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PARTIAL INTEGRABILITY OF ALMOST COMPLEX STRUCTURES*

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In this survey we review some old and new results on the problem for local and global existence of holomorphic functions on almost complex manifolds. The following topics are considered: Existence of holomorphic functions and IJ-bundles, Hypersurfaces in $\mathbb{R}^7$, Nearly Kähler manifolds, Homogeneous almost complex spaces, Partial integrability on twistor spaces.

1. Introduction. An almost complex structure on a smooth manifold M is an endomorphism J of the tangent bundle of M with $J^2 = -Id$. These structures have been introduced by C. Ehresmann and H. Hopf in the forties of the last century in the course of their study of the problem for existence of complex structures on smooth manifolds. The almost complex structures arise as a real interpretation of the multiplication by i ($i^2 = -1$) in the holomorphic tangent bundle of a complex manifold and provide a general framework for studying complex geometric structures as well as holomorphic objects like functions, maps, curves, vector and tensor fields etc. This more general point of view allows one to apply important ideas and techniques from complex analysis and complex geometry to other branches of mathematics like differential and symplectic geometry, algebraic topology and other. The increasing interest to almost complex manifolds in the last forty years has been also motivated by the fact that they are on the basis of several new theories with important applications in mathematics and mathematical physics- the twistor theory [42, 2], the theory of J-holomorphic curves [18, 32], the theory of generalized complex structures [24, 19] etc.

The study of almost complex manifolds from analytic point of view has been initiated by Spencer [45] and Kodaira-Spencer [28] who analyzed the possibilities to develop the classical potential theory on almost complex manifolds. An important result in this direction was the solution of the problem for integrability of almost complex structures given by Newlander and Nirenberg [41] in 1954. It shows in particular the main difficulty for direct applications of techniques from complex analysis to non-integrable almost complex manifolds- the lack of local holomorphic coordinates.

Given an almost complex manifold (M, J) one can define the operator $\overline{\partial}$ which maps the space of (p, q) forms on (M, J) into the space of $(p, q+1)$ forms [7]. It is well-known

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that the almost complex structure J is integrable if and only if $\overline{\partial}^2 = 0$. So in the general case $\overline{\partial}^2 \neq 0$ and this makes difficult even the local study of the $\overline{\partial}$-equation on non-integrable almost complex manifolds. The first step in this direction is to consider the problem for local existence of functions f satisfying the equation $\overline{\partial} f = 0$. Note that this is an overdetermined system of PDEs which generalizes the classical Cauchy-Riemann equations. Following [45] and [20] we call these functions holomorphic on (M, J). The theory of holomorphic functions on almost complex manifolds is not as well developed as in the complex case (for some initial results see [34, 11]). One of the main reasons for this is the fact that not much is known even for the local existence of holomorphic functions on almost complex manifolds. The most important result in this direction is the Newlander-Nirenberg theorem [41] which states that near each point of M there exist $1/2 \dim M$ functionally independent holomorphic functions iff the Nijenhuis tensor of the almost complex structure J vanishes. In sharp contrast to complex manifolds, there exist almost complex manifolds which even locally have no holomorphic functions except constants. The first example of such a manifold is the 6-sphere S^6 with the almost complex structure defined via octonionic multiplication [46] (unpublished result of Ehresmann). Latter we shall discuss some more general examples of almost complex manifolds without local holomorphic functions constructed by Hermann [20] and Calabi [6].

In the present survey we review some old and new results on local and global existence of holomorphic functions on almost complex manifolds with emphasis on some important classes of such manifolds which have been previously studied from other points of view.

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2. Existence of holomorphic functions and IJ-bundles. Given an almost complex manifold (M, J) of real dimension $2n$ and a point $x \in M$ denote by $m(x)$ the maximal number of functionally independent holomorphic functions at x. For every integer $0 \leq k \leq n$ set

$$\mathcal{F}_k(J) = \{x \in M | m(x) = k\}.$$

Then M is the disjoint union of its subsets $\mathcal{F}_k(J)$ for $k = 0, 1, \ldots, n$. In these terms the Newlander-Nirenberg theorem [41] says that the almost complex structure J is integrable if and on1y if $M = \mathcal{F}_n(J)$. On the other hand $M = \mathcal{F}_0(J)$ if and only if there are no non-constant local holomorphic functions on (M, J). If $M = \mathcal{F}_k(J)$ for some k then we say that the almost complex structure J is partially integrable and has type k [21].

Denote by N the Nijenhuis tensor of J defined by

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

for arbitrary vector fields X, Y on M. For any point $x \in M$ we define the Nijenhuis space $\mathcal{LN}(x)$ by

$$\mathcal{LN}(x) = \mathrm{Span}\{N_x(a, b) | a, b \in T_x(M)\}$$

and denote by $\mathrm{rank}_x N$ its real dimension. Notice that $\mathrm{rank}_x N$ is an even integer since $\mathcal{LN}(x)$ is a J-invariant subspace of $T_x(M)$. If $\mathrm{rank}\, N$ is a constant function then we denote by $\mathcal{LN}$ the subbundle of the tangent bundle TM with fibres $\mathcal{LN}(x)$ and call it the Nijenhuis bundle of (M, J).

Definition *A vector subbundle $\mathcal{V}$ of TM is called an IJ-bundle if:*

(i) $\mathcal{V}$ *is* J*-invariant;*
(ii) $\mathcal{LN}(x) \subsetneqq \mathcal{V}(x)$ *for all* $x \in M$*;*
(iii) $\mathcal{V}$ *is involutive;*
(iv) $[A, X] + J[JA, X]$ *is a section of* $\mathcal{V}$ *for all sections* A *of* $\mathcal{V}$ *and vector fields* X *on* M*.*

The following theorem gives a geometric criterion for local existence of holomorphic functions on an arbitrary almost complex manifold.

Theorem 1 ([35])**.** *Let (M, J) be an almost complex manifold. Then in a neighbourhood of a point* $x \in M$ *there exist* k *independent holomorphic functions if and only if on a neighbourhood of* x *there exists an* IJ*-bundle of rank* $\dim M - 2k$*.*

Note that Han and Kim [21] developed a dual approach to the problem for local existence of holomorphic functions on almost complex manifolds based on the Cartan-Gardner theory [14] of first integrals.

As consequences of Theorem 1, we obtain the following geometric conditions for an almost complex manifold to be of constant type.

Corollary 2. *Let* (M, J) *be an almost complex manifold with Nijenhuis tensor* N *and let* $\operatorname{rank} N = \dim M - 2k$ *be a constant function. Then* (M, J) *is of type* k *if and only if the Nijenhuis bundle* $\mathcal{LN}$ *has the following properties:*
(i) $\mathcal{LN}$ *is involutive;*
(ii) $[A, X] + J[A, JX]$ *is a section of* $\mathcal{LN}$ *whenever* A *is a section of* $\mathcal{LN}$ *and* X *is a vector field on* M*.*

Corollary 3. *Let* (M, J) *be an almost complex manifold such that* $\{x \in M | \operatorname{rank}_x N = \dim M\}$ *is a dense subset of* M*. Then* (M, J) *is of type* 0*.*

In the next sections we consider the problem for partial integrability on some important classes of almost complex manifolds which have been intensively studied from a differential geometric point of view.

3. Hypersurfaces in $\mathbb{R}^7$. In 1958, Calabi [6] showed that any orientable hypersurface M of $\mathbb{R}^7$ has a natural almost complex structure J defined by means of the octonions. For any point $p \in M$ denote by K_p the symmetric Hermitian transformation of the tangent space T_pM defined by $K_p = -A_p + J_pA_pJ_p$, where A_p is the Weingarten map of M at p [27]. Let k be the following function on M:

$$k(p) = \frac{1}{24}[(\operatorname{Trace} K_p)^3 - 4 \operatorname{Trace} K_p^3].$$

It is shown in [36] that the rank of the Nijenhuis tensor N of J and the function k are related by

$$\{p \in M | \operatorname{rank}_p N = \dim M\} = \{p \in M | k(p) \neq 0\}.$$

This together with Corollary 2 and the principle of analytic continuation implies that any compact orientable real-analytic hypersurface of $\mathbb{R}^7$ with the Calabi almost complex structure is of type 0. It is interesting to note that there are hypersurfaces of $\mathbb{R}^7$ which are of type 1 but it is still an open problem whether there exist hypersurfaces of type 2 [31]. A related open question is whether there are hypersurfaces of $\mathbb{R}^7$ with $\operatorname{rank} N = 2$.

4. Nearly Kähler manifolds. An almost Hermitian manifold (M, g, J) is said to be nearly Kähler if the covariant derivative of J with respect to the Levi-Civita connection of g is skew-symmetric [16]. A four dimensional nearly Kähler manifold is automatically Kähler, and there has been a strong belief that the only examples of compact non-Kähler nearly Kähler manifolds in dimension 6 are the 3-symmetric spaces $S^3 \times S^3$, the complex projective space $\mathbb{CP}^3$, the flag manifold $\mathbb{F}^3$ and the sphere S^6 . Indeed, J.-B. Butruille [5] has shown that there are no other homogeneous examples in dimension 6, but L. Foscolo and M. Haskins[13] have recently proved the existence of (inhomogeneous) cohomogeneity one nearly Kähler structures on S^6 and on $S^3 \times S^3$. Note also the structure result of P.A. Nagy [40] who has proved that every compact simply connected nearly Kähler manifold is isometric to a Riemannian product $M_1 \times \cdots \times M_k$, such that each M_i is either a Kähler manifold, a naturally reductive 3-symmetric space [48], the twistor space of a compact quaternion-Kähler manifold with positive scalar curvature [3] or a 6-dimensional nearly Kähler manifold.

Combining results of Kirichenko [26], Gray [16] and Corollary 1 one can prove [35] that the Nijenhuis tensor N of a connected nearly Kähler manifold has constant rank (this follows also by the result of P.A. Nagy mentioned above) and the type of (M, J) is equal to $1/2(\dim M - \operatorname{rank} N)$. In low dimensions the type of a non-Kähler nearly Kähler manifold can be determined completely. Namely, by the results of Gray [25] and the above formula it follows that if $\dim M = 6$, then (M, J) is of type 0. If $\dim M = 8$, then (M, J) is of type 1.

5. Homogeneous almost complex spaces. Every homogeneous space with an invariant almost complex structure is of constant type. Its computation can be reduced to a purely algebraic problem as follows. Let $M = G/K$ be a homogeneous space on which the connected Lie group G acts effectively. Fix a direct sum decomposition $\mathfrak{g} = \mathfrak{k} \bigoplus \mathfrak{m}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, respectively and $\mathfrak{m}$ is a vector subspace of $\mathfrak{g}$ which is identified with the tangent space to M at $o = \{K\} \in G/K$. The G-invariant almost complex structures on M are identified with $ad(\mathfrak{k})$-invariant endomorphisms J of $\mathfrak{m}$ such that $J^2X = -X$ for all $X \in \mathfrak{m}$. Denote by N the Nijenhuis tensor of J and by $\mathfrak{l}n$ the subspace $\mathcal{LN}(o)$ of $\mathfrak{m}$. Then $\operatorname{rank} N = \dim_{\mathbb{R}} \mathfrak{l}n$ since N is a G-invariant tensor field on M. For the sake of simplicity we assume that M is reductive in the sense that $\mathfrak{m}$ is an $ad(\mathfrak{k})$-invariant subspace of $\mathfrak{g}$.

Definition. *A vector subspace* $\mathfrak{h}$ *of* $\mathfrak{m}$ *is called* IJ*-subspace if:*
(*i*) $\mathfrak{h}$ *is both* $ad(\mathfrak{k})$*-invariant and* J*-invariant;*
(*ii*) $\mathfrak{l}n \subseteq \mathfrak{h}$*;*
(*iii*) $[\mathfrak{h}, \mathfrak{h}]_{\mathfrak{m}} \subseteq \mathfrak{h}$*;*
(*iv*) $[X, Y]_{\mathfrak{m}} + J[X, JY]_{\mathfrak{m}} \in \mathfrak{h}$ *for all* $X \in \mathfrak{h}, Y \in \mathfrak{m}$.
(*The subscript denotes the component in* $\mathfrak{m}$.)

Denote by $d(M, J)$ the real dimension of the intersection of al1 IJ-subspaces of $\mathfrak{m}$. Note that the almost complex structure J is integrable iff $d(M, J) = 0$ [27]. Moreover, it follows from Theorem 1 that the type of a reductive homogeneous almost complex space (M, J) is equal to $1/2(\dim M - d(M, J))$.

Now let $M = G/K$ be an isotropy irreducible homogeneous almost complex space. Since $\mathfrak{l}n$ is an invariant subspace of the linear isotropy representation of K at o it follows that either $\operatorname{rank} N = 0$ or $\operatorname{rank} N = \dim M$. Thus by Corollary 2 we obtain the following result of Hermann [20]:

Theorem 4. *Every isotropy irreducible homogeneous almost complex space is either integrable or of type* 0*.*

A typical example of type 0 is the sphere $S^6 = G_2/SU(3)$. Other examples of such spaces can be found in Wolf [49]. We also refer the reader to [36] and [17] where the partial integrability of invariant almost complex structures on the generalized Grassmann manifolds and the Ledger-Obata s-symmetric spaces [29] have been considered.

5.1. Lie groups of type T. In 1976, Thurston [47] constructed the first example of a compact symplectic manifold W with no Kähler structure. It is well-known [1] that W is a 4-dimensional nil-manifold and in [37] the author noticed that every left-invariant almost complex structure on the corresponding nilpotent Lie group has at least one non-constant local holomorphic function. Motivated by this example he introduced the class of real Lie groups, called Lie groups of type T, having the above property [38].

A simple algebraic condition for a Lie group to be of type T is given in the following

Theorem 5 ([38])**.** *Let G be a $2n$-dimensional Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$ such that $\dim \mathfrak{a} \leq n-1$ and*

$$[X, Y] \in \mathrm{Span}(X, Y) + \mathfrak{a}, \quad X, Y \in \mathfrak{g}.$$

Then G is of type T.

Corollary 6. *Every $2n$-dimensional Lie group G with $\dim[\mathfrak{g}, \mathfrak{g}] \leq n-1$ is of type T.*

Note that all 4-dimensional Lie groups of type T have been classified in [38]. In particular it follows that all of them satisfy the algebraic condition of Theorem 5 but it is still an open problem if the same is true in higher dimensions.

5.2. Nilpotent Lie groups of type T. Simple algebraic arguments show that for nilpotent Lie groups of type T the conditions of Theorem 5 and Corollary 6 are equivalent. This raises the question whether the condition $\dim[\mathfrak{g}, \mathfrak{g}] \leq n-1$ is also necessary for a nilpotent Lie group to be of type T.

Recall that a Lie group G is said to be *2-step nilpotent*, if its Lie algebra $\mathfrak{g}$ satisfies the identity $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$. The next result shows that the answer to the question above is positive for this class of nilpotent Lie groups.

Theorem 7 ([39])**.** *Let G be a 2-step nilpotent Lie group with real dimension $2n$ and Lie algebra $\mathfrak{g}$. Then G is of type T if and only if $\dim[\mathfrak{g}, \mathfrak{g}] \leq n-1$.*

Using the well-known classification of nilpotent Lie algebras of real dimensions 4 and 6 (see [15] and [44]) one can show that Theorem 7 is also true for nilpotent Lie groups in these dimensions. Moreover, it leads to the following classification results [39]:

Theorem 8. *A 4-dimensional nilpotent Lie group is of type T if and only if its Lie algebra is isomorphic to one of the following:*

$\mathfrak{k}_1$: *Abelian*; $\quad \mathfrak{k}_2 : [e_1, e_2] = e_3$.

Theorem 9. *A 6-dimensional nilpotent Lie group is of type T if and only if its Lie algebra is isomorphic to one of the following:*

$\mathfrak{l}_1$: *Abelian*; $\quad \mathfrak{l}_2 : [e_1, e_2] = e_3; \quad \mathfrak{l}_3 : [e_1, e_2] = e_5, [e_3, e_4] = e_5;$

$\mathfrak{l}_4 : [e_1, e_2] = e_3, [e_1, e_3] = e_4; \qquad \mathfrak{l}_5 : [e_1, e_2] = e_4, [e_1, e_3] = e_5;$

$\mathfrak{l}_6 : [e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5;$

$\mathfrak{l}_7 : [e_1, e_2] = e_3, [e_1, e_3] = e_6, [e_4, e_5] = e_6;$

$\mathfrak{l}_8(\epsilon) : [e_1, e_2] = e_5, [e_1, e_3] = e_6, [e_2, e_4] = \epsilon e_6, [e_3, e_4] = e_5, \ \epsilon = 0, \pm 1.$

Finally, let us note that in general the condition $\dim[\mathfrak{g}, \mathfrak{g}] \leq n-1$ is not necessary for a Lie group to be of type T. To see this consider the following

Example. Let G be the solvable Lie group whose Lie algebra $\mathfrak{g}$ has a basis $(e_1, \ldots, e_{2n})$, $n > 1$ such that

$$\mathfrak{g} : [e_i, e_{2n}] = e_i, \ 1 \leq i \leq 2n-1.$$

It has been proved in [38] (see also [44]) that these Lie groups can be characterized by the property that every left-invariant almost complex structure on G is integrable. In particular, G is of type T and $\dim[\mathfrak{g},\mathfrak{g}] = 2n-1 > n-1$.

Remark. J. Milnor [33] has also considered the Lie groups in the above example (of all dimensions) and characterized them by the property that every left-invariant Riemannian metric has sectional curvature of constant sign.

6. Partial integrability on twistor spaces. In this section we discuss the problem for local and global existence of holomorphic functions on the twistor spaces of oriented 4-manifolds endowed with a Riemannian or a neutral metric.

6.1. Twistor spaces of oriented Riemannian 4-manifolds. Let M be an oriented 4-manifold with a Riemannian metric g. Then g induces an inner product on the bundle Λ^2 of tangent bi-vectors by

$$\langle X \wedge Y, Z \wedge T \rangle = \frac{1}{2}[g(X,Z)g(Y,T) - g(X,T)g(Y,Z)].$$

The Hodge star operator $* : \Lambda^2 \to \Lambda^2$ is an involution and we denote by $\Lambda_\pm$ its eigenbundles corresponding to the eigenvalues ± 1. Let $\mathcal{R} : \Lambda^2 \longrightarrow \Lambda^2$ be the curvature operator of (M, g). It is related to the curvature tensor R by

$$g(\mathcal{R}(X \wedge Y), Z \wedge T) = g(R(X,Y)Z,T); \quad X,Y,Z,T \in TM.$$

The curvature operator $\mathcal{R}$ admits an $SO(4)$-irreducible decomposition

$$\mathcal{R} = \frac{\tau}{6}I + \mathcal{B} + \mathcal{W}_+ + \mathcal{W}_-$$

where τ is the scalar curvature, $\mathcal{B}$ represents the traceless Ricci tensor, $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$ corresponds to the Weyl conformal tensor, and $\mathcal{W}_\pm = \dfrac{1}{2}(\mathcal{W} \pm *\mathcal{W})$. The metric g is Einstein exactly when $\mathcal{B} = 0$ and is conformally flat when $\mathcal{W} = 0$. It is said to be self-dual, resp. anti-self-dual, if $\mathcal{W}_- = 0$, resp. $\mathcal{W}_+ = 0$ [3].

The twistor space $\mathcal{Z}$ of (M, g) is the 2-sphere bundle on M consisting of all unit bivectors in Λ_-. It can be identified with the space of all complex structures on the tangent spaces of M compatible with the metric g and the opposite orientation of M.

The Levi-Civita connection of g gives rise to a splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $\mathcal{Z}$ into horizontal and vertical components. Following [2] and [12] define two almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on $\mathcal{Z}$ by

$$\mathcal{J}_n V = (-1)^n \sigma \times V, \quad V \in \mathcal{V}_\sigma,$$

$$\pi_*(\mathcal{J}_n A) = J_\sigma(\pi_* A), \quad A \in \mathcal{H}_\sigma,$$

where $\times$ is the usual vector product in the oriented 3-dimensional vector space $(\Lambda_-)_p$ and J_σ is the complex structure on T_pM, $p \in \pi(\sigma)$, defined by σ. Recall [2, 12] that the almost complex structure $\mathcal{J}_1$ is integrable if and only if the metric g is self-dual, whereas $\mathcal{J}_2$ is never integrable. The next theorem generalizes these results.

Theorem 10 ([8]). *Let $\mathcal{Z}$ be the twistor space of an oriented Riemannian 4-manifold (M, g). Then*

$$\mathcal{Z} = \mathcal{F}_0(\mathcal{J}_1) \cup \mathcal{F}_3(\mathcal{J}_1) = \mathcal{F}_0(\mathcal{J}_2) \cup \mathcal{F}_1(\mathcal{J}_2).$$

Moreover

$$\mathcal{F}_3(\mathcal{J}_1) = \pi^{-1}(Int\{p \in M : (\mathcal{W}_-)_p = 0\}),$$

$$\mathcal{F}_1(\mathcal{J}_2) = \pi^{-1}(Int\{p \in M : \mathcal{R}_p = (\mathcal{W}_+)_p\}).$$

Examples. i) It is well-known that the manifolds $S^4, S^1 \times S^3$ and $\mathbb{CP}^2$ with their standard metrics are self-dual and with nowhere vanishing Ricci tensor. Hence in these cases the almost complex structure $\mathcal{J}_1$ is integrable, whereas $\mathcal{J}_2$ has no local holomorphic functions except constants.

ii) The equality $\mathcal{F}_1(\mathcal{J}_2) = \mathcal{Z}$ means that around each point of $\mathcal{Z}$ there exists a non-constant $\mathcal{J}_2$-holomorphic function. By Theorem 11 this is equivalent to (M, g) being a self-dual and Ricci flat manifold. In this case Λ_- is a flat vector bundle and the projection on the fibre is $\mathcal{J}_2$-holomorphic. Note that by a result of Hitchin [22] the universal covering of any compact oriented self-dual Ricci flat manifold is either a K3-surface, or M is flat.

Hitchin [23] has proved that when the almost complex structure $\mathcal{J}_1$ is integrable (i.e. (M, g) is self-dual) there are no non-constant global holomorphic functions on $\mathcal{Z}$. A simple reasoning yields the following slightly more general result:

Theorem 11 ([9])**.** *Let M be a connected oriented Riemannian 4-manifold. Then every holomorphic function on the almost complex manifold $(\mathcal{Z}, \mathcal{J}_n), n = 1, 2,$ is constant.*

6.2. Hyperbolic twistor spaces. A pseudo-Riemannian metric on a smooth 4-manifold M is called *neutral* if it has signature $(+, +, -, -)$. In contrast to the Riemannian case the existence of such metrics on compact manifolds imposes topological restrictions since it is equivalent to the existence of a field of 2-planes [30].

One can construct a twistor space of an oriented 4-manifold M with a neutral metric g as in the Riemannian case [4]. This time the fiber of the twistor bundle $\pi : \mathcal{Z} \to M$ is the two-sheeted hyperboloid

$$H = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : -y_1^2 + y_2^2 + y_3^2 = -1\}$$

and we can think of the $y_1 > 0$ branch as one of the standard models of the hyperbolic plane. Further, we consider the hyperboloid H with the complex structure S determined by the restriction to H of the metric $-dy_1^2 + dy_2^2 + dy_3^2$.

As in the Riemannian case one can define two almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on $\mathcal{Z}$ and it has been shown in [4] that Theorem 11 holds for the hyperbolic twistor spaces as well. However, if we consider the problem for global existence of holomorphic functions, then there is a considerable difference between the hyperbolic and classical twistor spaces.

Given a neutral almost hyperhermitian 4-manifold (M, g, J_1, J_2, J_3), denote by $\pi : \mathcal{Z} \to M$ its hyperbolic twistor bundle. Then the 2-vectors corresponding to J_1, J_2, J_3 form a global frame of Λ_- and we have a natural projection $p : \mathcal{Z} \to H$ defined by $p(\sigma) = (y_1, y_2, y_3)$, where $J_\sigma = y_1 J_1(x) + y_2 J_2(x) + y_3 J_3(x)$, $x = \pi(\sigma)$. Thus $\mathcal{Z}$ is diffeomorphic to $M \times H$ by the map $\sigma \to (\pi(\sigma), p(\sigma))$ and it is obvious that p maps any fibre of $\mathcal{Z}$ biholomorphically on H with respect to $\mathcal{J}_1$ and S.

Theorem 12 ([4])**.** *Let M be a neutral almost hyperhermitian 4-manifold and $\mathcal{Z}$ its hyperbolic twistor space. Then the natural projection $p : \mathcal{Z} \to H$ is $\mathcal{J}_1$-holomorphic (resp. $\mathcal{J}_2$-anti-holomorphic) if and only if M is neutral hyperhermitian (resp. neutral hyperkähler).*

Furthermore we have the following result in the compact case.

Theorem 13 ([4])**.** *Let M be a compact neutral hyperhermitian manifold with hyperbolic twistor space $\mathcal{Z}$ and let $p : \mathcal{Z} \to H$ be the natural projection. Then any $\mathcal{J}_1$-holomorphic function f on $\mathcal{Z}$ has the form $f = g \circ p$, where g is a holomorphic function on H. If M is* *neutral hyperkähler, any $\mathcal{J}_2$-holomorphic function f on $\mathcal{Z}$ has the form $f = g \circ p$, where g is an anti-holomorphic function on H.*

We show now that in the non-compact case the situation changes drastically. To see this consider the following neutral metrics on $\mathbb{R}^4$ introduced by J. Petean in [43]:

$$g = f(dx_1 \otimes dx_1 + dx_2 \otimes dx_2) + dx_1 \otimes dx_3 + dx_3 \otimes dx_1 + dx_2 \otimes dx_4 + dx_4 \otimes dx_2,$$

where (x_1, x_2, x_3, x_4) are the standard coordinates on $\mathbb{R}^4$ and f is a smooth positive function depending on x_1 and x_2 only. According to [10] the metrics g are neutral hyperkähler, i.e. self-dual and Ricci-flat. Hence, the almost complex structure $\mathcal{J}_1$ on the hyperbolic twistor space $\mathcal{Z}$ of $(\mathbb{R}^4, g)$ is integrable. Moreover, we have the following result [4], which shows that there can be an abundance of global holomorphic functions on a hyperbolic twistor space.

Theorem 14. *The hyperbolic twistor space $(\mathcal{Z}, \mathcal{J}_1)$ of $(\mathbb{R}^4, g)$ is biholomorphic to $\mathbb{C}^2 \times H$.*

This result suggests the following problem: *Characterize the non-compact neutral hyperkähler 4-manifolds whose hyperbolic twistor spaces $(\mathcal{Z}, \mathcal{J}_1)$ are Stein manifolds.*

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ЧАСТИЧНА ИНТЕГРУЕМОСТ НА ПОЧТИ КОМПЛЕКСНИ СТРУКТУРИ

Олег Мушкаров

В обзора се прави преглед на някоя стари и нови резултати за локално и глобално съществуване на холоморфни функции върху почти комплексни многообразия. Разгледани са следните теми: Съществуване на холоморфни функции и IJ-разслоения, Хиперповърхнини в $\mathbb{R}^7$, Приблизително Келерови многообразия, Хомогенни почти комплексни пространства, Частична интегруемост върху туисторни пространства.