objects. In the current context, unfortunately there is not general convergence result for all of these objects but instead all of them have to be treated separately "by hand". Furthermore, the bounds on the simplest "second-order" object unfortunately appear to break down at $\beta^2 = 6\pi$.

It is interesting to note that for $\beta^2 \in (0, 4\pi)$, one auxiliary process have to be constructed, and this construction does indeed involve a careful tracking of cancellations due to the grouping of terms into "dipoles", while for $\beta^2 \in [4\pi, \frac{16\pi}{3})$, a second auxiliary process should be built which requires to keep track of cancellations obtained by considering "quadrupoles".

The limiting process u is a continuous function of time, taking values in a suitable space of spatial distributions. Regarding the right hand side of the equation however, it only makes sense as a random distribution at fixed time when $\beta^2 < 4\pi$. For $\beta^2 \geq 4\pi$ however, it exists only as a random space-time distribution.

The article [1] appears in principle to cover (1) as part of a larger class of nonlinearities. It is however unclear what the meaning of the solutions constructed there is and how they relate to the construction given in the present article. The interpretation of the solutions in [1] is that of a random Colombeau generalised function and it is not clear at all whether this generalised function represents an actual distribution. In particular, the construction given there is completely impervious to the presence of the Kosterlitz-Thouless transition and the collapse of multipoles which clearly transpire in our analysis.

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НОВИТЕ ЛИДЕРИ В МАТЕМАТИКАТА. ФИЙЛДСОВИТЕ МЕДАЛИСТИ ЗА 2014

Анжела Славова

На Световния конгрес на математиците 13-21 Август 2014 г., Сеул бяха връчени Фийлдсовите медали за 2014 г. Те по традиция се дават на млади математици до 40 г. Тази година носителите бяха: Артур Авила, Манжул Баргава, Мартин Хайрер, Мариам Мирзакани. В този доклад ще бъдат представени основните постижения, за които младите математици получиха Фийлдсови медали. По-подробно ще бъдат разгледани някои работи по динамични системи на Артур Авила и по стохастични диференциални уравнения на Мартин Хайрер.