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BOURGAIN ALGEBRAS OF SOME SUBALGEBRAS OF THE DISK ALGEBRA*

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Let ψ be a finite Blaschke product and $A\left(\bar{D}\right)$ be the disk algebra. In this paper we prove that the Bourgain algebra of $\psi A\left(\bar{D}\right)$ relative to $H^{\infty}(D)$ coincides with the algebra generated by the Blaschke products having only a finite number of singularities in the unit circle.

1. Introduction. Let Y be a commutative Banach algebra with an identity and let X be a linear subspace of Y. J. Cima and R. Timoney [1] introduced the notion of the Bourgain algebra based on ideas of J. Bourgain [2]. The Bourgain algebra $X_b = (X, Y)_b$ of X relative to $Y$is defined to the set of all $f \in Y$such that:

$$\text{if } f_n \to 0 \text{ weakly in } X, \text{ then } \operatorname{dist}(f.f_n, X) \to 0.$$

The distance, $\operatorname{dist}(f.f_n, X)$ between $f.f_n$ and X is the quotient norm of the coset $f.f_n + X$ in the space Y/X. The proof in [1] shows that X_b is a closed subalgebra of $Y$and contains the constant functions.

Let $H^{\infty}(D)$ be the Banach algebra of all bounded analytic functions on the open unit disk $D = \{z \in \mathrm{C} : |z| < 1\}$ with the supremum norm. The spectrum of $H^{\infty}(D)$ is the space $M(H^{\infty}(D))$ of all nonzero multiplicative linear functionals on endowed with weak-* topology. Then $M(H^{\infty}(D))$ is a compact Hausdorf space and Carleson's corona theorem says that D is dense in $M(H^{\infty}(D))$ [3]. By considering boundary functions on $T = \partial D$, we can consider that $H^{\infty}(D) = H^{\infty}(T)$ is an essentially supremum-norm closed subalgebra of $L^{\infty} = L^{\infty}(T)$.

Let $C\left(\bar{D}\right)$ be the space of all continuous functions on the closed unit disk $\bar{D}$ and let $A\left(\bar{D}\right) = H^{\infty}(D) \cap C\left(\bar{D}\right)$ denote the disk algebra, i.e. the algebra of all continuous functions on $\bar{D}$ which are analytic on D. There are various alternative descriptions of $A\left(\bar{D}\right)$. For example, $A\left(\bar{D}\right)$ is the uniform closure in $C\left(\bar{D}\right)$ of the polynomials, also consists of the continuouse functions on the unit circle whose Fourier coefficients vanish on the negative integers. Every $\lambda \in \bar{D}$ determines the evaluation homomorphism $\phi_{\lambda} \in M\left(A\left(\bar{D}\right)\right)$ defined by

$$\phi_{\lambda}(f) = f(\lambda), \text{ for every } f \in A\left(\bar{D}\right).$$

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The correspondence $\lambda \to \phi_\lambda$ embeds the closed unit disk $\bar{D}$ as a closed subset of $M\left(A\left(\bar{D}\right)\right)$. Suppose $\phi \in M\left(A\left(\bar{D}\right)\right)$ is arbitrary and $\lambda = \phi\left(z\right)$, where z is the coordinate function. Then ϕ coincides with ϕ_λ. Consequently the spectrum of $A\left(\bar{D}\right)$ coincides whit $\bar{D}$ [3].

We denote the space of continuous functions on the unit circle T by $C = C\left(T\right)$. In [4] J. Cima, Sv. Janson and K. Yale showed that the Bourgain algebra H_b^∞ of $H^\infty\left(T\right)$ relative to $L^\infty = L^\infty\left(T\right)$ is $H^\infty\left(T\right) + C$. K. Izuchi [5] proved that the Bourgain algebra of a closed subalgebra between disk algebra $A\left(T\right) = H^\infty\left(T\right) \cap C$ and H^∞ relative L^∞ is always contained in $H^\infty\left(T\right) + C$. Some results for Bourgain algebras of subalgebras at $H^\infty(D)$ are proven in [6, 7].

In [8] Cima, Stroethoff and Yale show that $\left(A\left(\bar{D}\right), H^\infty\left(D\right)\right)_b$ contains every Blaschke product whose zeros cluster only at a finite number of points, i.e. which have only a finite number of singularities. J. Cima and R. Mortini prove that $\left(A\left(\bar{D}\right), H^\infty\left(D\right)\right)_b$ is the algebra B generated by the Blaschke products having only a finite number of singularities [9].

In this paper we prove that this is true and for the Bourgain algebra of $\psi A\left(D\right)$ where ψ is a finite Blaschke product. If ψ have only a finite number of singularities we prove that $\left(\psi A\left(\bar{D}\right), H^\infty\left(D\right)\right)_b \subset B$.

2. Preliminaries. A sequence $\left\{z_n\right\}_n$ in D is called interpolating if for every bounded sequence $\left\{a_n\right\}_n$ of complex numbers, there is a function $f \in H^\infty(D)$ such that $f\left(z_n\right) = a_n$ for all n. For a sequence $\left\{z_n\right\}_n$ in D with $\sum\limits_{n=1}^{\infty}\left(1 - \left|z_n\right|\right) < \infty$, the function:

$$B\left(z\right) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{\left|z_n\right|} \frac{z - z_n}{1 - \bar{z}_n z}, \quad z \in D,$$

is called a Blaschke product with zeros $\left\{z_n\right\}_n$. If $\left\{z_n\right\}_n$ is an interpolating sequence, then $B\left(z\right)$ is also called interpolating. More information about the interpolating Blaschke product can be found in [10, 11, 12].

Lemma 2.1 [3]**.** *If $\left\{z_n\right\}_n \subset D$ is interpolating sequence, then there exist functions $\left\{f_n\right\}_n \subset H^\infty\left(D\right)$ and positive number M such that $f_n\left(z_n\right) = 1$ for all n, $f_n\left(z_k\right) = 0$ for $n \neq k$ and $\sum\limits_{n=1}^{\infty}\left|f_n\left(z\right)\right| \leq M$ for $z \in D$.*

Lemma 2.2 [4]**.** *Suppose that $\left\{f_n\right\}_n$ is a sequence in $H^\infty\left(D\right)$ such that $\sum\limits_{n=1}^{\infty}\left|f_n\left(z\right)\right| \leq M$ for $z \in D$. Then $f_n \to 0$ weakly in $H^\infty\left(D\right)$.*

Let ψ be an inner function, i.e. $\psi \in H^\infty\left(D\right)$ such that $\left|\psi\left(e^{i\theta}\right)\right| = 1$ almost everywhere on T. Then ψ has the form: $\psi\left(z\right) = B\left(z\right).S\left(z\right)$, $z \in D$ where $B\left(z\right)$ is a Blaschke product

$$B\left(z\right) = \prod_{n=1}^{\infty} \frac{-\bar{a}_n}{\left|a_n\right|} \frac{z - a_n}{1 - \bar{a}_n z}$$

with zeros $\left\{a_n\right\}_n \subset D$, and

$$S\left(z\right) = \exp\left[-\int\limits_T \frac{\lambda + z}{\lambda - z} d\mu\left(\lambda\right)\right],$$

where μ is a finite, nonnegative measure on T, singular with respect to Lebesgue measure.

The support of ψ is the set of points $\lambda \in T$ for which there is a sequence $\{z_n\}_n \subset D$ of points such that $z_n \to \lambda$ and $\psi(z_n) \to 0$. This set, denoted by $\operatorname{supp}\psi$, is known to be the union of the support of the measure μ and the cluster set of the sequence $\{a_n\}_n$ [3]. Note that if ψ is a Blaschke product, then $\operatorname{supp}\psi$ coincides with the cluster set of $\{a_n\}_n$. If $\operatorname{supp}\psi$ is a finite set, then ψ is called a Blaschke product with a finite number of singularities.

As usual, the cluster set $\mathrm{Cl}(f,\xi)$ of a function $f \in H^\infty(D)$ at a point $\xi \in T$ is defined to be the set of all points $\omega \in \mathrm{C}$ for which there exists a sequence $\{z_n\}$ in D converging to ξ such that $f(z_n) \to \omega$. Let $\mathcal{A}$ denote the set:

$$\mathcal{A} = \{f \in H^\infty\text{: for every } \varepsilon > 0 \text{ the set } \{\xi \in T : \mathrm{diamCl}(f,\xi) \geq \varepsilon\} \text{ is finite}\},$$

where, as usual, $\operatorname{diam} E = \sup\{|a-b| : a, b \in E\}$ is the diameter of a bounded subset E of C. Let B be the algebra generated by the set of Blaschke products which have a finite number of singularities. In [9] it is shown that $\mathcal{A}$ is a closed sualgebra of $H^\infty(D)$ and that $\mathcal{A} = B$.

3. The main result.

Lemma 3.1. *If $f \in H^\infty(D)$ and $\xi \in T$, then:*

1) $\mathrm{diamCl}(f,\xi) \leq 2\|f\|$;

2) $\mathrm{diamCl}(f,\xi) = \mathrm{diamCl}(f-g,\xi)$, *where $g \in H^\infty(D)$ and $\lim\limits_{z\to\xi} g(z) = \alpha \in \mathrm{C}$;*

3) $\mathrm{diamCl}(fg,\xi) = \mathrm{diamCl}(f,\xi)$, *where $g \in H^\infty(D)$ and $\lim\limits_{z\to\xi} g(z) = \beta$, $|\beta| = 1$.*

Proof. 1) Let ω_1 and ω_2 belong to $\mathrm{Cl}(f,\xi)$ and $\varepsilon > 0$. There exist two sequences $\{z'_n\}_n$, $\{z''_n\}_n$ in D such that $z'_n \to \xi$, $z''_n \to \xi$ and $f(z'_n) \to \omega_1$, $f(z''_n) \to \omega_2$. Therefore we can find n_0 such that for $n > n_0$ is fulfilled $|f(z'_n) - \omega_1| < \varepsilon$ and $|f(z''_n) - \omega_2| < \varepsilon$. Then for every $n > n_0$ we have:

$$|\omega_1 - \omega_2| \leq |\omega_1 - f(z'_n)| + |\omega_2 - f(z''_n)|\text{''} + |f(z'_n)| + |f(z''_n)| < 2\varepsilon + 2\|f\|,$$

$$\text{i.e. } \mathrm{diamCl}(f,\xi) = \sup\{|\omega_1 - \omega_2| : \omega_1, \omega_2 \in \mathrm{Cl}(f,\xi)\} \leq 2\|f\|.$$

2) We have the following equivalences:

$\omega \in \mathrm{Cl}(f,\xi) \Leftrightarrow$ there exists a sequence $\{z_n\}_n$, $z_n \to \xi$ such that $f(z_n) \to \omega \quad \Leftrightarrow$ there exists a sequence $\{z_n\}_n$, $z_n \to \xi$ such that $(f-g)(z_n) \to \omega - \alpha \quad \Leftrightarrow \omega - \alpha \in \mathrm{Cl}(f-g,\xi)$.

3) Therefore $\mathrm{diamCl}(f,\xi) = \mathrm{diamCl}(f-g,\xi)$, because $\mathrm{Cl}(f-g,\xi)$ is the translation of $\mathrm{Cl}(f,\xi)$ determined by the vector α.

We have the following equivalences:

$\omega \in \mathrm{Cl}(f,\xi) \Leftrightarrow$ there exists a sequence $\{z_n\}_n$, $z_n \to \xi$ such that $f(z_n) \to \omega \quad \Leftrightarrow$ there exists a sequence $\{z_n\}_n$, $z_n \to \xi$ such that $(f.g)(z_n) \to \beta.\omega \quad \Leftrightarrow \beta.\omega \in \mathrm{Cl}(f.g,\xi)$.

Therefore $\mathrm{diamCl}(f,\xi) = \mathrm{diamCl}(f.g,\xi)$, because $\mathrm{Cl}(f.g,\xi)$ is the rotation of the set $\mathrm{Cl}(f,\xi)$ determined by $\arg\beta$. $\square$

Theorem 3.2. *If ψ is a finite Blaschke product, then*

$$\left(\psi.A\left(\bar{D}\right), H^\infty(D)\right)_b = B = \mathcal{A},$$

i.e. the Bourgain of the algebra $\psi.A\left(\bar{D}\right)$ with respect to $H^\infty(D)$ is generated by the set of Blaschke products which have a finite number of singularities.

Proof. If $f \notin \mathcal{A}$, then there exist $\varepsilon > 0$ and a sequence $\{\xi_n\}_n \subset T$ such that $\mathrm{Cl}(f,\xi_n) \geq \varepsilon$ for all n. Without loss of generality we can consider that $\xi_n \to \xi \in T$

and $\xi_n \neq \xi$ for every n. As in [3] and [8] there exist functions $f_n \in A\left(\bar{D}\right)$ such that $f_n(\xi_n) = 1$ and $\sum\limits_{n=1}^{\infty} |f_n(z)| \leq 2$ for all $z \in D$. By Lemma 2.2 we obtain that $f_n \to 0$ weakly in $A(D)$.

Therefore for the sequence $\{\psi f_n\}_n \subset \psi A\left(\bar{D}\right)$ we have $(\psi f_n)(\xi_n) = \psi(\xi_n)$ for all n, and $\sum\limits_{n=1}^{\infty} |(\psi f_n)(z)| \leq 2$ for every $z \in D$. By Lemma 2.2 (with $\psi A\left(\bar{D}\right)$ instead of $H^\infty(D)$) it follows that $\psi f_n \to 0$ weakly in $\psi A\left(\bar{D}\right)$. Let $\{\psi g_n\}_n \subset \psi A\left(\bar{D}\right)$. Then by Lemma 3.1. we have that for every n:

$$2\left\|f.\psi f_n - \psi g_n\right\| \geq \mathrm{diamCl}(f.\psi f_n - \psi g_n, \xi_n) = \mathrm{diamCl}(f.\psi f_n, \xi_n) = \mathrm{diamCl}(f, \xi_n) \geq \varepsilon,$$

because $\lim\limits_{z\to\xi_n} (\psi f_n)(z) = \psi(\xi_n) f_n(\xi_n) = \psi(\xi_n)$ and $|\psi(\xi_n)| = 1$ for every n. Thus $f \notin \left(\psi A\left(\bar{D}\right), H^\infty(D)\right)_b$ and we obtain $\left(\psi A\left(\bar{D}\right), H^\infty(D)\right)_b \subset \mathcal{A}$.

Let f be a Blaschke product with a finite number of singularities $\xi_1, \xi_2, \ldots, \xi_k$ on T. Without loss of generality we can consider that $k = 1$, $\xi_1 = 1$ and $\|f\| \leq 1$. If $\psi f_n \to 0$ weakly in $\psi A\left(\bar{D}\right)$ then $\psi f_n \to 0$ weakly in $A\left(\bar{D}\right)$, because $\psi A\left(\bar{D}\right) \subset A\left(\bar{D}\right)$. Since $M\left(A\left(\bar{D}\right)\right) = \bar{D}$ we obtain that $\psi(1).f_n(1) = \phi_1(\psi.f_n) \to 0$, where $\phi_1 \in M\left(A\left(\bar{D}\right)\right)$ is the point evaluation $\phi_1(g) = g(1)$ for every $g \in A\left(\bar{D}\right)$. But $\psi(1) \neq 0$ and we have $f_n(1) \to 0$. Let $\varepsilon > 0$. Then by exactly the same arguments as in [8] there exists a sequence $\{h_n\}_n \subset A\left(\bar{D}\right)$ such that $\|f.f_n - h_n\| \leq \varepsilon$ for large enough n. Since $|\psi(z)| < 1$ for every $z \in D$ we see that

$\mathrm{dist}\left(f.\psi f_n, \psi A\left(\bar{D}\right)\right) \leq \|f.\psi f_n - \psi g_n\| \leq \|f.f_n - g_n\| \leq \varepsilon$, i.e. $f \in \left(\psi.A\left(\bar{D}\right), H^\infty(D)\right)_b$. Since $\left(\psi A\left(\bar{D}\right), H^\infty(D)\right)_b$ is a closed algebra, we obtain that $B \subset \left(\psi A\left(\bar{D}\right), H^\infty(D)\right)_b$.

The theorem is proved. □

Remark. Let ψ be a Blaschke product, with a finite number of singularities $E = \{\eta_1, \eta_2, \ldots, \eta_k\}$ on T. If $f \notin \mathcal{A}$, then there exist $\varepsilon > 0$ and a sequence $\{\xi_n\}_n \subset T$ such that $\mathrm{Cl}(f, \xi_n) \geq \varepsilon$ for all n. Without loss of generality we can consider that $E \cap \{\xi_n\}_n = \emptyset$, $\xi_n \to \xi \in T$ and $\xi_n \neq \xi$ for every n. Note that then ψ extends to be continuous on $\bar{D}\backslash E$ and therefore in ξ_n with $|\psi(\xi_n)| = 1$ for every n. As in Theorem 3.2. there exists a sequence of functions $\{f_n\}_n \subset A\left(\bar{D}\right)$ such that $f_n(\xi_n) = 1$, $\psi f_n \to 0$ weakly in $\psi A\left(\bar{D}\right)$ and

$$2\left\|f.\psi f_n - \psi g_n\right\| \geq \mathrm{diamCl}(f.\psi f_n - \psi g_n, \xi_n) = \mathrm{diamCl}(f.\psi f_n, \xi_n) = \mathrm{diamCl}(f, \xi_n) \geq \varepsilon$$

for every n, where $\{\psi g_n\}_n \subset \psi A\left(\bar{D}\right)$. Consequently $\left(\psi.A\left(\bar{D}\right), H^\infty(D)\right)_b \subset \mathcal{A}$ and when ψ is a Blaschke product, with a finite number of singularities.

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АЛГЕБРИ НА БУРГЕН НА НЯКОИ ПОДАЛГЕБРИ НА ДИСК АЛГЕБРАТА

Мирослав Колев Христов

Нека ψ е крайно произведение на Блашке и $A\left(\bar{D}\right)$ е диск алгебрата. В тази работа ние доказваме, че алгебрата на Бурген на $\psi A(D)$ относно $H^{\infty}(D)$ съвпада с алгебрата, породена от произведенията на Блашке, които имат само краен брой особени точки върху единичната окръжност.