МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2016
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2016
Proceedings of the Forty Fifth Spring Conference
of the Union of Bulgarian Mathematicians
Pleven, April 6–10, 2016

ON THE GEOMETRY OF THE PONTRYAGIN MAXIMUM PRINCIPLE IN INFINITE DIMENSIONAL SPACES*

Mikhail I. Krastanov, Nadezhda K. Ribarska

Two new concepts of uniform approximating cones are discussed. The main result is a theorem for nonseparability of two closed sets. As application of this result, an abstract Lagrange multiplier rule and a necessary optimality condition of Pontryagin maximum principle type for an optimal control problem in infinite-dimensional state space are obtained.

1. Introduction. Since the Pontryagin maximum principle was discovered, various versions of this result have been established under different technical assumptions and with different proofs. In a series of papers (cf., for example, the bibliography of [14]), the corresponding proofs are based on separation theorems for sets, which assert that a necessary condition for two sets A, B containing a point x_0 to be locally separated at x_0 (in the sense that there exists a neighborhood Ω of x_0 so that $\Omega \cap A \cap B = \{x_0\}$) is that C^A and C^B are not strongly transversal (the cones C^A and C^B are strongly transversal iff $C^A - C^B = R^n$ and $C^A \cap C^B \neq \{0\}$, cf. Definition 3.2 from [14]), where C^A and C^B are "tangent cones" to A and B, respectively, at the point x_0. These separation theorems are true if "tangent cones" are interpreted to mean "Boltyanskii approximating cones" and also if they are taken to mean "Clarke tangent cones". A. Bressan has constructed an example (cf. [1]) of two four-dimensional separated sets A and B at x_0, where (surprisingly) the approximating cones C^A and C^B are strongly transversal (C^A and C^B approximate the sets A and B in the Boltyanskii, respectively, in the Clarke sense at x_0). In the infinite dimensional setting the things become even worse: there exist convex sets A and B that are locally separated at a common point x_0 such that the corresponding approximating tangent cones C^A and C^B of the sets A and B, respectively, at x_0 are strongly transversal. Indeed, let A be the canonical example of a Hilbert cube: a closed convex bounded set with empty interior whose closed affine hull is the whole space (set $A := \{(x_n) \in l_2 : \ |x_n| \leq 1/n\} \subset l_2$) and let B be a ray whose intersection with A is the origin, for example take $B = \{\lambda y : \ \lambda \geq 0\}$, where $y = (1/n^{3/4})_{n=1}^{\infty}$). Then the Clarke tangent cone C^A to A at the origin is l_2, the Clarke tangent cone C^B to B at the origin

---

***2010 Mathematics Subject Classification:** 49K27, 35F25, 46N10.

Key words: infinite-dimensional problems, approximating cone, nonseparation of sets, Lagrange multiplier rule, Pontryagin maximum principle.

This work was supported by by the Bulgarian Ministry of Science and Higher Education – National Fund for Science Research under grant DFNI-IO2/10 and by the Sofia University "St. Kliment Ohridski" under contract No. 08/26.03.2015.

coincides with B, and thus C^A and C^B are strongly transversal, while the sets A and B are locally separated. This motivates the search for a smaller approximating cone than the Clarke one such that the strong transversality of two approximating cones at a common point of two sets implies local nonseparation of these sets.

The organization of this paper is as follows: In section 2 we introduce and discuss two definitions of uniform approximating cones. Moreover, a nonseparation result is proved in Theorem 2.6. In Section 3 we use this theorem to obtain an abstract Lagrange multipliers rule and a necessary optimality condition of Pontryagin maximum principle type for an optimal control problem in infinite-dimensional state space.

2. A non-separation property. One of the starting points of our study is the paper [2], where (among others) it is proved that if $\bar{x}$ belongs to a closed set S and the Bouligand tangent cone $T_S(x)$ at each point x of the intersection of a small neighborhood of $\bar{x}$ and S covers a ball with fixed positive radius, then $\bar{x}$ belongs to the interior of S. In [5] the concept of uniform Clarke tangent cone to a set is introduced and a corresponding non separation result is proved (cf. Definition 3.6 and Corollary 3.8). These results as well as Examples 2.5 and 4.11 in [5] motivate the importance of "uniformity of approximation" with respect to all elements of the tangent cone. Definition 2.3 given below extends Definition 3.6 in [5] and explains what we mean by "uniformity of the approximation".

Throughout this section X is a Banach space, $\mathbf{B}$ ($\bar{\mathbf{B}}$) is its open (closed) unit ball centered at the origin.

Definition 2.1. *Let S be a closed subset of X and x_0 belong to S. We say that the bounded set D is an uniform tangent set to S at the point x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $v \in D$ and for each point $x \in S \cap (x_0 + \delta\bar{\mathbf{B}})$ there exists $\lambda > 0$ such that for each $t \in [0, \lambda]$ the set $S \cap (x + t(v + \varepsilon\bar{\mathbf{B}}))$ is non empty.*

Definition 2.2. *Let S be a closed subset of X and x_0 belong to S. We say that the bounded set D is a sequence uniform tangent set to S at the point x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $v \in D$ and for each point $x \in S \cap (x_0 + \delta\bar{\mathbf{B}})$ there exists a sequence of positive reals $t_m \longrightarrow 0$ such that for each positive integer m the set $S \cap (x + t_m(v + \varepsilon\bar{\mathbf{B}}))$ is non empty.*

Definition 2.3. *Let S be a closed subset of X and x_0 belong to S. We say that the cone C is an uniform tangent cone to S at the point x_0 if $C \cap \bar{\mathbf{B}}$ is an uniform tangent set to S at the point x_0. We say that the cone C is a sequence uniform tangent cone to S at the point x_0 if $C \cap \bar{\mathbf{B}}$ is a sequence uniform tangent set to S at the point x_0.*

Remark 2.4. It is remarkable that in a finite-dimensional space X the two notions introduced in Definition 2.3 coincide. Moreover, the usual Clarke tangent cone is a (sequence) uniform cone (cf. Theorem 1 of [12] and Theorem 6.26 on page 217 of [13]).

The next Lemma shows some of the properties of the uniform tangent sets:

Lemma 2.5. *The closed hull of an uniform tangent set D to S at x_0 is an uniform tangent set to S at x_0. Moreover, the convex hull of an uniform tangent set D is an uniform tangent set to S at x_0. Also, the closed hull of a sequence uniform tangent set D to S at x_0 is a sequence uniform tangent set to S at x_0.*

The next theorem is the main result in this note. It is based on the same ideas used in the proof of Theorem 3.3 and Corollary 3.8 in [5]. One of its advantages is that one of the sets is no longer assumed to be convex.

Theorem 2.6. *Let A and B be closed subsets of X with $x_0 \in A \cap B$. Let C^A be an uniform tangent cone to the set A at the point x_0 and C^B be a sequence uniform tangent cone to the set B at the point x_0, such that the following conditions hold true:*

(A1) *There exist $M > 1$ and $\varepsilon \in (0, 1/(9 + 8M))$ such that the set $\left(C^A \cap M\bar{\mathbf{B}}\right) - \left(C^B \cap M\bar{\mathbf{B}}\right)$ is ε-dense in the unit sphere $\{w \in X : \|w\| = 1\}$, that is for every $v \in X$ with $\|v\| = 1$ there exist $v_1 \in C^A \cap M\bar{\mathbf{B}}$ and $v_2 \in C^B \cap M\bar{\mathbf{B}}$ such that $\|v - (v_1 - v_2)\| < \varepsilon$;*

(A2) *there exists v_0 with unit norm which is ε-close to C^A and to C^B (that is, there are $\tilde{v}_0^A \in C^A$ and $\tilde{v}_0^B \in C^B$ of norm one such that $\|\tilde{v}_0^A - v_0\| < \varepsilon$ and $\|\tilde{v}_0^B - v_0\| < \varepsilon$).*

Then for each positive integer n there exists $\bar{x} \neq x_0$ that belongs to the set $A \cap B$ and $\|\bar{x} - x_0\| < 1/n$.

The next corollary is a well-known result (cf. Proposition 4.2 of [14]), but it is not easy to find a simple self-contained proof in the literature:

Corollary 2.7. *Let X be a finite-dimensional vector space, A and B be closed subsets of X with $x_0 \in A \cap B$. Let C^A and C^B be the Clarke tangent cones to A and B, respectively, at x_0. Let $C^A - C^B = X$ and $C^A \cap C^B \neq \{0\}$ (i.e. the cones C^A and C^B are strongly transversal). Then for each positive integer n there exists $\bar{x} \neq x_0$ which belongs to the set $A \cap B$ and $\|\bar{x} - x_0\| < 1/n$ (i.e. the sets A and B are not locally separated at x_0).*

3. Applications. We apply the main result (Theorem 2.6) from the previous section to obtain an abstract Lagrange multiplier rule and a Pontryagin maximum principle for an infinite-dimensional optimal control problem.

Let Y be a Banach space. We consider further $Y \times R$ equipped with the uniform norm $\|(y, r)\| := \max\{\|y\|, |r|\}$. Throughout this section we denote by $\bar{\mathbf{B}}$ and by $\tilde{\mathbf{B}}$ the closed unit balls in Y and in $Y \times R$, respectively.

The next definition is well known (cf. [10]):

Definition 3.1. *A mapping $\varphi : Y \to R$ is said to be strictly Fréchet differentiable at $\bar{y} \in Y$ if there exists a continuous linear operator $\varphi'(\bar{y}) : Y \to R$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ so that*

$$|\varphi(y) - \varphi(x) - \varphi'(\bar{y})(y - x)| \leq \varepsilon\|y - x\|$$

whenever $\|y - \bar{y}\| < \delta$ and $\|x - \bar{y}\| < \delta$.

First we prove three lemmas we need further:

Lemma 3.2. *Let A be a closed subset of Y, $\bar{y} \in A$, $\varphi : Y \to R$ be strictly Fréchet differentiable at $\bar{y}$, C^A be a closed cone in Y, $\tilde{A} := \{(y, \varphi(y)) : \ y \in A\}$ and $C^{\tilde{A}} := \{(v, \varphi'(\bar{y})v) : \ v \in C^A\}$. Then C^A is a (sequence) uniform tangent cone to A at $\bar{y}$ iff $C^{\tilde{A}}$ is a (sequence) uniform tangent cone to $\tilde{A}$ at $(\bar{y}, \varphi(\bar{y}))$.*

Lemma 3.3. *Let $\bar{y} \in Y$ and $\varphi(y) := \|y - \bar{y}\|^2$ for each $y \in Y$. Then φ is strictly Fréchet differentiable at $\bar{y}$ with zero derivative.*

Proof. It is straightforward. □

Lemma 3.4. *Let A be a closed subset of Y, $\bar{y} \in A$ and $\tilde{A} := A \times (-\infty, \bar{r}]$ for $\bar{r} \in R$. Then C^A is a (sequence) uniform tangent cone to A at $\bar{y}$ iff $C^{\tilde{A}} := C^A \times (-\infty, 0]$ is a (sequence) uniform tangent cone to $\tilde{A}$ at $(\bar{y}, \bar{r})$.*

Lemma 3.5. *Let $\tilde{A} := A \times (-\infty, r_0]$, where $r_0 \in R$ and A is a closed subset of Y. Let $\tilde{B}$ be a closed subset of $Y \times R$ with $(x_0, r_0) \in \tilde{A} \cap \tilde{B}$. Let $C^{\tilde{A}}$ be an uniform tangent cone (a sequence uniform tangent cone) to the set $\tilde{A}$ at the point (x_0, r_0) and $C^{\tilde{B}}$ be a sequence uniform tangent cone (an uniform tangent cone) to the set $\tilde{B}$ at the point (x_0, r_0). If there exist $M > 1$ and $\varepsilon_M > 0$ such that*

$$D := \left(C^{\tilde{A}} \cap \ M\tilde{\mathbf{B}}\right) - \left(C^{\tilde{B}} \cap \ M\tilde{\mathbf{B}}\right)$$

is ε_M-dense in the unit sphere $\tilde{\Sigma}$ in $Y \times R$ centered at the origin, then there exists $w_0 \in Y \times R$ with unit norm which is $2\varepsilon_M$-close to $C^{\tilde{A}}$ and to $C^{\tilde{B}}$.

Lemma 3.6. *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Let C^A be an uniform tangent cone or a convex sequence uniform tangent cone to A at the point x_0 and C^B be a convex sequence uniform tangent cone or an uniform tangent cone to the set B at the point x_0. Let $\rho\tilde{\mathbf{B}} \subset \overline{co}D$ for some $\rho \in (0,1)$, where $\tilde{\mathbf{B}}$ is the closed unit ball of X and*

$$D := \left(C^A \cap \ \tilde{\mathbf{B}}\right) - \left(C^B \cap \ \tilde{\mathbf{B}}\right).$$

Then there exist $M > 1$, an uniform tangent cone or a convex sequence uniform tangent cone $\tilde{C}^A$ to A at x_0 and an uniform tangent cone or a convex sequence uniform tangent cone $\tilde{C}^B$ to B at x_0 such that the set

$$\tilde{D} := \left(\tilde{C}^A \cap \ M\tilde{\mathbf{B}}\right) - \left(\tilde{C}^B \cap \ M\tilde{\mathbf{B}}\right)$$

is ε_M-dense in the unit sphere Σ of X centered at the origin, where $\varepsilon_M = \dfrac{1}{3(9+8M)}$.

We are going to use a concept introduced in [4]:

Definition 3.7. *Let Y be a Banach space and S be a subset of Y. The set S is said to be quasisolid if its closed convex hull $\overline{\text{co}}\ S$ has nonempty interior in its closed affine hull, i.e. if there exists a point $y_0 \in \overline{\text{co}}\ S$ such that $\overline{\text{co}}\ \{S - y_0\}$ has nonempty interior in $\overline{\text{span}}\ (S - y_0)$ (the closed subspace spanned by $S - y_0$).*

Let us consider the following optimization problem

$$\varphi(G(x)) \to \min \tag{1}$$

subject to:

$$G(x) \in S. \tag{2}$$

Here X is a complete metric space, Y is a Banach space, S is a closed subset of Y, $G : X \to Y$ and $\varphi : Y \to R$ are maps. We consider again the Banach space $Y \times R$ equipped with the uniform norm $\|(y,r)\| := \max\{\|y\|, |r|\}$.

Theorem 3.8. *Let $\bar{x}$ be the solution of the problem (1)–(2). We set $\bar{y} = G(\bar{x})$, $\tilde{S} := S \times (-\infty, \varphi(\bar{y})]$ and*

$$\tilde{\mathcal{R}} := \{(G(x), \varphi(G(x)) + \|G(x) - \bar{y}\|^2) : \ x \in X\} \subset Y \times R.$$

Let $C^{\tilde{S}}$ be an uniform tangent cone to $\tilde{S}$ at the point $\tilde{y} := (\bar{y}, \varphi(\bar{y}))$ and $C^{\tilde{\mathcal{R}}}$ be an uniform tangent cone to the set $\tilde{\mathcal{R}}$ at the point $\tilde{y}$. We assume that the set

$$\left(C^{\tilde{S}} \cap \ \tilde{\mathbf{B}}\right) - \left(C^{\tilde{\mathcal{R}}} \cap \ \tilde{\mathbf{B}}\right)$$

is quasisolid, where $\tilde{\mathbf{B}}$ *is the closed unit ball in* $Y \times R$*. Then there exists a nontrivial pair* $(\xi, \eta) \in Y^* \times R$ *such that*

(*i*) $(\xi, \eta) \neq (\mathbf{0}, 0)$;

(*ii*) $\eta \in \{0, 1\}$;

(*iii*) ξ *belongs to the polar cone of the cone* C^S;

(*iv*) $-(\xi, \eta)$ *belongs to the polar cone of the cone* $C^{\tilde{\mathcal{R}}}$.

Remark 3.9. The assertion of Theorem 3.8 remains true if one of the approximating cones is a convex sequence uniform tangent cone. The same remark applies also to Corollary 3.10, Theorem 3.12 and Corollary 3.13.

Corollary 3.10. *Let* $\bar{x}$ *be a solution of the problem* (1)–(2) *and let* φ *be strictly differentiable at* $\bar{y}$*. Let* C^S *be an uniform tangent cone to* S *at the point* $\bar{y}$ *with* $\bar{y} = G(\bar{x})$ *and* $C^{\mathcal{R}}$ *be an uniform tangent cone to the set* $\mathcal{R} := \{G(x) : \ x \in X\}$ *at the point* $\bar{y}$*. Setting* $C^{\tilde{\mathcal{R}}} := \{(v, \varphi'(\bar{y})v) : \ v \in C^{\mathcal{R}}\}$ *and* $C^{\tilde{S}} := \{(v, r) : \ v \in C^S, \ r \leq 0\}$*, we assume that the set*

$$\left(C^{\tilde{S}} \cap \ \tilde{\mathbf{B}}\right) - \left(C^{\tilde{\mathcal{R}}} \cap \ \tilde{\mathbf{B}}\right)$$

is quasisolid. Then there exist a nontrivial pair $(\xi, \eta) \in Y^* \times R$ *such that*

(*i*) $(\xi, \eta) \neq (\mathbf{0}, 0)$;

(*ii*) $\eta \in \{0, 1\}$;

(*iii*) ξ *belongs to the polar cone of the cone* C^S;

(*iv*) $-\xi - \eta\varphi'(\bar{y})$ *belongs to the polar cone of the cone* $C^{\mathcal{R}}$.

Remark 3.11. Another starting point of the present study were the papers [8] and [9]. Theorem 3.8 is very similar to the abstract result (Theorem 6) in [8]. Unfortunately, the proof of the latter result contains a lacuna. In fact, the problem is that uniformity with respect to the direction is not assumed in Definition 2(c) of [8]. Indeed, the example in the introduction provides a counter example to the multiplier rule in Theorem 6: We take $\mathcal{W}$ to be the Hilbert cube $\{(x_n)_{n=1}^{\infty} \in l_2 : \ |x_n| \leq 1/n\} \subset l_2$, $Q := \{\lambda y : \ \lambda \in R\}$ with $y = (1/n^{3/4})_{n=1}^{\infty}$, $S : l_2 \to l_2$ to be the the identity map and $J : l_2 \to R$ to be defined as follows: $J(x_1, x_2, \dots) := x_1$. Then the solution of the optimization problem $\min_{S(x) \in Q, \ x \in \mathcal{W}} J(x)$ is 0 because the origin of l_2 is the only point belonging to the intersection of $\mathcal{W}$ and Q. The gradient of J coincides with J itself (because of the linearity of J). On the other hand-side, every element of the Clarke tangent cone to the set $\mathcal{W}$ at the origin is a sequential strict derivative of S at the same point (because S is the identity map). Since the Clarke tangent cone to the set $\mathcal{W}$ at the origin is the whole space l_2, it is easy to check that $(J(z), z)$ is a sequential strict derivative of (J, S) at the origin for each $z \in l_2$. Then Theorem 6 of [8] would yield the existence of a non trivial pair $(\psi^0, \psi) \in R \times l_2$ so that

$$\psi^0 J(z) + \langle \psi, z \rangle \geq 0 \text{ for each } z \in l_2 \text{ and } \langle \psi, \eta \rangle \leq 0 \text{ for each } \eta \in Q.$$

The first inequality yields $\langle \psi, z\rangle = -\psi^0 J(z)$ on l_2 and $\psi^0 \neq 0$. This and the second inequality (in fact it is an equality) gives $0 = \langle \psi, y\rangle = -\psi^0 J(y) \neq 0$, a contradiction.

Nevertheless, we think that the statement of the problem in [8] and [9] is very interesting and could be used successfully in the future.

Let us consider the following optimal control problem

$$\varphi(x(T)) \to \min \tag{3}$$

subject to the semilinear dynamics:

$$\begin{aligned} \dot{x}(t) = {} & Ax(t) + f(t, x(t), u(t)) \text{ a.e. in } [0, T], \\ x(0) = {} & x_0 \in Y, \quad x(T) \in S \\ u(\cdot) \in {} & \mathcal{U} := \{u(\cdot) : [0, T] \to U \mid u(\cdot) \text{ is strongly measurable}\}. \end{aligned} \tag{4}$$

Here Y is a Banach space, U is a separable complete metric space, the linear operator A is the infinitesimal generator of a strongly continuous semigroup $\{\mathcal{S}(t) \mid t \geq 0\}$. The function $f : [0, T] \times Y \times U \to Y$ is strongly measurable for any fixed $(x, u) \in Y \times U$, the function f is Fréchet differentiable in x for any fixed (t, u), the function φ is strictly Fréchet differentiable, the functions $f(t, \cdot, \cdot)$ and $f'_x(t, \cdot, \cdot)$ are jointly continuous, $f'_x(t, \cdot, u)$ is a locally uniformly continuous function uniformly with respect to $u \in U$ and $t \in [0, T]$. Moreover, there is $M > 0$ such that

$$\|f'_x(t, x, u)\| \leq M \quad \text{and} \quad \|f(t, 0, u)\| \leq M$$

for each $(t, x, u) \in [0, T] \times Y \times U$.

It is standard to consider the set $\mathcal{U}$ endowed with the metric

$$\text{dist}\ (u_1(\cdot), u_2(\cdot)) := \text{meas}\ \{t \in [0, T] : \ u_1(t) \neq u_2(t)\}.$$

It turns out that ($\mathcal{U}$, dist) is a complete metric space (cf. Lemma 7.2 in [3]).

Let $\tilde{\mathbf{B}}$ denote the unit ball in $Y \times R$ centered at the origin and $G(u)$ denote the mild solution of the control system (4) corresponding to the control $u \in \mathcal{U}$ at the moment $t = T$. Let us fix $\bar{u} \in \mathcal{U}$ and set $\bar{x} := G(\bar{u})$.

In the formulation of the next theorem we use a class of popular control variations, namely the so called diffuse variations, introduced to our knowledge by Xunjing Li and his co-workers in the early 1980's (cf., for example [6]). It is proved in [4] that the set of all diffuse variations is an uniform tangent set to the reachable set. It is important to note that the cone generated by the set of all diffuse variations is not obliged to be an uniform tangent cone to the reachable set (cf. Example 2.5 from [5]).

Theorem 3.12. *Let us fix $\bar{u} \in \mathcal{U}$. Let C^S be an uniform cone tangent to the target set S at the point $\bar{x}$ and $C^{\mathcal{R}}$ be an uniform cone tangent to the reachable set $\mathcal{R} := \{G(u) : u \in U\}$ at the point $\bar{x}$. We set*

$$C^{\tilde{\mathcal{R}}} := \{(v, \varphi'(\bar{x})v) : \ v \in C^{\mathcal{R}}\} \text{ and } C^{\tilde{S}} := \{(v, r) : \ v \in C^S, \ r \leq 0\}.$$

(*i*) *If there exists $\rho > 0$ such that the set*

$$co\left(\left(C^{\tilde{\mathcal{R}}} \cap \ \tilde{\mathbf{B}}\right) - \left(C^{\tilde{S}} \cap \ \tilde{\mathbf{B}}\right)\right)$$

is dense in $\rho\tilde{B}$, then $\bar{u}$ is not optimal;

(*ii*) *If the cone $co\left(C^{\tilde{\mathcal{R}}} - C^{\tilde{S}}\right)$ is not dense in $Y \times R$ and $C^{\mathcal{R}}$ contains all diffuse variations at $\bar{u}$, then the following necessary condition of Pontryagin maximum* *principle type holds true: there exist a nontrivial* $(\psi(\cdot),\psi^0) \in C_{w^*}([0,T],Y^*) \times (-\infty,0]$ *such that*

$$\psi(t) = \mathcal{S}(t-T)\psi(T) + \int_t^T \mathcal{S}(t-s)\left(f'_x(s,\bar{x}(s),\bar{u}(s))\right)^* \psi(s)\,ds,$$

$$H(t,\bar{x}(t),\bar{u}(t),\psi(t)) = \max_{u\in U} H(t,\bar{x}(t),u,\psi(t)) \ \text{a.e. in } [0,T],$$

$$\left\langle \psi(T) + \psi^0\varphi'(\bar{x}(T)), v\right\rangle \geq 0 \ \ \text{for each } v \in C^S$$

and

$$\langle \psi(T), v\rangle \leq 0 \ \ \text{for each } v \in C^{\mathcal{R}},$$

where

$$H(t,y,u,\psi) = \langle \psi, f(t,y,u)\rangle\,.$$

Corollary 3.13. *Let* $\bar{u} \in \mathcal{U}$ *be a solution of the optimal control problem (3)–(4) and let* $\bar{x} = G(\bar{u})$*. Let* C^S *be an uniform cone tangent to the target set* S *at the point* $\bar{x}$ *and* $C^{\mathcal{R}}$ *be an uniform cone tangent to the reachable set* $\mathcal{R} := \{G(u) :\ u \in U\}$ *at the point* $\bar{x}$*. We set*

$$C^{\tilde{\mathcal{R}}} := \{(v,\varphi'(\bar{x})v) :\ v \in C^{\mathcal{R}}\} \text{ and } C^{\tilde{S}} := \{(v,r) :\ v \in C^S,\ r \leq 0\}.$$

If the set

$$\left(C^{\tilde{\mathcal{R}}} \cap\ \tilde{\mathbf{B}}\right) - \left(C^{\tilde{S}} \cap\ \tilde{\mathbf{B}}\right)$$

is quasisolid and $C^{\mathcal{R}}$ *contains the set of all diffuse variations at* $\bar{u}$*, then the following necessary condition of Pontryagin maximum principle type holds true: there exist a nontrivial* $(\psi(\cdot),\psi^0) \in C_{w^*}([0,T],Y^*) \times (-\infty,0]$ *such that*

$$\psi(t) = \mathcal{S}(t-T)\psi(T) + \int_t^T \mathcal{S}(t-s)\left(f'_x(s,\bar{x}(s),\bar{u}(s))\right)^* \psi(s)\,ds,$$

$$H(t,\bar{x}(t),\bar{u}(t),\psi(t)) = \max_{u\in U} H(t,\bar{x}(t),u,\psi(t)) \ \text{a.e. in } [0,T],$$

$$\left\langle \psi(T) + \psi^0\varphi'(\bar{x}(T)), v\right\rangle \geq 0 \ \ \text{for each } v \in C^S$$

and

$$\langle \psi(T), v\rangle \leq 0 \ \ \text{for each } v \in C^{\mathcal{R}},$$

where

$$H(t,y,u,\psi) = \langle \psi, f(t,y,u)\rangle\,.$$

REFERENCES

[1] A. Bressan. On the Intersection of a Clarke Cone with a Boltyanskii Cone. *Siam Journal on Control and Optimization*, **45**, No 6 (2007), 2054–2064.

[2] H. Frankowska. Some inverse mapping theorems. *Annales de l'Institut Henri Poincaré, section C*, **7**, No 3 (1990), 183–234.

[3] I. Ekeland. On the variational principle. *J. Math. Anal. Appl.*, **47** (1974), 324—353.

[4] M. I. Krastanov, N. K. Ribarska, Ts. Y. Tsachev. A Pontryagin maximum principle

for infinite-dimensional problems. *SIAM Journal on Control and Optimization*, **49**, No 5 (2011), 2155–2182, DOI: 10.1137/100799009.

[5] M. I. Krastanov, N. K. Ribarska, Ts. Y. Tsachev. On the Geometry of the Pontryagin Maximum Principle in Banach Spaces. *Set-Valued and Variational Analysis*, **23**, No 3 (2015), 443–463, DOI 10.1007/s11228-015-0316-9.

[6] X. J. Li, Y. Yao. Maximum principle of distributed parameter systems with time lags. *Springer Lecture Notes in Control and Inform. Sci.*, **75** (1985), 410-–427.

[7] X. J. Li, J. Yong. Optimal control theory for infinite dimensional systems. Basel, Birkhäuser, 1994.

[8] M. McAsey, L. Mou. A multiplier rule on a metric space. *J. Math. Anal. Appl.*, **337** (2008), 1064–1071.

[9] M. McAsey, L. Mou. A proof of a general maximum principle for optimal controls via a multiplier rule on a metric space. *J. Math. Anal. Appl.*, **337** (2008), 1072–1088.

[10] *G. Peano.* Sur la définition de la dérivée. Mathesis, (2), II (1892), 12–14.

[11] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko. Selected works, vol. 4. The mathematical theory of optimal processes. Ed. and with a preface by R. V. Gamkrelidze. Transl. from the Russian by K. N. Trirogoff. Transl. ed. by L. W. Neustadt. With a preface by L. W. Neustadt and K. N. Trirogoff. Reprint of the 1962 Engl. translation. Classics of Soviet Mathematics. New York, NY: Gordon and Breach Science Publishers, 1986, 360 pp.

[12] R. T. Rockafellar. Clarke's tangent cones and the boundaries of closed sets in R^n. *Nonlinear Analysis: Theory, Methods & Applications*, **3**, No 1 (1979), 145 –154.

[13] R. T. Rockafellar, R. J.-B. Wets. Variational Analysis. Springer, 2009.

[14] H. Sussmann. On the validity of the transversality condition for different concepts of tangent cone to a set. Proceedings of the 45-th IEEE CDC, San Diego, CA, December 13-15, 2006, 241–246.

Mikhail I. Krastanov
e-mail: krastanov@fmi.uni-sofia.bg
Nadezhda K. Ribarska
e-mail: ribarska@fmi.uni-sofia.bg
Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, J. Bourchier Blvd
1164 Sofia, Bulgaria
and
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria

ВЪРХУ ГЕОМЕТРИЯТА НА ПРИНЦИПА НА МАКСИМУМ НА ПОНТРЯГИН В БЕЗКРАЙНОМЕРНИ ПРОСТРАНСТВА

Михаил Кръстанов, Надежда Рибарска

Обсъждат се две нови понятия за апроксимация на множества чрез равномерни конуси.Основният резултат е теорема за неотделимост на две затворени множества. Като приложение на този резултат са получени правилото на множителите на Лагранж и необходимо условие за оптималност от типа на принципа на максимума на Понтрягин в безкрайномерни пространства.