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RECENT DEVELOPMENTS IN EXPONENTIAL FUNCTIONALS OF LÉVY PROCESSES*

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In this document we review the chronology of the developments of a currently active area of research, i.e. the exponential functionals of Levy processes. The interest in these objects starts with applications in finance but it was quickly realized that their importance spreads into several other area of modern probability. This triggered a number of papers on the topic which aim to describe the law and the properties of the exponential functionals of Lévy processes. Here we make an exposition of all key developments and techniques that preceded and led to the latest results on the topic and we provide references for all major papers in the field.

1. Introduction. In this report we deal with exponential functionals of Lévy processes, i.e. we consider quantities of the type

$$I_q := \int_0^{\infty} e^{\xi_s} 1_{\{\mathbf{e}_q < s\}} ds = \int_0^{\mathbf{e}_q} e^{\xi_s} ds \text{ and } I_t := \int_0^t e^{\xi_s} ds,$$

provided they exist, where $\xi := (\xi_s)_{s\geq 0}$ is a Lévy process, $t > 0$ and $\mathbf{e}_q \sim Exp\,(q)$ is an independent of ξ exponential random variable with parameter $q \geq 0$ with the convention $\mathbf{e}_0 = \infty$. These three random variables have attracted the huge interest of many researchers over the last two decades, which is mostly due to the prominent role played by their laws in the study of important processes, such as spectral theory of self-similar Markov processes, fragmentation and branching processes but also in various settings ranging from astrophysics, biology to financial and insurance mathematics, see the survey paper [2] (note that in view of the latest results this survey is outdated). The progress has been predominantly concerning the quantities I_q and it has involved a mixture of techniques from complex analysis, probability theory, special functions, etc. The purpose of this document is to acquaint the reader with the tools available for the study of I_q; to formulate, provide reference and furnish metaproofs for the latest results; to discuss the difficulties behind the understanding of I_t and to point towards future possible developments and applications.

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1.1. Lévy processes and their Lévy-Khintchine exponent. A stochastic process $\xi = (\xi_s)_{s\geq 0} : [0,\infty) \mapsto \mathcal{D}(0,\infty)$, in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the space $\mathcal{D}[0,\infty)$ is the space of right-continuous functions with left-limits endowed with the Skorohod topology, $\mathcal{F}$ is an adapted filtration and $\mathbb{P} : \mathcal{F} \mapsto [0,1]$ is a probability measure, is called a Lévy process if it satisfies the following requirements

1) ξ has *independent increments*, i.e. $\xi_t - \xi_s$ is independent of $(\xi_v)_{v\leq s}, \forall s < t$

2) ξ has *stationary increments*, i.e. the processes $(\xi_t - \xi_s)_{t\geq s} \stackrel{w}{=} \xi, \forall s < t$, where $\stackrel{w}{=}$ stands for identity in law between stochastic processes in the Skorohod topology.

ξ is called a killed Lévy process if for some independent $\mathbf{e}_q \sim Exp(q)$ the trajectory of ξ is terminated at the random time $\mathbf{e}_q$. When $q = 0$ the process lives forever. We denote by Ψ_q the Lévy-Khintchine exponent of a possibly killed Lévy process. It takes the form, for any $z \in i\mathbb{R}$,

$$(1.1)\quad \Psi_q(z) = \Psi(z) - q = \ln\left(\mathbb{E}\left[e^{z\xi_1} 1_{\{\mathbf{e}_q > 1\}}\right]\right) = bz + \frac{\sigma^2}{2}z^2 + \int_{-\infty}^{\infty}\left(e^{zy} - 1 - zy\mathbb{I}_{\{|y|<1\}}\right)\Pi(dy) - q,$$

where $q \geq 0$ is the killing rate, $\sigma \geq 0, b \in \mathbb{R}$ and Π is a sigma-finite positive measure satisfying the condition $\int_{\mathbb{R}}(y^2 \wedge 1)\Pi(dy) < \infty$. We recall that σ^2 is called the Brownian component of ξ, $b \in \mathbb{R}$ the linear drift and Π the Lévy measure since it determines the intensity and the size of the jumps of ξ. For a thorough account on the basic theory of Lévy processes, see [1].

1.2. Wiener-Hopf factorization of Lévy processes. A basic tool throughout our work is the celebrated Wiener-Hopf factorization of a Lévy process ξ. The terminology Wiener-Hopf factorization stems from the factorization of functions defined along curves in the complex plain as a product of two functions respectively analytic on the two sides of the curve. Although strictly speaking the factorization of Ψ_q, see (1.1), is precisely the same, it bears a very beautiful probabilistic flavour.

First, we recall that the reflected processes of ξ, i.e. $\left(\sup_{0\leq s\leq t}\xi_s - \xi_t\right)_{t\geq 0}$ and $\left(\xi_t - \inf_{0\leq s\leq t}\xi_s\right)_{t\geq 0}$ are Feller processes in $[0,\infty)$ which possess local times $(L^{\pm}_t)_{t\geq 0}$ at level 0, see [1, Chapter IV]. The ascending and descending ladder times are defined as the right-continuous inverses of $L^{\pm}$, viz. $(L^{\pm}_t)^{-1} = \inf\{s > 0;\ L^{\pm}_s > t\}$ and the ladder height processes H^+ and H^- by

$$H^+_t = \xi_{(L^+_t)^{-1}} = \sup_{0\leq s\leq (L^+_t)^{-1}} \xi_s\,, \qquad \text{whenever } (L^+_t)^{-1} < \infty,$$

$$H^-_t = \xi_{(L^-_t)^{-1}} = \inf_{0\leq s\leq (L^-_t)^{-1}} \xi_s\,, \qquad \text{whenever } (L^-_t)^{-1} < \infty.$$

Here, we use the convention that $\inf\varnothing = \infty$ and $H^{\pm}_t = \infty$, when $L^{\pm}_{\infty} \leq t$. From [5, p. 27], we have for $q \geq 0$, $s \geq 0$,

$$(1.2)\quad \log\mathbb{E}\left[e^{-q(L^+_1)^{-1} - sH^+_1}\right] = -\Phi_+(q,s) = - k_+ - \eta_+ q - \delta_+ s - \int_0^{\infty}\int_0^{\infty}\left(1 - e^{-(qy_1 + sy_2)}\right)\mu_+(dy_1, dy_2),$$

where η_+ (resp. δ_+) is the drift of the subordinator $(L^+)^{-1}$ (resp. H^+) and $\mu_+(dy_1, dy_2)$ is the Lévy measure of the bivariate subordinator $\left((L^+)^{-1}, H^+\right)$. Similarly, for $q \geq 0$, $s \geq 0$,

$$\begin{aligned} \text{(1.3)} \quad \log \mathbb{E}\left[e^{-q(L_1^-)^{-1}+sH_1^-}\right] &= -\Phi_-(q,s) \\ &= -k_- - \eta_- q - \delta_- s - \int_0^\infty \int_0^\infty \left(1 - e^{-(qy_1+sy_2)}\right)\mu_-(dy_1, dy_2), \end{aligned}$$

where η_- (resp. δ_-) is the drift of the subordinator $(L^-)^{-1}$ (resp. $-H^-$) and $\mu_-(dy_1, dy_2)$ is the Lévy measure of the bivariate subordinator $\left((L^-)^{-1}, -H^-\right)$. The celebrated Wiener-Hopf factorization then reads off as

$$\Psi_q(z) = \Psi(z) - q = -\Phi_+(q, -z)\Phi_-(q, z), \tag{1.4}$$

where we set $\Phi_+(1,0) = \Phi_-(1,0) = 1$ as the normalization of the local times. We point out that while it can happen that $((L^+)^{-1}, H^+)$ $\left(\text{resp. } \left((L^-)^{-1}, -H^-\right)\right)$ can be increasing renewal processes, see [3, Section 1], this does not affect our definitions.

2. Exponential functionals: history, developments and results. The history of exponential functionals originates from the work of Dufresne, see [6], with his work on perpetuities and from Urbanick's work on multiplicative infinite divisibility, see [16], which extends the notion of additive multiplicative divisibility on the product group $\mathbb{R}^+$. However, the latter work is better explained and further developed by Hirsch and Yor in [8]. This two quite different studies already hint at the variety of potential applications of the exponential functionals.
In this section we only deal with I_q and we set $I := I_0$, i.e.

$$I_q = \int_0^\infty e^{\xi_s} 1_{\{\mathbf{e}_q < s\}} ds = \int_0^{\mathbf{e}_q} e^{\xi_s} ds. \tag{2.1}$$

2.1. Exponential functionals of Brownian motion. Let $\xi_s = \sigma^2 B_s - \mu s, \forall\, s \geq 0$, where $\mu > 0$ and $B := (B_s)_{s \geq 0}$ is a standard Brownian motion started from 0. Then we have that

Theorem 2.1 (Dufresne 90)**.** *For any $\mu > 0$ and $\sigma > 0$ we have that*

$$I = \int_0^\infty e^{\xi_s} ds \stackrel{d}{=} \frac{1}{G}, \tag{2.2}$$

where $G \sim \Gamma\left(\frac{2\mu}{\sigma^2}, \frac{\sigma^2}{2}\right)$ is a Gamma random variable.

Remark 2.2. The proof of (2.2) utilizes the very special features of the three dimensional Bessel process, see [17]. However, this result together with the distribution of I_q, see (2.1), is a mere consequence of our claims below, see Theorem 2.11, but also has been developed in [17].

This initial work depends entirely on the special features of the three dimensional Bessel process.

2.2. Exponential functionals of Lévy processes with no positive jumps. Another case that can be tackled in some generality is the scenario when ξ is a Lévy process with no positive jumps, i.e. in (1.1) $\Pi(dx) 1_{\{x>0\}} \equiv 0$. Then we have the following result

Theorem 2.3 (Patie 2012)**.** *Let ξ be a Lévy process with no positive jumps and $\mathbb{E}[\xi_1] < 0$. Then the density of I, i.e. $f(x) = \mathbb{P}(I \in dx)/dx, x > 0$, extends to an-*

alytic function on $\mathbb{C} \setminus \mathbb{R}^-$ and $f(x)$ can be established as infinite series which depends explicitly on Ψ.

Remark 2.4. The results in [13] depend on the solution of a specific Cauchy problem arising from the semigroup of a positive self-similar Markov process. It thus utilizes general potential theory of Markov processes and requires significant efforts. We point out the effortless application of our Theorem 2.11 (*note that* $\pi(x) \equiv 0, x > 0$) below to recover this result and extend it to I_q. However, this must not be understood as an understatement of [13] which develops a very useful approach.

Remark 2.5. The main result Theorem 2.3 in [13] provides much more information than Theorem 2.3, e.g. one can derive the asymptotic of $f(x)$ for large values of x, etc.

In a number of subsequent works various results on the real and complex moments of I_q were obtained. The will serve as a basis of the thorough understanding of the exponential functionals I_q for general Lévy processes.

2.3. Complex moments of exponential functionals. Let denote by

$$\mathcal{M}_q(s) := \mathbb{E}\left[I_q^{s-1}\right], \quad s \in \mathbb{C} \tag{2.3}$$

$\forall s \in \mathbb{C} : |\mathcal{M}_q(s)| < \infty$. The latter holds at least for $s \in 1 + i\mathbb{R}$ since then $|\mathcal{M}_q(s)| \leq 1$. One way to obtain information as to the law of I_q, in particular, and to study its properties, in general, is to compute its moments. The following result then holds:

Theorem 2.6. *Let ξ be a Lévy process such that $\mathbb{E}[|\xi_1|] < \infty$. Then, $\forall q > 0$,*

$$\mathcal{M}_q(s+1) = -\frac{s}{\Psi_q(s)}\mathcal{M}(s), \; s \in \{\Re(s) > 0\} \cap \{s : \Psi_q(s) < 0\}. \tag{2.4}$$

When $\mathbb{E}[\xi_1] < 0$, (2.4) extends to $q = 0$ on

$$\{\Re(s) > 0 : \Psi(s) < 0\} \cup \{\Re(s) \leq 0 : |\Psi(\Re(s))| < \infty\}.$$

Remark 2.7. The final version of Theorem 2.6 has been gradually obtained. First results in this direction can be found in [4] whereas the most general case for $I = I_0$, i.e. $\mathbb{E}[\xi_1] < 0$, can be found in [11, Lemma 2.1]. The latter result allows for almost immediate extension to the case $q > 0$.

Remark 2.8. It may happen that when $q > 0$ (2.4) is nowhere defined on $\mathbb{C}$ and thus (2.4) is immaterial. However, by choosing a suitable sequence of Lévy processes $\xi^n \stackrel{w}{\to} \xi$ for which (2.4) holds, we can study $\left\{\mathcal{M}_q^{(n)}(s)\right\}_{n\geq 1}$ and find its limit, i.e. $\mathcal{M}_q(s)$, on an appropriate strip, see [15].

The equation (2.4) cannot be immediately solved in generality since it cannot be analytically continued to either $\mathbb{C}^+ = \mathbb{C} \cap \{\Re(s) \geq 0\}$ or $\mathbb{C}^- = \mathbb{C} \cap \{\Re(s) \leq 0\}$. Notable exceptions are the classes of possibly killed decreasing Lévy processes (negative subordinators), i.e. (2.4) is solved for all I_q, and the class of unkilled Lévy processes with positive jumps only, i.e. (2.4) is solved for I only, see e.g. [11, Proposition 2.2 and Proposition 2.3] and [8, Theorem 3.2].

As the most general example of classes when (2.4) can be solved without an appeal to the Wiener-Hopf factorization, the following case is considered.

2.4. Exponential functionals of meromorphic Lévy processes. A Lévy process ξ is called meromorphic if it has the following structure for its jump measure $\Pi(dx)$:

$$\Pi(dx) = \pi(x)dx = 1_{\{x>0\}} \sum_{n\geq 1}^{\infty} a_n \rho_n e^{-\rho_n x} + 1_{\{x<0\}} \sum_{n\geq 1}^{\infty} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x}, \tag{2.5}$$

where the sequences $\{a_n\}_{n\geq 1}$, $\{\hat{a}_n\}_{n\geq 1}$, $\{\rho_n\}_{n\geq 1}$, $\{\hat{\rho}_n\}_{n\geq 1}$ are positive, $\lim_{n\to\infty}\rho_n = \lim_{n\to\infty}\hat{\rho}_n = \infty$ and $\sum_{n=1}^{\infty}\left(a_n\rho_n^{-2} + \hat{a}_n\hat{\rho}_n^{-2}\right) < \infty$, see [9] for details. The function $\Psi_q(z)$ then extends to a meromorphic function for $z \in \mathbb{C}$ and then the following result in [7, Theorem 2] obtained via the solution of (2.4) in the complex plane holds:

Theorem 2.9 (Kuznetsov et al. 2014)**.** *Let ξ be a mermorphic Lévy process. Then, for any $q > 0$, there exist positive sequences $\{\zeta_n(q)\}_{n\geq 1}$ and $\left\{\hat{\zeta}_n(q)\right\}_{n\geq 1}$ representing the simple real solutions of $\Psi(z) = q$ such that $\cdots < -\hat{\rho}_1 < -\hat{\zeta}_1 < 0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 \cdots$ and*

(2.6) $\mathcal{M}_q(s) =$

$$C^{s-1}\prod_{n\geq 1}\frac{\Gamma\left(1+\hat{\zeta}_n\right)\Gamma\left(s+\hat{\rho}_{n-1}\right)}{\Gamma\left(s+\hat{\zeta}_n\right)\Gamma\left(1+\hat{\rho}_{n-1}\right)}\left(\frac{1+\hat{\zeta}_n}{1+\hat{\rho}_{n-1}}\right)^{s}\frac{\Gamma\left(\rho_n\right)}{\Gamma\left(\zeta_n\right)}\frac{\Gamma\left(1+\zeta_n-s\right)}{\Gamma\left(1+\rho_n-s\right)}\left(\frac{\zeta_n}{\rho_n}\right)^{s-1},$$

for $s:\ 0 < \Re(s) < 1+\zeta_1$, where $\hat{\rho}_0 := 0$, $C = C(q) = \frac{1}{q}\prod_{n\geq 1}\frac{1+\hat{\rho}_n^{-1}}{1+\hat{\zeta}_n^{-1}}$. The result extends to the case $q = 0$ provided that $\mathbb{E}\left[\xi_1\right] < 0$.

Remark 2.10. Equation (2.6) can be shown to yield that I_q is a ratio of two infinite products of independent Beta random variables with parameters depending on the sequences in the theorem. This specifies the law of I_q. This result can be derived from Theorem 2.14 below but in [7] is proved via a lucky guess for the form of $\mathcal{M}_q(s)$ together with a verification procedure, see [7]. The strategy "lucky" guess and verification that the guessed solution for $\mathcal{M}_q$ is indeed the solution for the Mellin transform of I_q had been the main approach for some time.

The latest approaches to studying I_q combine complex anaysis or general theory of Markov processes with the Wiener-Hopf factorization described in Section 1.2. We start chronologically.

2.5. Exponential functionals of Lévy processes with monotonely decreasing density of the Lévy measure. Assume next that $\Pi(dx)1_{\{x>0\}} = \pi(x)1_{\{x>0\}}dx$ and $\pi(x)$ is non-increasing on $x > 0$. Then the following result holds

Theorem 2.11. *Let ξ be a Lévy process such that $\pi(x)$ is non-increasing on $x > 0$. Then there exist two independent Lévy processes: an independent copy of the descending ladder height process H^- with Lévy-Khintchine exponent $\Phi_-(q,s)$ and unkilled spectrally positive Lévy process with Lévy-Khintchine exponent $\Psi_q^+(s) := s\Phi_+(q,-s)$, say ξ^+, such that*

$$I_q = I_{q^-}^{H^-} \times I^{\xi^+}, \tag{2.7}$$

where $q^- = \Phi_-(q,0)$.

Remark 2.12. We note that the proof of (2.7) is based upon the successful solution of (2.4). In fact we prove that $\mathcal{M}_q(s) = \mathcal{M}_{q^-}^{(H^-)}(s)\mathcal{M}^{(\xi^+)}(s)$ and we are even able to compute $\mathcal{M}_q, \mathcal{M}_{q^-}^{(H^-)}, \mathcal{M}^{(\xi^+)}$ on the domains where they are respectively well-defined.

Remark 2.13. We note that this is perhaps the first work which established such a general factorization of the law of I_q. It allows for a wealth of information about its law which is otherwise seemingly out of reach, see relevant corollaries and discussions in [12] and [14].

Later, after the development of these results in [12] and [14], the authors of the latter result managed to obtain (2.7) and to compute $\mathcal{M}_q(s)$ in a complete generality.

2.6. Exponential functionals of Lévy processes: the general case. In this section we work with general Lévy processes. For a function $\phi : \mathbb{C} \to \mathbb{C}$, we write formally the generalized Weierstrass product

$$W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)} z} \tag{2.8}$$

where

$$\gamma_\phi = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\phi'(k)}{\phi(k)} - \log \phi(n) \right).$$

We observe that if $\phi(z) = z$, then W_ϕ corresponds to the Weierstrass product representation of the Gamma function Γ, valid on $\mathbb{C}/\{0, -1, -2, \ldots\}$, and γ_ϕ is the Euler-Mascheroni constant, see e.g. [10], justifying both the terminology and notation. Then we have that the following general result announced in [15, Theoren 2.1] holds true:

Theorem 2.14. *For any Lévy process and $q > 0$ we have that*

$$\mathcal{M}_q(s) = \Phi_+(q,0)\,\Gamma(s)\,\frac{W_{\Phi_+(q,.)}(1-s)}{W_{\Phi_-(q,.)}(s)}, \qquad \Re(s) \in (0, 1+d_\Psi), \tag{2.9}$$

where $d_\Psi = \sup\{u \geq 0 : \Psi_q(u) < 0\}$. If $\mathbb{E}[\xi_1] < 0$ then (2.9) extends to the case $q = 0$.

Remark 2.15. We note that it is part of the statement that all quantities are well-defined, e.g. the formal products $W_{\Phi_\pm(q,.)}$ are always well defined. The quantities $W_{\Phi_\pm(q,.)}$ can be viewed as natural extensions of the celebrated Gamma function. We refer to [15] for more information.

Remark 2.16. We stress that (2.9) provides an explicit computation of $\mathcal{M}_q$ in terms of the functions $\Phi_{pm}(q,s)$ thanks to the explicit form of $W_{\Phi_\pm(q,.)}$. This came as a surprise as previous efforts for establishing $\mathcal{M}_q$ always necessitated a specific structure for Ψ_q, see e.g. Theorem 2.9.

As an immediate corollary we can obtain the following general factorization of the law of I_q.

Corollary 2.17. *For $q > 0$ let ξ be any Lévy process and when $q = 0$ assume additionally that $\mathbb{E}[\xi_1] < 0$. Then, we have that*

$$I_q = I_{q^-}^{H^-} \times X_{\Phi_+(q,.)}, \tag{2.10}$$

where $q^- = \Phi_-(q,0)$ and H^- is the possibly killed descending ladder height of ξ, and $X_{\Phi_+(q,.)}$ is a random variable uniquely determined by its negative moments.

We refer to which can be found in [15, Corollary 2.2] and the discussion there for more detailed information on the presented factorization and its implications.

3. Some ideas for the proofs. Utilizing (1.4) then (2.4) could be rewritten as

$$\mathcal{M}_q(s+1) = -\frac{s}{\Psi_q(s)}\mathcal{M}_q(s) = \frac{s}{\Phi_+(q,-s)\Phi_-(q,s)}\mathcal{M}_q(s), \tag{3.1}$$

provided it holds at least on one imaginary line. We note that when $q > 0$ and $\Psi_q(z) = \Psi(z) - q$ is well-defined only for $z \in i\mathbb{R}$. In this case we can modify the Lévy measure in the following fashion $\Pi_\epsilon(dx)1_{\{x>0\}} = e^{-\epsilon x 1_{\{x>1\}}}\Pi(dx)1_{\{x>0\}}$ and $\Pi_\epsilon(dx)1_{\{x<0\}} = \Pi(dx)1_{\{x<0\}}$. Denoting $\Psi_q^\epsilon(z) = \Psi^\epsilon(z) - q$ we have that $\Psi_q^\epsilon(z)$ is defined at least on

$\{s : \Re(s) \in (0,\epsilon)\}$ and $\Psi_q^{\epsilon}(x) < 0$ at least for some interval $x \in (0, x_\epsilon)$ and thus (3.1) is valid at least on $\Re(s) \in (0, x_\epsilon)$. Since $\lim_{\epsilon\to 0} \Psi_q^{\epsilon}(z) = \Psi_q(z)$ for the possibly killed Lévy processes corresponding to $\Psi_q^{\epsilon}(z)$ we get $\xi^\epsilon \xrightarrow[\epsilon\to 0]{} \xi$ and all claims valid for ξ^ϵ and $\mathcal{M}_q^{\epsilon}$ follow after considerable but not too hard efforts. Therefore from now on in the metaproofs we assume that (3.1) holds.

Proof of Theorem 2.11. Splitting formally the equation (3.1) we get formally two auxiliary equations

$$\mathcal{M}_q^{(1)}(s+1) = \frac{s}{\Phi_-(q,s)} \mathcal{M}_q^{(1)}(s)\,; \quad \mathcal{M}_q^{(2)}(s+1) = \frac{-s}{-s\Phi_+(q,-s)} \mathcal{M}^{(2)}(s)\,. \tag{3.2}$$

One can show that when $\pi(x)$ is non-increasing on $x > 0$ that $\Psi^+(s) = -s\Phi_+(q,-s)$ is a Lévy-Khintchine exponent of unkilled Lévy process ξ^+ with no negative jumps such that $\mathbb{E}\left[\xi_1^+\right] < 0$ and under the assumption that (3.1) holds on some strip $\Re(s) \in (0, d_\Psi)$, where $d_\Psi = \sup\{u \geq 0 : \Psi_q(u) < 0\}$, we get that

$$\mathcal{M}^{\xi^+}(s+1) = \frac{-s}{\Psi^+(s)} \mathcal{M}^{\xi^+}(s)\,,\ \Re(s) \in (-\infty, d_\Psi), \tag{3.3}$$

where we have put $\mathcal{M}^{(2)}(s) = \mathcal{M}^{\xi^+}(s) = \mathbb{E}\left[\left(I^{\xi^+}\right)^{s-1}\right]$. Next, the equation

$$\mathcal{M}_{q^-}^{H^-}(s+1) = \frac{s}{\Phi_-(q,s)} \mathcal{M}_{q^-}^{H^-}(s) \tag{3.4}$$

is defined on $\Re(s) > 0$, where $q^- = \Phi_-(q,0)$ and $\mathcal{M}_q^{(2)}(s) = \mathcal{M}_{q^-}^{H^-}(s) = \mathbb{E}\left[\left(I_{q^-}^{H^-}\right)^{s-1}\right]$. To prove that

$$\mathcal{M}_q(s) = \mathcal{M}^{\xi^+}(s) \times \mathcal{M}_{q^-}^{H^-}(s) \tag{3.5}$$

we use the following steps: first we reduce to the case when $q = 0$, i.e. ξ is unkilled Lévy process via suitable transformation, see [14]; then from [12] we tackle this case via a detailed study of the stationary Ornstein-Uhlenbeck processes from which we conclude that I and $I^{\xi^+} \times I^{H^-}$ being both the stationary distributions of the same Ornstein-Uhlenbeck process have the same distribution. While it is true that we could deduce this theorem from our more general case below, the approach employed for this proof is very instructive and conforms to the chronology of the results obtained. $\square$

Proof of Theorem 2.14. In the general case $\Psi^+(s) = -s\Phi_+(q,-s)$ is not a Lévy exponent of a spectrally positive Lévy process and therefore we reformulate (3.3) as follows

$$\mathcal{M}^{\xi^+}(s+1) = \frac{1}{\Phi_+(q,s)} \mathcal{M}^{\xi^+}(s)\,,\ \Re(s) \in (-\infty, d_\Psi), \tag{3.6}$$

where $d_\Psi = \sup\{u \geq 0 : \Psi_q(u) < 0\}$. The two equations (3.6) (resp. (3.4)) can be solved in terms of infinite products of the type (2.8) on $(-\infty, d_\Psi)$ (resp. on $(0,\infty)$) which in turn can be extended to analytical functions on $\{s \in \mathbb{C} : \Re(s) < d_\Psi\}$ (resp. $\{s \in \mathbb{C} : \Re(s) > 0\}$). Thus, like in [15], emphasising that in the notation therein $\Phi_\pm(q,s) = \phi_\mp(s)$ we get that

$$\mathcal{M}^{\xi^+}(1-s) = C_1 W_{\Phi_+(q,.)}(s)\,; \quad \mathcal{M}_{q^-}^{H^-}(s) = C_2 \Phi_+(q,0) \frac{\Gamma(s)}{W_{\Phi_-(q,.)}(s)}.$$

It can be shown that $C_1 C_2 = \Phi_+(q,0)$. Since $\mathcal{M}^{\xi^+}(s) \times \mathcal{M}^{H^-}_{q^-}(s)$ solves (3.1) we will show that $\mathcal{M}_q(s) = \mathcal{M}^{\xi^+}(s) \times \mathcal{M}^{H^-}_{q^-}(s)$ provided we show uniqueness of the solution in the class of Mellin transforms of random variables. This can be done after proving that

$$\Phi_+(q,0)\left|\frac{\Gamma(s)\, W_{\Phi_+(q,.)}(1-s)}{W_{\Phi_-(q,.)}(s)}\right| \geq C_3 e^{-\pi|\Im(s)|},$$

when $s \in a + i\mathbb{R}, \Im(s) >> 1$, and $C_3 > 0$ is an absolute constant, and applying standard properties of the decay along imaginary lines of periodic analytical functions to show that

$$\frac{\mathcal{M}^{\xi^+}(s) \times \mathcal{M}^{H^-}_{q^-}(s)}{\mathcal{M}_q(s)} = H(s)$$

cannot be any other periodic function but the constant 1. This proves the theorem. □

Proof of Corollary 2.17. The proof of the corollary is immediate in view of the previous proof since therein we show that

$$\mathcal{M}_q(s) = \mathcal{M}^{\xi^+}(s) \times \mathcal{M}^{H^-}_{q^-}(s).$$

By uniqueness of the Mellin transforms of positive random variables we then conclude that

$$I_q \stackrel{d}{=} I^{\xi^+} \times I^{H^-}.$$ □

4. Exponential functional of Lévy process at fixed time. Recall that for a given Lévy process we have that $I_t = \int_0^t e^{\xi_s} ds$, $t > 0$. Noting that

$$\mathbb{P}(I_q \in dx) = q \int_0^\infty e^{-qt} \mathbb{P}(I_t \in dx)\, dt$$

we conclude that the law of $I_q, q > 0$ is the Laplace transform of the law of I_t. Therefore one could try to obtain information as to the law of I_t by inverting its Laplace transform. However, in generality this is not a trivial task at all. We refer to corollaries and discussions in [14] for particular cases when $\mathbb{P}(I_q \in dx)$ can be explicitly computed and yet from those expressions it seems impossible to analytically derive information for $\mathbb{P}(I_t \in dx)$ beyond the case of Brownian motion which has been tackled in [18]. However, all the information about I_q be its law or its Mellin transform can be used for numerically approximating the law of I_t. To a good deal of precision this has been carried out in [7] for the case when ξ belongs to the class of so-called meromorphic Lévy processes which have been introduced in Subsection 2.4.

It is curious to note that besides pricing of Asian options the applications for I_t are not numerous as opposed to I_q which appears in spectral theory, astrophycis, insurance mathematics, self-similarity, etc.

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ЕКСПОНЕНЦИАЛНИ ФУНКЦИОНАЛИ НА ПРОЦЕСИ НА ЛЕВИ

Пиер Пати, Младен Савов

В този доклад ще разгледаме историята и модерните резултати в областта на експоненциалните функционали на процеси на Леви. Тази област претърпя голямо развитие през последните години благодарение на усилията на няколко научни групи. Оказа се, че освен, че имат много приложения, експоненциалните функционали на Леви се нуждаят от техники от различни области на математиката за тяхното изучаване – комлексен анализ, специални функции и прочее. Основният резултат на авторите на този доклад е факторизацията на експоненциалния функционал като произведение на две независими по-лесни за изучаване случайни величини и представянето на трансформациите на Мелин на тези количества във вид на обобщено произведение на Вайерщрас. Това дава възможност за усилване на много резултати в областта.