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PARAMETER ESTIMATION FOR THE DISCRETE KOLMOGOROV POPULATION DYNAMICS SYSTEM*

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In this paper a discrete modification of the population dynamics Kolmogorov system (n-competing species problem) is considered. An exact formula about statistical estimate for the parameters of the system by means of the least squares method is presented.

Introduction. The differential system of logistic type for collaboration or concurrence among n species (or among n resources) is given by the equations (e.g. [1], [2], [3])

$$\begin{array}{l} \dot{x}_1 = x_1 \left(\alpha_1 + \beta_{11} x_1 + \beta_{12} x_2 + \cdots + \beta_{1n} x_n \right) \\ \dot{x}_2 = x_2 \left(\alpha_2 + \beta_{21} x_1 + \beta_{22} x_2 + \cdots + \beta_{2n} x_n \right) \\ \cdots \\ \dot{x}_n = x_n \left(\alpha_n + \beta_{n1} x_1 + \beta_{n2} x_2 + \cdots + \beta_{nn} x_n \right) \end{array} \tag{1}$$

where the positive variables x_k, $k = 1, 2, \ldots, n$, describe the amount of the corresponding species, the parameters α_k and β_{kk} are connected with the self-growth and the inner interaction, and the parameters β_{ij}, $i \neq j$, describe the state of the interaction among the species. Usually we have $\alpha_k > 0$ and $\beta_{kk} < 0$ which defines a logistic type dynamic for the separate species with missing external effects. If $\beta_{ij} > 0 \quad (\beta_{ij} < 0)$, $i \neq j$, then the particular species j effects positively (negatively) on the growth of the particular species i. In the typical case system (1) has a stable limit behavior.

System (1) has a discrete analogue

$$\begin{array}{l} x_1^{(t)} = x_1^{(t-1)} + x_1^{(t-1)} \left(\alpha_1 + \beta_{11} x_1^{(t-1)} + \beta_{12} x_2^{(t-1)} + \cdots + \beta_{1n} x_n^{(t-1)} \right) \\ x_2^{(t)} = x_2^{(t-1)} + x_2^{(t-1)} \left(\alpha_2 + \beta_{21} x_1^{(t-1)} + \beta_{22} x_2^{(t-1)} + \cdots + \beta_{2n} x_2^{(t-1)} \right) \\ \cdots \\ x_n^{(t)} = x_n^{(t-1)} + x_n^{(t-1)} \left(\alpha_n + \beta_{n1} x_1^{(t-1)} + \beta_{n2} x_2^{(t-1)} + \cdots + \beta_{nn} x_n^{(t-1)} \right) \end{array} \tag{2}$$

where $x_k^{(t)}$ defines the amount of the species k at the moment t. Assume that we are given the data with observations at the moments $t = 1, 2, \ldots, N$. Then we have an opportunity to check out the model validity of (2). The parameters α_k and β_{ij} will be estimated according to the data by means of the least squares method.

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Parameter estimation . Introduce the notations

$$x_t = \begin{pmatrix} x_1^{(t)} \\ x_2^{(t)} \\ \vdots \\ x_n^{(t)} \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{pmatrix},$$

$$L_t = \begin{pmatrix} x_1^{(t)} & 0 & \cdots & 0 \\ 0 & x_2^{(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n^{(t)} \end{pmatrix}.$$

Then the system (2) can be written in the concise form

(3) $$x_t = x_{t-1} + L_{t-1}\left(\alpha + Bx_{t-1}\right)$$

and after the setting

$$y_t = \begin{pmatrix} 1 \\ x_t \end{pmatrix}, \quad B = (\alpha\,\mathrm{B}) = \begin{pmatrix} \alpha_1 & \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \alpha_2 & \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_n & \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{pmatrix}$$

the system (3) accepts the form

(4) $$x_t = x_{t-1} + L_{t-1} B y_{t-1}.$$

The least squares estimation is connected with the minimization of the following cost function

$$\begin{aligned} \varphi &= \sum_{t=2}^{N} \left|x_t - x_{t-1} - L_{t-1}By_{t-1}\right|^2 \\ &= \sum_{t=2}^{N} \left(x_t - x_{t-1} - L_{t-1}By_{t-1}\right)^T \left(x_t - x_{t-1} - L_{t-1}By_{t-1}\right), \\ \varphi &= \sum_{t=2}^{N} \left(L_{t-1}^{-1}\left(x_t - x_{t-1}\right) - By_{t-1}\right)^T L_{t-1}^T L_{t-1} \left(L_{t-1}^{-1}\left(x_t - x_{t-1}\right) - By_{t-1}\right), \end{aligned}$$

which after the setting

$$S_t = L_t^T L_t, \quad ret_t = L_{t-1}^{-1}\left(x_t - x_{t-1}\right) = \begin{pmatrix} \left(x_1^{(t)} - x_1^{(t-1)}\right)\Big/ x_1^{(t-1)} \\ \left(x_2^{(t)} - x_2^{(t-1)}\right)\Big/ x_2^{(t-1)} \\ \vdots \\ \left(x_n^{(t)} - x_n^{(t-1)}\right)\Big/ x_n^{(t-1)} \end{pmatrix},$$

accepts the form

(5) $$\varphi = \sum_{t=2}^{N} \left(ret_t - By_{t-1}\right)^T S_{t-1} \left(ret_t - By_{t-1}\right).$$

Before going further let us solve the following auxiliary task. Suppose we are given the vectors $a_{(k)} \in R^p$, $b_{(k)} \in R^q$ and the symmetric positively defined $(p \times p)$ matrices D_k, $k = 1, 2, \ldots, K$. We are looking for a $(p \times q)$ matrix A which minimizes the function

$$\psi = \sum_{k=1}^{K} \left(a_{(k)} - Ab_{(k)}\right)^T D_k \left(a_{(k)} - Ab_{(k)}\right).$$

Using straightforward calculations one can find that the matrix derivative $\frac{\partial \psi}{\partial A}$ is given by the formula

$$\frac{1}{2}\frac{\partial \psi}{\partial A} = -\sum_{k=1}^{K} D_k a_{(k)} b_{(k)}^T + \sum_{k=1}^{K} D_k A b_{(k)} b_{(k)}^T.$$

To find A we shall annul the gradient by solving the matrix equation $\frac{\partial \psi}{\partial A} = 0$ which leads to

$$\sum_{k=1}^{K} D_k a_{(k)} b_{(k)}^T = \sum_{k=1}^{K} D_k A b_{(k)} b_{(k)}^T.$$

One can transform the last by means of the vectorization $vec\,(\cdot)$ and the Kronecker product $\otimes$ (e.g. [4], [5]). Then it turns into the form

$$\left(\sum_{k=1}^{K} \left(\left(b_{(k)} a_{(k)}^T\right) \otimes D_k\right)\right) vec\,(E) = \left(\sum_{k=1}^{K} \left(\left(b_{(k)} b_{(k)}^T\right) \otimes D_k\right)\right) vec\,(A)\,,$$

which has a solution

$$vec\,(A) = \left(\sum_{k=1}^{K} \left(\left(b_{(k)} b_{(k)}^T\right) \otimes D_k\right)\right)^{-1} \left(\sum_{k=1}^{K} \left(\left(b_{(k)} a_{(k)}^T\right) \otimes D_k\right)\right) vec\,(E)\,. \tag{6}$$

Now we are able to apply the result of (6) immediately to the cost function defined in (5) and in this way to prove the validity of the following theorem.

Theorem. *Let we are given the data* $\left(x^{(t)}\right)$. *Then the least squares estimate for the parameters*

$$B = \begin{pmatrix} \alpha_1 & \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \alpha_2 & \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_n & \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{pmatrix}$$

of the system (2) *is obtained from the formula*

$$vec\,(B) = \left(\sum_{t=2}^{N} \left(\left(y_{t-1} y_{t-1}^T\right) \otimes S_{t-1}\right)\right)^{-1} \left(\sum_{t=2}^{N} \left(\left(y_{t-1} ret_t^T\right) \otimes S_{t-1}\right)\right) vec\,(E) \tag{7}$$

where

$$y_t = \begin{pmatrix} 1 \\ x_t \end{pmatrix}, \quad S_t = L_t^T L_t, \quad L_t = \begin{pmatrix} x_1^{(t)} & 0 & \cdots & 0 \\ 0 & x_2^{(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n^{(t)} \end{pmatrix}, \quad ret_t = L_{t-1}^{-1}\left(x_t - x_{t-1}\right).$$

and E is the corresponding identity matrix.

Example. Obviously the model (2) can be applied to the dynamics of items with various background (different from biological species).

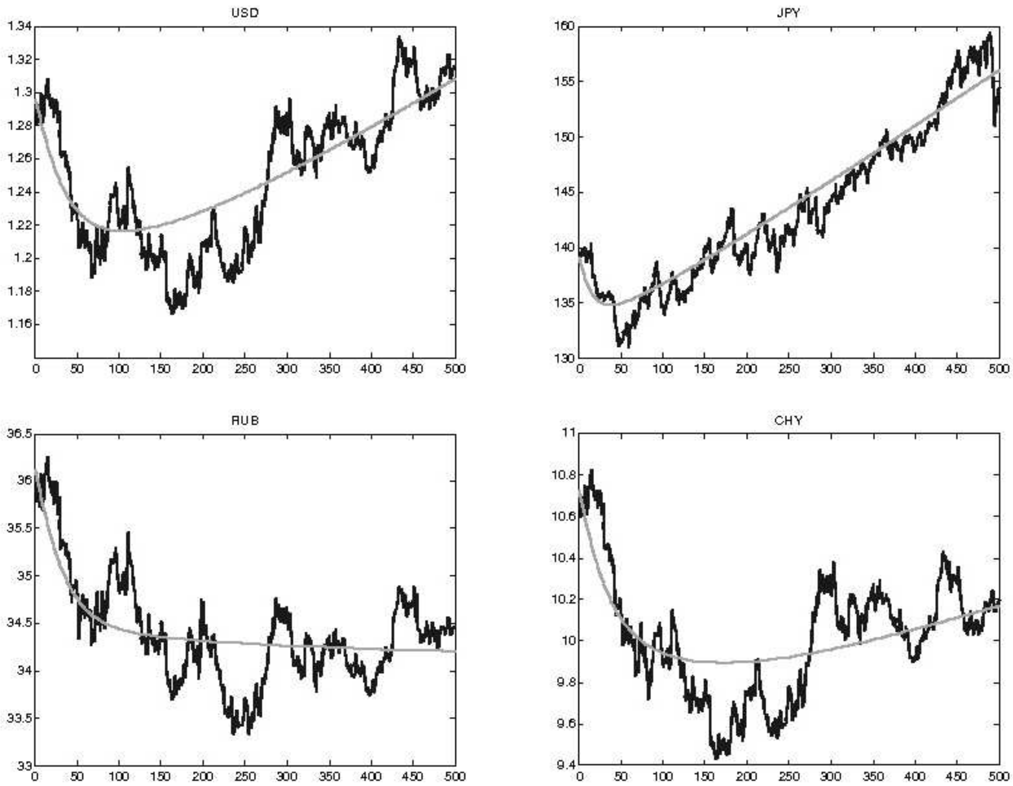

Fig. 1. Data vs model values plots about four currencies (500 days period)

We apply our model to financial data – four currencies: USD (United States Dollar), JPY (Japan Yen), RUB (Russian Ruble) and CHY (China Yuan). Initially we shall use data from the Central European Bank for 500 working days (approximately 2 years) starting from 1 April 2005 and ending in 14 March 2007. This period is characterized by a relatively stable USD, robust global economy without intensive military conflicts. In this case the model looks adequate to the data (see Figure 1). The estimated parameters are given below.

Table 1. The values of the estimated parameters (500 days period)

	$\alpha_\bullet$	$\beta_{\bullet 1}$	$\beta_{\bullet 2}$	$\beta_{\bullet 3}$	$\beta_{\bullet 4}$
USD	0.0244	-0.0341	0.0001	-0.0008	0.0027
JPY	0.0775	0.0909	-0.0004	-0.0017	-0.0080
RUB	0.0216	-0.0261	0.0001	-0.0008	0.0027
CHY	0.0284	-0.0169	0.0001	-0.0008	0.0010

The signs say that the RUB rivals to all the rest while the JPY cooperate with them. Also the CHY shows a positive inner push ($\beta_{44} = 0.001$) which is a possible indicator to an instable growth.

Now we shall increase the data period until the model looks to fit the data well enough (see Figure 2).

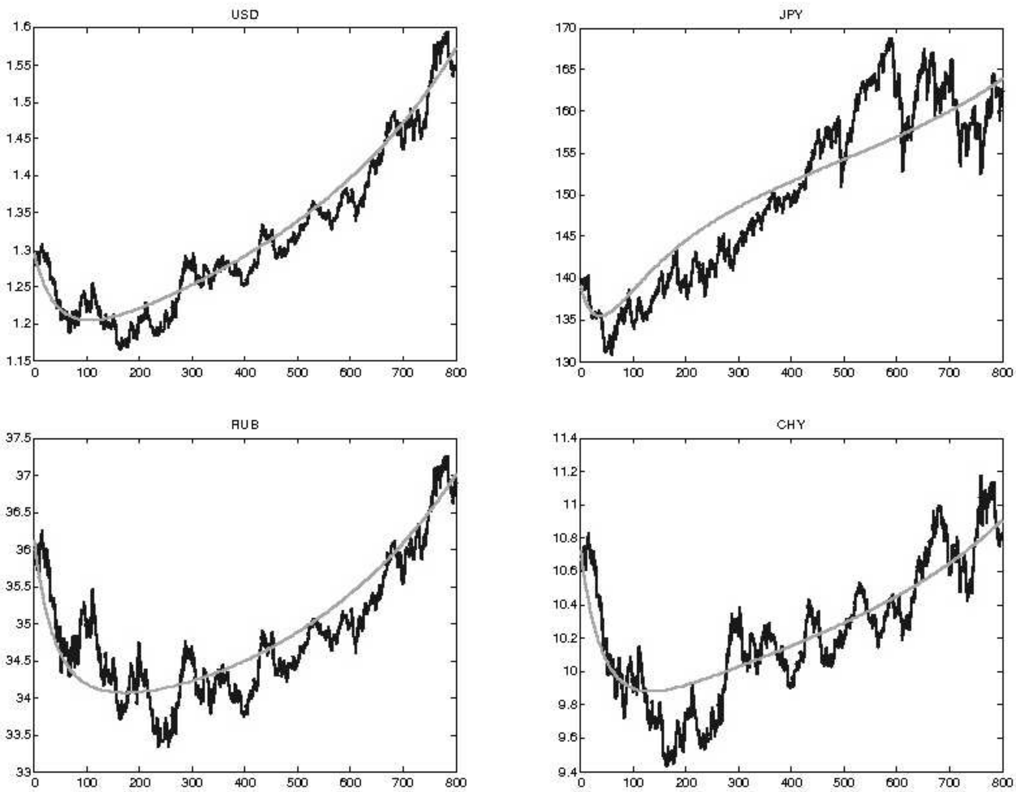

Fig. 2. Data vs model values plots about four currencies (800 days period)

It includes 800 working days starting again from 1 April 2005 and ending in 20 May 2008. This period ends around the beginning of another crisis. The estimated parameters are given below.

The USD now shows a positive inner push ($\beta_{11} = 0.0138$) and meets the rival of the RUB and CHY. The impact of the JPY is minor.

Finally let us include two another currencies: GBP (Great Brittan Pound) and CHF (Swiss Franc) for the same 800 days period.

The USD confirms either the positive inner push ($\beta_{11} = 0.0183$) and the rivalry almost of all the rest but the CHF, which cooperates with all.

The next period faces with a very poor model data fit. More detailed explanation of the results stays beyond of the scope of the economical skills of the authors.

Conclusions. The model applicability depends essentially on the relative stability

Table 2. The values of the estimated parameters (800 days period)

	$\alpha_\bullet$	$\beta_{\bullet 1}$	$\beta_{\bullet 2}$	$\beta_{\bullet 3}$	$\beta_{\bullet 4}$
USD	0.0295	0.0138	0.0000	-0.0005	-0.0029
JPY	0.0581	0.0241	-0.0001	-0.0016	-0.0015
RUB	0.0248	0.0110	-0.0000	-0.0007	-0.0014
CHY	0.0308	0.0130	0.0000	-0.0006	-0.0027

Table 3. The values of the estimated parameters (800 days period, six currencies)

	$\alpha_\bullet$	$\beta_{\bullet 1}$	$\beta_{\bullet 2}$	$\beta_{\bullet 3}$	$\beta_{\bullet 4}$	$\beta_{\bullet 5}$	$\beta_{\bullet 6}$
USD	0.0209	0.0183	-0.0001	-0.0015	0.0136	-0.0007	-0.0032
JPY	0.0778	0.0538	-0.0003	-0.0383	0.0191	-0.0018	-0.0043
GBP	0.0443	0.0441	-0.0002	-0.0461	0.0133	-0.0004	-0.0046
CHF	0.0356	0.0080	0.0000	-0.0125	-0.0185	-0.0003	-0.0004
RUB	0.0087	0.0105	-0.0001	0.0060	0.0143	-0.0008	-0.0012
CHY	0.0270	0.0192	-0.0001	-0.0052	0.0113	-0.0007	-0.0032

of the macro environment. The important model information is hidden in the signs and the magnitudes of the estimated parameters. One additional option of the model is the existence of limit amount values for the species, which are obtained from the formula $x_{\lim} = -B^{-1}\alpha$. Conceivably these values may be used for prediction purposes but our experiments show that they appear non-realistic.

Perhaps usual environment along with a positive USD inner push and total rivalry of the rest may serve as a specific indicator for the soon upcoming "up-down" trend change.

The model described permits generalization by adding another autoregressive terms

$$x_t = x_{t-1} + L_{t-1}\left(\alpha + B_1 x_{t-1} + B_2 x_{t-2} + \cdots + B_p x_{t-p}\right)$$

but probably it does not lead to the efficient improvement of the basic model (3).

REFERENCES

[1] A. Tineo. An iterative scheme for the n-competing species problem. *J. Differential Equations*, **116**, No 1 (1995), 1–15.

[2] F. Zanolin. Permanence and positive periodic solutions for Kolmogorov competing species systems. *Results Math.*, **21**, No 1–2 (1992), 224–250.

[3] S. Ahmad, A. Lazer. On the nonautonomous n-competing species problems. *Appl. Anal.*, **57**, No 3-4 (1995), 309–323.

[4] J. Hamilton. Time series analysis. Princeton, NJ, Princeton University Press, 1994.

[5] H. Lütkepohl. New introduction to multiple time series analysis. Berlin, Springer-Verlag, 2005.

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ОЦЕНКА НА ПАРАМЕТРИТЕ ЗА ДИСКРЕТНА СИСТЕМА НА КОЛМОГОРОВ ОТ ПОПУЛАЦИОННАТА ДИНАМИКА

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В тази статия се разглежда дискретна модификация на система на Колмогоров от популационната динамика (система за състезание между n-вида). Представена е точна формула за статистическа оценка на параметрите на системата посредством метода на най-малките квадрати.