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HOW NOT TO FAIL PROVING CEVA'S THEOREM*

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Vectors can be extremely effective in proving theorems and carrying out calculations in plane geometry. Their current use is limited due to vector arithmetic being underdeveloped and vector-based problem solving methods almost non-existing. In previous publications we have drawn attention to the utility of employing vectors as a natural computational tool for plane geometry. We continue exploring the topic by presenting a more extensive example: proofs of the classic Ceva's and Menelaus' theorems, along with a number of others. Ceva's theorem is particularly worth considering as, amazingly, its proofs, including those by most respected authors, are much more often flawed than not.

Introduction. Vectors are a natural and powerful tool for calculating in geometry, but a neglected one. Neglection is especially apparent in plane geometry, where an adequate, practical vector apparatus is generally lacking and vector-based methods of proving theorems and solving problems are only rudimentary. In previous publications [1, 2] we have marked some ways in which vectors can be profitably employed to do calculations and solve problems in plane geometry, illustrating them by a number of small examples. Here we work out a more detailed example: a series of small theorems from the elementary geometry, related to cevians and similar configurations, including Ceva's, Menelaus', Routh's and other theorems, mostly well known, perhaps with a few exceptions.

Ceva's theorem and, to some extent, that of Menelaus, are good examples to point at because, although a very popular topic in elementary geometry, too often they are given incorrect proofs. Ceva's theorem is a simple statement about simple objects, yet it is amazing in how many ways its formulation and proof turn out to be challenging.

H. Coxeter – both in *Introduction to geometry* and in *Geometry revisited*, D. Pedoe – in *Geometry, a comprehensive course*, I. Sharygin – in an article discussing precisely Ceva's and Menelaus' theorems, and many others indeed present flawed proofs of Ceva's theorem. Theorem formulations themselves are at times erroneous or unclear or unnecessarily limited. The same holds for most school, college and higher-level textbooks on plane geometry. Not surprisingly then, online reference sources, such as *Wikipedia*, *Cut The Knot!*, *Art of Problem Solving*, and *PlanetMath* all currently (November 2014) contain flawed – in more than one way! – 'proofs' of Ceva's theorem.

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We believe that this unfortunate situation should be attributed to the imperfection of the traditional so called synthetic language of geometry which makes it too easy for imprecise, vague, and plain wrong reasoning to sneak in the mathematical literature. Ceva's theorem just happens to be one of the foci where this imperfection shows up clearly. An account of what can be incorrect in proving Ceva's theorem is beyond the scope of this paper, but for a correct synthetic proof and pointers to some of the pitfalls thus avoided one can refer e. g. to [3].

Our purpose here is to show how an analytic method – that of calculating with vectors – leads to proofs of Ceva's and a number of other theorems that are not only correct, but also straightforward, general, and short.

In the following, bold letters are used to denote vectors, all vectors, and only vectors. A position vector is named by a capital letter – the name of the respective point. A juxtaposition of such letters denotes the vector, represented by the directed line segment between the two points. Small letters denote vector values without relation to representing points.

Throughout the text we make heavy use of the area product of planar vectors, denoted by $\times$, analogously to the cross product of spatial vectors, with which it shares many properties. Also used is the 'perp' operation ($^{\perp}$) which rotates a vector at a right angle counterclockwise. The most essential properties and applications of $\times$ and $^{\perp}$ are discussed in [1, 2].

Proofs of propositions are enclosed in ⌜ and ⌟.

Basic calculations with vectors. For any three vectors $\mathbf{p}$, $\mathbf{u}$ and $\mathbf{v}$ where $\mathbf{u}\times\mathbf{v}\neq 0$, $\mathbf{p}$ resolves with respect to $\mathbf{u}$ and $\mathbf{v}$ [1]:

$$\mathbf{p} \quad = \quad \frac{(\mathbf{p}\times\mathbf{v})\,\mathbf{u} + (\mathbf{u}\times\mathbf{p})\,\mathbf{v}}{\mathbf{u}\times\mathbf{v}}\,, \tag{1}$$

and by substituting $\mathbf{u}^{\perp}$ for $\mathbf{v}$, $\mathbf{p}$ resolves with respect to $\mathbf{u}$ and $\mathbf{u}^{\perp}$:

$$\mathbf{p} \quad = \quad \frac{(\mathbf{u}\cdot\mathbf{p})\,\mathbf{u} + (\mathbf{u}\times\mathbf{p})\,\mathbf{u}^{\perp}}{\mathbf{u}^2}\,. \tag{2}$$

We will be considering lines, each one determined by a point on it and a vector to which the line is parallel. As a matter of convenience, if A is the point and $\mathbf{u}$ is the directional vector, we denote such a line by $(A, \mathbf{u})$.

Any point P on a line $(A, \mathbf{u})$, and only such points, satisfy

$$\mathbf{AP}\times\mathbf{u} = 0 \qquad \text{or in the form} \qquad \mathbf{P}\times\mathbf{u} = \mathbf{A}\times\mathbf{u}\,, \tag{3}$$

which therefore is an equation of that line. Also useful is the parametric equation

$$\mathbf{P} = \mathbf{A} + t\,\mathbf{u}\,. \tag{4}$$

The point X of intersection of line $(A, \mathbf{u})$ with a line $(B, \mathbf{v})$ can be found by resolving $\mathbf{X}$ into $\mathbf{u}$ and $\mathbf{v}$ ($\mathbf{u}\times\mathbf{v}\neq 0$) using (1) and replacing $\mathbf{u}\times\mathbf{X}$ and $\mathbf{X}\times\mathbf{v}$ according to (3). This leads to

$$\mathbf{X} = \frac{(\mathbf{B}\times\mathbf{v})\,\mathbf{u} + (\mathbf{u}\times\mathbf{A})\,\mathbf{v}}{\mathbf{u}\times\mathbf{v}}\,. \tag{5}$$

Another expression for X follows from solving the parametric equations (4) simultaneously for the two lines:

$$\mathbf{X} \;=\; \mathbf{A} + \frac{(\mathbf{AB}\times\mathbf{v})\,\mathbf{u}}{\mathbf{u}\times\mathbf{v}} \;=\; \mathbf{B} + \frac{(\mathbf{AB}\times\mathbf{u})\,\mathbf{v}}{\mathbf{u}\times\mathbf{v}}\,. \tag{6}$$

For any three points $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ we adopt the abbreviations $\mathbf{a}=\mathbf{BC}$, $\mathbf{b}=\mathbf{CA}$, and $\mathbf{c}=\mathbf{AB}$, and denote the doubled oriented area of $\triangle ABC$ by S.

S can be expressed in a number of ways that follow easily from the definition of $\times$ and appear to be useful in deriving vector-related equalities:

$$\begin{aligned} S &= \mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a} \\ &= \mathbf{A}\times\mathbf{B}+\mathbf{B}\times\mathbf{C}+\mathbf{C}\times\mathbf{A} \\ &= \mathbf{A}\times\mathbf{b}+\mathbf{B}\times\mathbf{c}+\mathbf{C}\times\mathbf{a} \\ &= -(\mathbf{a}\times\mathbf{B}+\mathbf{b}\times\mathbf{C}+\mathbf{c}\times\mathbf{A}) \\ &= \frac{1}{2}\,(\mathbf{a}\times\mathbf{A}+\mathbf{b}\times\mathbf{B}+\mathbf{c}\times\mathbf{C})\,, \end{aligned} \tag{7}$$

Line triads, triangles, and transversals.

Proposition. The lines $(P_i,\mathbf{u_i})$, $i=1,2,3$ are concurrent, including being parallel to each other, or two or all three coincide, if and only if

$$(\mathbf{P_1}\times\mathbf{u_1})(\mathbf{u_2}\times\mathbf{u_3}) + (\mathbf{P_2}\times\mathbf{u_2})(\mathbf{u_3}\times\mathbf{u_1}) + (\mathbf{P_3}\times\mathbf{u_3})(\mathbf{u_1}\times\mathbf{u_2}) = 0\,. \tag{8}$$

⌜ Let the three lines have a common point X. Being the point of intersection of lines 1 and 2, X satisfies (5), and as X is also on line 3, $\mathbf{u_3}\times\mathbf{X}=\mathbf{u_3}\times\mathbf{P_3}$ holds. Multiplying the first equality by $\mathbf{u_3}\times$ and using the second, we arrive at (8). That (8) holds also for parallel or coinciding ($\mathbf{u}_i\times\mathbf{u}_j=0$) lines is immediately obvious.

Conversely, if (8) holds, and two lines, say 1 and 2, are of the same attitude, from $\mathbf{u_1}\times\mathbf{u_2}=0$ and (8) it follows $(\mathbf{P_1P_2}\times\mathbf{u_1})(\mathbf{u_3}\times\mathbf{u_1})=0$, which means that either P_2 belongs to line 1 ($\mathbf{P_1P_2}\times\mathbf{u_1}=0$) and therefore the lines 1 and 2 coincide, or the line 3 is of the same attitude as 1 and 2 ($\mathbf{u_3}\times\mathbf{u_1}=0$), or both are true. If no two lines are parallel or coincide, the lines 1 and 2 meet at X as above, thus

$$(\mathbf{u_3}\times\mathbf{X})(\mathbf{u_1}\times\mathbf{u_2}) = (\mathbf{P_1}\times\mathbf{u_1})(\mathbf{u_2}\times\mathbf{u_3}) + (\mathbf{P_2}\times\mathbf{u_2})(\mathbf{u_3}\times\mathbf{u_1})\,.$$

From this, along with (8), there follows

$$(\mathbf{u_3}\times\mathbf{P_3} - \mathbf{u_3}\times\mathbf{X})(\mathbf{u_1}\times\mathbf{u_2}) = (\mathbf{u_3}\times\mathbf{XP_3})(\mathbf{u_1}\times\mathbf{u_2}) = 0\,,$$

and as $\mathbf{u_1}\times\mathbf{u_2}\neq 0$, it must be that $\mathbf{u_3}\times\mathbf{XP_3}=0$, so X belongs to line 3. ⌟

Making use of (6) in place of (5) for expressing X, by an argument similar to the above, we also obtain the following necessary and sufficient condition:

$$(\mathbf{u_1}\times\mathbf{u_2})(\mathbf{u_3}\times\mathbf{P_3P_1}) + (\mathbf{u_3}\times\mathbf{u_1})(\mathbf{u_2}\times\mathbf{P_2P_1}) = 0 \tag{9}$$

(as well as two other forms resulting from cyclically replacing the indices 1, 2, and 3).

A number of theorems that are often mentioned as ‘immediate’ applications of Ceva’s theorem can be shown to follow even more naturally from (8) or (9). For example, to show that the altitudes of $\triangle ABC$ meet at a point, it suffices to note that they are parallel to $\mathbf{a}^{\perp}$, $\mathbf{b}^{\perp}$, and $\mathbf{c}^{\perp}$ (and never parallel to each other), (8) then amounting to $(\mathbf{A}\times\mathbf{a}^{\perp}+\mathbf{B}\times\mathbf{b}^{\perp}+\mathbf{C}\times\mathbf{c}^{\perp})\,S = (\mathbf{A}\cdot\mathbf{a}+\mathbf{B}\cdot\mathbf{b}+\mathbf{C}\cdot\mathbf{c})\,S = 0$, which indeed holds whatever A, B, and C.

The case of three intersecting lines is treated in more details in the next proposition.

Proposition. Given three lines $(P_i, \mathbf{u_i})$, $i=1,2,3$, each two of which intersect, the area of the triangle formed by these lines is

$$(10)\quad \frac{1}{2}\,\frac{((\mathbf{P_1}\times\mathbf{u_1})(\mathbf{u_2}\times\mathbf{u_3})+(\mathbf{P_2}\times\mathbf{u_2})(\mathbf{u_3}\times\mathbf{u_1})+(\mathbf{P_3}\times\mathbf{u_3})(\mathbf{u_1}\times\mathbf{u_2}))^2}{(\mathbf{u_1}\times\mathbf{u_2})\,(\mathbf{u_2}\times\mathbf{u_3})\,(\mathbf{u_3}\times\mathbf{u_1})}\,.$$

⌜ The area can be found using (7), after obtaining the points of intersection of the lines by means of (5) or (6). ⌟

The numerator of (10), being the l. h. s. of (8), vanishes when the area does, and when the latter is non-zero, its sign is that of the expression in the denominator. This expression deserves special attention, as its sign is meaningful even when the area itself is zero, i. e. for lines concurring at a single point. The sign does not depend on the actual direction of any of $\mathbf{u}_i$ taken alone, as replacing $\mathbf{u}_i$ with $-\mathbf{u}_i$ leaves the expression unchanged. However, the sign is inverted if any two vectors exchange their places. Therefore, the expression has a positive value if and only if the lines are encountered in the order 1–2–3 walking in the *clockwise* direction along any circle, large enough to contain the three (or one) points of intersection.

It is clear that the concrete lines are inessential to this result: any three lines with the same planar attitudes give the same expression. Precisely the attitudes are that really matter, in the order in which they are taken.

From now on we adopt, for whatever different points A, B, and C, and points $A_1\in BC$, $B_1\in CA$, and $C_1\in AB$, the following denotations:

$$(11)\qquad \lambda=\mathbf{BA_1}:\mathbf{BC},\qquad \mu=\mathbf{CB_1}:\mathbf{CA},\qquad \nu=\mathbf{AC_1}:\mathbf{AB}$$

(the ratios being positive or negative according to the respective vectors being codirectional or not).

We also denote

$$\mathbf{k_A}=\mathbf{AA_1},\qquad \mathbf{k_B}=\mathbf{BB_1},\qquad \mathbf{k_C}=\mathbf{CC_1}$$

and it follows immediately that

$$\begin{aligned}\mathbf{k_A}&=\mathbf{AB}+\mathbf{BA_1}=\mathbf{c}+\lambda\,\mathbf{a},\\ \mathbf{k_B}&=\mathbf{BC}+\mathbf{CB_1}=\mathbf{a}+\mu\,\mathbf{b},\\ \mathbf{k_C}&=\mathbf{CA}+\mathbf{AC_1}=\mathbf{b}+\nu\,\mathbf{c},\end{aligned}$$

and

$$(12)\quad \mathbf{k_A}\times\mathbf{k_B}=(1-\mu+\lambda\mu)S,\ \mathbf{k_B}\times\mathbf{k_C}=(1-\nu+\mu\nu)S,\ \mathbf{k_C}\times\mathbf{k_A}=(1-\lambda+\nu\lambda)S.$$

One proposition that easily follows is:

Proposition. Whatever the points A, B, C and A_1, B_1, and C_1 as above, the vectors $\mathbf{k_A}$, $\mathbf{k_B}$, and $\mathbf{k_C}$ form a triangle, i. e. $\mathbf{k_A}+\mathbf{k_B}+\mathbf{k_C}=\mathbf{0}$, if and only if $\lambda=\mu=\nu$.

⌜ Indeed, $\mathbf{k_A}+\mathbf{k_B}+\mathbf{k_C}=\lambda\,\mathbf{a}+\mu\,\mathbf{b}+\nu\,\mathbf{c}=(\lambda-\nu)\,\mathbf{a}+(\mu-\nu)\,\mathbf{b}$, which, in view of $\mathbf{a}\nparallel\mathbf{b}$, can only be $\mathbf{0}$ when $\lambda-\nu=\mu-\nu=0$. ⌟

Proposition: Cheva's theorem. For any triangle $\triangle ABC$ and points $A_1\in BC$, $B_1\in CA$, $C_1\in AB$ the lines AA_1, BB_1, CC_1 are concurrent if and only if

$$(13)\qquad \lambda\,\mu\,\nu-(1-\lambda)(1-\mu)(1-\nu)=0\,.$$

⌜ Substituting $P_1\equiv A$, $P_2\equiv B$, $P_3\equiv C$ and $\mathbf{u_1}\equiv\mathbf{k_A}$, $\mathbf{u_2}\equiv\mathbf{k_B}$, $\mathbf{u_3}\equiv\mathbf{k_C}$ in (9), after carrying out the multiplications, it reduces to $((1-\lambda)(1-\mu)(1-\nu)-\lambda\,\mu\,\nu)\,S^2=0$, and as $S\neq 0$, it turns to be equivalent to (13), so the theorem holds. ⌟

NOTE 1. Ceva's theorem ensures that (13) always holds if the three cevians are parallel or some of them coincide. Additionally, from (12) it follows that for such cevians $1-\mu+\lambda\mu=0$, $1-\nu+\mu\nu=0$, $1-\lambda+\nu\lambda=0$, and $1+\lambda\mu\nu=0$, where each two of these can be derived from the other two.

NOTE 2. As formulated here, Ceva's theorem does not preclude the point of concurrence to be a vertex of the triangle. Let for example this vertex be C – then the cevians from A and B are the lines AC and BC, respectively ($\lambda=1$, $\mu=0$), and the cevian from C can meet AB at any point (ν can take any value). In particular, the cevian from C can pass through A ($\nu=0$) or B ($\nu=1$), thus coinciding with one of the other cevians.

NOTE 3. Rewriting (13) in the form $\lambda\,\mu-(1-\lambda)(1-\mu)(\frac{1}{\nu}-1)=0$ and letting C_1 move indefinitely away from the triangle, i. e. considering $\nu\to\pm\infty$, the line CC_1 becomes parallel to AB, and (13) reduces to $\lambda\,\mu+(1-\lambda)(1-\mu)=0$. And indeed, the latter is easily shown to be the necessary and sufficient condition that cevians through A and B meet at C or another point of the line $(C,\mathbf{AB})$. In the $\lambda\,\mu+(1-\lambda)(1-\mu)=0$ equality, in turn, we can consider e. g. $\mu\to\pm\infty$ (the cevian through B becomes parallel to CA), so that the equality reduces to $\lambda=1/2$ – the condition for the cevian from A to intersect the cevians $(B,\mathbf{CA})$ and $(C,\mathbf{AB})$. That is why Ceva's theorem can be considered to hold also when one or two of the lines through the triangle's vertices are parallel to their opposite sides: for two parallel lines we say that they 'concur at an infinite point'. If one cevian is parallel to the opposite side and the other two pass through that side, then all of them meet at an infinite point: for example, if the cevian from C is parallel to AB, while the other two are the line AB itself, then $\lambda=0$, $\mu=1$, $\nu=\infty$, and $\lambda\,\mu+(1-\lambda)(1-\mu)=0$ holds. Thus Ceva's theorem holds without exceptions, including all cases of concurrence.

Proposition: Routh's theorem about the inscribed triangle. For any three points A, B, C and points $A_1\in BC$, $B_1\in CA$, and $C_1\in AB$, the oriented area of $\triangle A_1B_1C_1$ is related to that of $\triangle ABC$ by a coefficient of $\lambda\,\mu\,\nu+(1-\lambda)(1-\mu)(1-\nu)$.

⌜ From $\mathbf{A_1}=\mathbf{B}+\lambda\,\mathbf{a}$, $\mathbf{B_1}=\mathbf{C}+\mu\,\mathbf{b}$, and $\mathbf{C_1}=\mathbf{A}+\nu\,\mathbf{c}$ it follows

$$\begin{aligned}2\,S_{A_1B_1C_1} &= \mathbf{A_1B_1}\times\mathbf{A_1C_1}=((1-\lambda)\,\mathbf{a}+\mu\,\mathbf{b})\times(-\lambda\,\mathbf{a}-(1-\nu)\,\mathbf{c})\\ &= (1-\lambda)(1-\nu)\,\mathbf{c}\times\mathbf{a}+\lambda\,\mu\,\mathbf{a}\times\mathbf{b}-\mu\,(1-\nu)\,\mathbf{b}\times\mathbf{c}\\ &= (\lambda\,\mu\,\nu+(1-\lambda)(1-\mu)(1-\nu))\cdot(2\,S_{ABC})\,. \end{aligned}$$ ⌟

Proposition: Menelaus' theorem. For any triangle $\triangle ABC$ the points $A_1\in BC$, $B_1\in CA$, $C_1\in AB$ are collinear if and only if $\lambda\,\mu\,\nu+(1-\lambda)(1-\mu)(1-\nu)=0$.

⌜ The theorem is an immediate corollary of the previous one. ⌟

As with Ceva's theorem, one can observe that Menelaus's theorem holds even in the extreme cases. Indeed, the equality $\lambda\,\mu\,\nu+(1-\lambda)(1-\mu)(1-\nu)=0$, rewritten e. g. as $\lambda\,\mu\,\frac{\nu}{1-\nu}+(1-\lambda)(1-\mu)=\lambda\,\mu\,\frac{1}{1-\nu}+(1-\lambda-\mu)=0$, suggests that as C_1 moves indefinitely away from $\triangle ABC$, i. e. at $\nu\to\pm\infty$, that equality transforms into $\lambda+\mu=1$. The latter holds precisely when $A_1B_1\|AB$, thus it can be assumed that a line parallel to AB meets not only BC and CA but also AB, at its 'point at infinity'. And if e. g. $A_1\equiv B_1$, which means that both points coincide with C, $\lambda=1$, and $\mu=0$, then ν can take an arbitrary value: C_1 is anywhere along AB.

The following theorem generalizes that of Ceva for the case of intersecting cevians.

Proposition: Routh's theorem about the inner triangle. For any three points A, B, C and points $A_1\in BC$, $B_1\in CA$, and $C_1\in AB$, the oriented area of the triangle, formed by the lines AA_1, BB_1, and CC_1 is related to that of $\triangle ABC$ by the coefficient

$$\frac{(\lambda\,\mu\,\nu - (1-\lambda)(1-\mu)(1-\nu))^2}{(1-\lambda+\nu\,\lambda)\,(1-\mu+\lambda\,\mu)\,(1-\nu+\mu\,\nu)}\,.$$

⌜ As the line AA_1 is $(A, \mathbf{k_A})$, BB_1 is $(B, \mathbf{k_B})$, and CC_1 is $(C, \mathbf{k_C})$, the statement of the theorem follows immediately from (10) and (12). ⌟

Metric relations for intersecting cevians. Let A, B, and C be the vertices of a triangle, and X is a point such that $AX \cap BC = A_1$, $BX \cap CA = B_1$, and $CX \cap AB = C_1$ – this is the geometric situation from Ceva's theorem in the case of intersecting cevians. A_1, B_1, and C_1 divide the segments BC, CA, and AB, while X divides any of AA_1, BB_1, and CC_1 internally or externally in some ratios. Clearly, these ratios are not independent, but, in general, picking any two of them, the other four can be determined in terms of these two. Besides, each of the segments AA_1, BB_1, CC_1, and its length in particular, is determined from the ratio in which the segment opposite to A, B, and C, respectively, is divided by the transversal.

To find how the six ratios are related, we continue using λ, μ, ν, $\mathbf{k_A}$, $\mathbf{k_B}$, and $\mathbf{k_C}$ as already defined, add

$$p = \mathbf{AX} : \mathbf{AA_1}, \qquad q = \mathbf{BX} : \mathbf{BB_1}\,, \qquad r = \mathbf{CX} : \mathbf{CC_1}\,,$$

and use (12) where applicable.

Ceva's theorem lets us express any of λ, μ, and ν in terms of the other two, e. g. $\nu = \dfrac{(1-\lambda)(1-\mu)}{\lambda\mu + (1-\lambda)(1-\mu)}$.

To relate λ, μ, and ν, on the one hand, and p, q, and r, on the other, act with $\times\mathbf{k_B}$ and $\mathbf{k_A}\times$ on the equality $\mathbf{c} = \mathbf{AB} = \mathbf{AX} + \mathbf{XB} = p\,\mathbf{k_A} - q\,\mathbf{k_B}$ and substitute $\mathbf{k_A}$, $\mathbf{k_B}$, and $\mathbf{k_A}\times\mathbf{k_B}$. For reasons of symmetry, from each of the resulting two equalities another two follow by changing p, q, λ, and μ to either q, r, μ, and ν or r, p, ν, and λ. Thus we arrive at

$$\begin{aligned} &(1-\mu+\lambda\,\mu)\,p = 1-\mu\,, &&(1-\mu+\lambda\,\mu)\,q = \lambda\,,\\ &(1-\nu+\mu\,\nu)\,q = 1-\nu\,, &&(1-\nu+\mu\,\nu)\,r = \mu\,,\\ &(1-\lambda+\nu\,\lambda)\,r = 1-\lambda\,, &&(1-\lambda+\nu\,\lambda)\,p = \nu\,. \end{aligned} \tag{14}$$

The left-hand sides of the first pair of equalities can be made equal by multiplying with q and p to obtain another equality for the r. h. s., and similarly for the other two pairs. Thus:

$$\begin{aligned} \lambda\,p &= (1-\mu)\,q\,,\\ \mu\,q &= (1-\nu)\,r\,,\\ \nu\,r &= (1-\lambda)\,p \end{aligned} \tag{15}$$

or, in another form:

$$\begin{aligned} \lambda\,p + \mu\,q &= q\,,\\ \mu\,q + \nu\,r &= r\,,\\ \nu\,r + \lambda\,p &= p\,, \end{aligned} \tag{16}$$

and solving (16) with respect to λp, μq, and νr:

$$\begin{aligned}\lambda p &= (p+q-r)/2\,,\\ \mu q &= (-p+q+r)/2\,,\\ \nu r &= (p-q+r)/2\,.\end{aligned} \tag{17}$$

Another expression of λp, μq, and νr is obtained by rewriting the first pair of equalities in (14) as

$$\lambda\,\mu\,p-(1-p)(1-\mu)=0\,,\quad (\mu\,q-1)\,\lambda+(1-\mu)\,q=0\,,$$

multiplying the first by $\mu\,q-1$, the second by $\mu\,p$, and subtracting one from the other. This brings us to $(1-\mu)\,(\mu\,q-(1-p))=0$. If $\mu\neq 1$ holds, then obviously also does $\mu\,q=1-p$. And if $\mu=1$, then $X\in AB$ and

$$q=\mathbf{BX}:\mathbf{BA}=(\mathbf{BA}+\mathbf{AX}):\mathbf{BA}=1+\mathbf{AX}:\mathbf{BA}=1-p,$$

therefore again $\mu\,q=1-p$. Now, rewriting the second of the initial equalities in the form $\mu\,q\,(\lambda-1)-\lambda+q=0$, by substituting $1-p$ for $\mu\,q$, we get $\lambda\,p=p+q-1$. The other two pairs of equalities can be similarly transformed, and bringing all together, the result is:

$$\begin{aligned}\lambda p &= p+q-1 = 1-r\,,\\ \mu q &= q+r-1 = 1-p\,,\\ \nu r &= r+p-1 = 1-q\,.\end{aligned} \tag{18}$$

From any of the above it immediately follows

$$p+q+r=2\,, \tag{19}$$

and by adding up (16) or (17) and taking into account (19), we also find

$$\lambda\,p+\mu\,q+\nu\,r=1\,. \tag{20}$$

The equalities (19) and (20) can be considered analogous to (13) for the case of cevians concurring at a finite point.

By adding up each pair of equalities in (14) and using (19), we get yet another set of relations:

$$\begin{aligned}(1-\nu+\mu\,\nu)\,p &= \mu\,\nu+(1-\mu)(1-\nu)\,,\\ (1-\lambda+\nu\,\lambda)\,q &= \nu\,\lambda+(1-\nu)(1-\lambda)\,,\\ (1-\mu+\lambda\,\mu)\,r &= \lambda\,\mu+(1-\lambda)(1-\mu)\,.\end{aligned} \tag{21}$$

Other relations also follow more or less immediately, e. g. from (16) and (15):

$$\begin{aligned}p &= (1-\mu)\,q+\nu\,r\,,\\ q &= (1-\nu)\,r+\lambda\,p\,,\\ r &= (1-\lambda)\,p+\mu\,q\,.\end{aligned}$$

The (14) or (21) equalities can be used to find p, q, and r in different ways, according to which two of λ, μ, and ν are known. Conversely, one can use (17) or (18) for finding λ, μ, and ν in terms of p, q, and r. Actual calculation can only be done when the corresponding equality is not degenerate as a linear equation, i. e. does not contain indefinite values or require dividing by 0. But it is not hard to convince oneself that, provided the point of concurrence of the cevians is finite, there always exist non-degenerate equations among the mentioned, which, along with the notes to Ceva's theorem, make finding any unknown possible. For example, a cevian from C parallel to AB means $\nu=\infty$ but also $r=0$, and any three of λ, μ, p, and q can be expressed in terms of the remaining one.

Equality (19), together with the same in the form $(1-p)+(1-q)+(1-r)=1$, expressed in terms of the segments involved, namely:

$$\begin{aligned}\mathbf{AX}:\mathbf{AA_1}+\mathbf{BX}:\mathbf{BB_1}+\mathbf{CX}:\mathbf{CC_1} &= 2\,,\\ \mathbf{XA_1}:\mathbf{AA_1}+\mathbf{XB_1}:\mathbf{BB_1}+\mathbf{XC_1}:\mathbf{CC_1} &= 1\end{aligned}$$

are known as a *theorem of Gergonne*, and generalize the property of the triangle's centroid to divide each median in parts with ratio 2 : 1.

The third equality in (21) can be restructured into

$$\frac{r}{1-r}=\frac{1-\lambda}{\lambda}+\frac{\mu}{1-\mu}$$

which, again expressed in terms of the segments involved, namely

$$\mathbf{CX}:\mathbf{XC_1}=\mathbf{CA_1}:\mathbf{A_1B}+\mathbf{CB_1}:\mathbf{B_1A}$$

is known as *van Aubel's theorem of triangles*.

A similar relation from (19), rewritten as $r=(1-p)+(1-q)$, is

$$\mathbf{CX}:\mathbf{CC_1}=\mathbf{XA_1}:\mathbf{AA_1}+\mathbf{XB_1}:\mathbf{BB_1}\,.$$

How does the length of a cevian-segment depend on the way it cuts $\triangle ABC$? For CC_1, as

$$CC_1^2=\mathbf{CC_1}^2=(\mathbf{b}+\nu\,\mathbf{c})^2=\mathbf{b}^2+2\nu\mathbf{bc}+\nu^2\mathbf{c}^2,$$

by substituting for $2\,\mathbf{b}\cdot\mathbf{c}$ from $\mathbf{a}^2=(-\mathbf{b}-\mathbf{c})^2=\mathbf{b}^2+\mathbf{c}^2+2\,\mathbf{b}\cdot\mathbf{c}$, we get

$$CC_1^2=\nu\,\mathbf{a}^2+(1-\nu)\,\mathbf{b}^2-\nu\,(1-\nu)\,\mathbf{c}^2\,.$$

This equality in the form $m\,a^2+n\,b^2=c\,(d^2+m\,n)$, where $m=AC_1>0$, $n=C_1B>0$ and $d=CC_1$, and it is assumed that C_1 is between A and B, is known as *Stewart's theorem*. The above form is both a cleaner and more general (admitting $\nu<0$) expression of the respective fact.

Conclusions. Using vector arithmetics, we have presented proofs for a number of theorems in elementary plane geometry. Most of the results are well known but may appear here in a new form. It is even possible – although, given the attention triangle geometry has received in the centuries, not highly probable – that some of them are mentioned here for the first time. Such is the case e. g. with (20), which, simple though it is, we do not recall seing elsewhere. However, our main goal is to provide further evidence of the usefulness of vector-based methods for streamlining plane geometry.

Traditional expositions of elementary geometry rely too much on the synthetic approach, often leading to imprecise and faulty formulations and proofs of propositions. Ceva's theorem is a notorious example.

Introducing and developing a vector-based analytic method to accompany the synthetic one attracts in a number of ways. First of all, ensuring correctness becomes much more easier, and correctness is easily examined and validated: roughly speaking, a proof is correct if a calculation is correct. Another advantage is that proofs are, as a rule, much more straightforward: they require less guessing and additional constructions, and tend to favour symmetric arguments over asymmetric ones. Most often, proofs are derived systematically, their basic building blocks being computation of length, area and other geometric values, vector resolution with respect to a basis, using equations etc.

Very importantly, collinearity, perpendicularity, and other properties are computable, i. e. readily verifiable. Representing formulations and proofs analytically not only makes their domains and limits of application explicit, but may also suggest fruitful abuse of notation.

There is an unfortunate split between elementary and higher-level analytic geometry. The split is both by subject and by method. Analytic geometry adopts analytic method(s) foreign to elementary geometry, but it also abandons the latter geometry's traditional domain: the realm of triangles, circles and their interrelations. Vectors, especially planar vector methods, can effectively lessen the split by providing analytic apparatus suitable for both kinds of geometries. Unlike coordinates, matrices etc. requisite of linear algebra, vectors are a genuinely geometric language, with truly geometric meaning. Both elementary and analytic geometries gain: the former receives a proper tool for carrying out calculations, and the latter becomes less attached to coordinates and more open to dealing with classic geometry problems.

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КАК ДА НЕ СБЪРКАМЕ ДОКАЗВАНЕТО НА ТЕОРЕМАТА НА ЧЕВА

Бойко Бл. Банчев

Векторите могат да бъдат твърде ефикасни при доказване на теореми и провеждане на пресмятания в планиметрията. Понастоящем използването им е ограничено поради недоразвитост на векторната аритметика и почти пълно отсъствие на векторни методи за решаване на задачи. В предишни публикации обърнахме внимание на ползата от прилагане на вектори като естествен инструмент за смятане в планиметрията. Като продължение на тази тема представяме по-обширен пример: доказателства на класическите теореми на Чева и Менелай, наред с някои други. Теоремата на Чева заслужава особено внимание, тъй като, за голяма изненада, сред нейните доказателства, включително от най-уважавани автори, грешните са много повече от правилните.