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# ROTATIONAL SURFACES WITH CONSTANT MEAN CURVATURE IN PSEUDO-EUCLIDEAN 4-SPACE WITH NEUTRAL METRIC*

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We give the classification of constant mean curvature rotational surfaces of elliptic, hyperbolic, and parabolic type in the four-dimensional pseudo-Euclidean space with neutral metric.

**1. Introduction.** In the Minkowski 4-space $\mathbb{E}^4_1$ there exist three types of rotational surfaces with two-dimensional axis – rotational surfaces of elliptic, hyperbolic or parabolic type, known also as surfaces invariant under spacelike rotations, hyperbolic rotations or screw rotations, respectively. A *rotational surface of elliptic type* is an orbit of a regular curve under the action of the orthogonal transformations of $\mathbb{E}^4_1$ which leave a timelike plane point-wise fixed. Similarly, a *rotational surface of hyperbolic type* is an orbit of a regular curve under the action of the orthogonal transformations of $\mathbb{E}^4_1$ which leave a spacelike plane point-wise fixed. A *rotational surface of parabolic type* is an orbit of a regular curve under the action of the orthogonal transformations of $\mathbb{E}^4_1$ which leave a degenerate plane point-wise fixed.

The marginally trapped surfaces in Minkowski 4-space which are invariant under spacelike rotations (rotational surfaces of elliptic type) were classified by S. Haesen and M. Ortega in [6]. The classification of marginally trapped surfaces in $\mathbb{E}^4_1$ which are invariant under boost transformations (rotational surfaces of hyperbolic type) was obtained in [5] and the classification of marginally trapped surfaces which are invariant under screw rotations (rotational surfaces of parabolic type) is given in [7].

Motivated by the classification results of S. Haesen and M. Ortega about marginally trapped rotational surfaces in the Minkowski space, in [4] G. Ganchev and the second author considered three types of rotational surfaces in the four-dimensional pseudo-Euclidean space $\mathbb{E}^4_2$, namely rotational surfaces of elliptic, hyperbolic, and parabolic type, which are analogous to the three types of rotational surfaces in $\mathbb{E}^4_1$. They classified all quasi-minimal rotational surfaces of elliptic, hyperbolic, and parabolic type.

Constant mean curvature surfaces in arbitrary spacetime are important objects for the special role they play in the theory of general relativity. The study of constant mean curvature surfaces (CMC surfaces) involves not only geometric methods but also PDE

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and complex analysis, that is why the theory of CMC surfaces is of great interest not only for mathematicians but also for physicists and engineers. Surfaces with constant mean curvature in Minkowski space have been studied intensively in the last years. See for example [1, 2, 10, 11, 12]. Classification results for rotational surfaces in three-dimensional space forms satisfying some classical extra conditions have also been obtained. For example, a classification of all timelike and spacelike hyperbolic rotational surfaces with non-zero constant mean curvature in the three-dimensional de Sitter space $\mathbb{S}^3_1$ is given in [8] and a classification of the spacelike and timelike Weingarten rotational surfaces of the three types in $\mathbb{S}^3_1$ is found in [9]. Chen spacelike rotational surfaces of hyperbolic or elliptic type are described in [3].

In the present paper we study Lorentz rotational surfaces of elliptic, hyperbolic, and parabolic type and give the classification of all such surfaces with non-zero constant mean curvature in $\mathbb{E}^4_2$.

**2. Preliminaries.** Let $\mathbb{E}^4_2$ be the pseudo-Euclidean 4-space endowed with the canonical pseudo-Euclidean metric of index 2 given by $g_0 = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$, where $(x_1, x_2, x_3, x_4)$ is a rectangular coordinate system of $\mathbb{E}^4_2$. As usual, we denote by $\langle\, , \rangle$ the indefinite inner scalar product with respect to $g_0$. A non-zero vector $v$ is called *spacelike* (respectively, *timelike*) if $\langle v, v\rangle > 0$ (respectively, $\langle v, v\rangle < 0$). A vector $v$ is called *lightlike* if it is nonzero and satisfies $\langle v, v\rangle = 0$.

A surface $M^2_1$ in $\mathbb{E}^4_2$ is called *Lorentz* if the induced metric $g$ on $M^2_1$ is Lorentzian, i.e. at each point $p \in M^2_1$ we have the following decomposition $\mathbb{E}^4_2 = T_pM^2_1 \oplus N_pM^2_1$ with the property that the restriction of the metric onto the tangent space $T_pM^2_1$ is of signature $(1, 1)$, and the restriction of the metric onto the normal space $N_pM^2_1$ is of signature $(1, 1)$.

Denote by $\nabla$ and $\nabla'$ the Levi Civita connections of $M^2_1$ and $\mathbb{E}^4_2$, respectively. Let $x$ and $y$ denote vector fields tangent to $M^2_1$ and $\xi$ be a normal vector field. The formulas of Gauss and Weingarten are given respectively by

$$\nabla'_x y = \nabla_x y + \sigma(x, y);$$
$$\nabla'_x \xi = -A_\xi x + D_x \xi,$$

where $\sigma$ is the second fundamental form, $D$ is the normal connection, and $A_\xi$ is the shape operator with respect to $\xi$. In general, $A_\xi$ is not diagonalizable.

The mean curvature vector field $H$ of the surface $M^2_1$ is defined as $H = \dfrac{1}{2}\operatorname{tr}\sigma$. A surface $M^2_1$ is called *minimal* if its mean curvature vector vanishes identically, i.e. $H = 0$. A surface $M^2_1$ is *quasi-minimal* if its mean curvature vector is lightlike at each point, i.e. $H \neq 0$ and $\langle H, H\rangle = 0$. In this paper we consider Lorentz surfaces in $\mathbb{E}^4_2$ for which $\langle H, H\rangle = const \neq 0$.

**3. Lorentz rotational surfaces with constant mean curvature in $\mathbb{E}^4_2$.** Let $Oe_1e_2e_3e_4$ be a fixed orthonormal coordinate system in the pseudo-Euclidean space $\mathbb{E}^4_2$ such that $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 1$, $\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = -1$. Lorentz rotational surfaces of elliptic, hyperbolic, and parabolic type are defined in [4]. Here we shall present shortly the construction.

First we consider rotational surfaces of elliptic type. Let $c : \widetilde{z} = \widetilde{z}(u),\ u \in J$ be a smooth spacelike curve lying in the three-dimensional subspace $\mathbb{E}^3_1 = \operatorname{span}\{e_1, e_2, e_3\}$ and parameterized by $\widetilde{z}(u) = (x_1(u), x_2(u), r(u), 0)\,;\ u \in J$. Without loss of generality we assume that $c$ is parameterized by the arc-length, i.e. $(x_1')^2 + (x_2')^2 - (r')^2 = 1$, and

$r(u) > 0,\ u \in J$.

Let $\mathcal{M}'$ be the surface in $\mathbb{E}^4_2$ defined by

$$(1) \qquad \mathcal{M}' : z(u,v) = (x_1(u), x_2(u), r(u)\cos v, r(u)\sin v)\,; \quad u \in J,\ v \in [0; 2\pi).$$

The surface $\mathcal{M}'$, defined by (1), is a Lorentz surface in $\mathbb{E}^4_2$, obtained by the rotation of the spacelike curve $c$ about the two-dimensional Euclidean plane $Oe_1e_2$. It is called a *rotational surface of elliptic type*.

One can also obtain a rotational surface of elliptic type in $\mathbb{E}^4_2$ using rotation of a timelike curve about the two-dimensional plane $Oe_3e_4$ (see [4]).

Next, we consider rotational surfaces of hyperbolic type. Let $c : \widetilde{z} = \widetilde{z}(u),\ u \in J$ be a smooth spacelike curve, lying in the three-dimensional subspace $\mathbb{E}^3_1 = \operatorname{span}\{e_1, e_2, e_4\}$ of $\mathbb{E}^4_2$ and parameterized by $\widetilde{z}(u) = (r(u), x_2(u), 0, x_4(u))\,;\ u \in J$. Without loss of generality we assume that $c$ is parameterized by the arc-length, i.e. $(r')^2 + (x_2')^2 - (x_4')^2 = 1$, and $r(u) > 0,\ u \in J$.

Let $\mathcal{M}''$ be the surface in $\mathbb{E}^4_2$ defined by

$$(2) \qquad \mathcal{M}'' : z(u,v) = (r(u)\cosh v, x_2(u), r(u)\sinh v, x_4(u))\,; \quad u \in J,\ v \in \mathbb{R}.$$

The surface $\mathcal{M}''$, defined by (2), is a Lorentz surface in $\mathbb{E}^4_2$, obtained by hyperbolic rotation of the spacelike curve $c$ about the two-dimensional Lorentz plane $Oe_2e_4$. $\mathcal{M}''$ is called a *rotational surface of hyperbolic type*.

Similarly, one can obtain a rotational surface of hyperbolic type using hyperbolic rotation of a timelike curve lying in $\operatorname{span}\{e_2, e_3, e_4\}$ about the two-dimensional Lorentz plane $Oe_2e_4$. Rotational surfaces of hyperbolic type can also be obtained by hyperbolic rotations of spacelike or timelike curves about the two-dimensional Lorentz planes $Oe_1e_3$, $Oe_1e_4$ and $Oe_2e_3$.

Now, we shall consider rotational surfaces of parabolic type in $\mathbb{E}^4_2$. For convenience, in the parabolic case we use the pseudo-orthonormal base $\{e_1, e_4, \xi_1, \xi_2\}$ of $\mathbb{E}^4_2$, such that $\xi_1 = \dfrac{e_2 + e_3}{\sqrt{2}}$, $\xi_2 = \dfrac{-e_2 + e_3}{\sqrt{2}}$. Note that $\langle \xi_1, \xi_1 \rangle = 0$; $\langle \xi_2, \xi_2 \rangle = 0$; $\langle \xi_1, \xi_2 \rangle = -1$.

Let $c$ be a spacelike curve lying in the subspace $\mathbb{E}^3_1 = \operatorname{span}\{e_1, e_2, e_3\}$ of $\mathbb{E}^4_2$ and parameterized by $\widetilde{z}(u) = x_1(u)\, e_1 + x_2(u)\, e_2 + x_3(u)\, e_3;\ u \in J$, or equivalently,

$$\widetilde{z}(u) = x_1(u)\, e_1 + \frac{x_2(u) + x_3(u)}{\sqrt{2}}\, \xi_1 + \frac{-x_2(u) + x_3(u)}{\sqrt{2}}\, \xi_2; \quad u \in J.$$

Denote $f(u) = \dfrac{x_2(u) + x_3(u)}{\sqrt{2}}$, $g(u) = \dfrac{-x_2(u) + x_3(u)}{\sqrt{2}}$. Then $\widetilde{z}(u) = x_1(u)\, e_1 + f(u)\, \xi_1 + g(u)\, \xi_2$. Without loss of generality we assume that $c$ is parameterized by the arc-length, i.e. $(x_1')^2 + (x_2')^2 - (x_3')^2 = 1$, or equivalently $(x_1')^2 - 2f'g' = 1$.

A rotational surface of parabolic type is defined in the following way:

$$(3) \quad \mathcal{M}''' : z(u,v) = x_1(u)\, e_1 + f(u)\, \xi_1 + (-v^2 f(u) + g(u))\, \xi_2 + \sqrt{2}\, v f(u)\, e_4;\ u \in J,\ v \in \mathbb{R}.$$

The rotational axis is the two-dimensional plane spanned by $e_1$ (a spacelike vector field) and $\xi_1$ (a lightlike vector field).

Similarly, one can obtain a rotational surface of parabolic type using a timelike curve lying in the subspace $\operatorname{span}\{e_2, e_3, e_4\}$ (see [4]).

In what follows, we find all CMC Lorentz rotational surfaces of elliptic, hyperbolic, and parabolic type.

**3.1. *Constant mean curvature rotational surfaces of elliptic type.*** Let us consider the surface $\mathcal{M}'$ in $\mathbb{E}^4_2$, defined by (1). The tangent space of $\mathcal{M}'$ is spanned by the vector fields $z_u = (x_1', x_2', r'\cos v, r'\sin v)\,;\ z_v = (0, 0, -r\sin v, r\cos v)$. Hence, the coefficients of the first fundamental form of $\mathcal{M}'$ are $E = \langle z_u, z_u\rangle = 1$; $F = \langle z_u, z_v\rangle = 0$; $G = \langle z_v, z_v\rangle = -r^2(u)$. Since the generating curve $c$ is a spacelike curve parameterized by the arc-length, i.e. $(x_1')^2 + (x_2')^2 - (r')^2 = 1$, then $(x_1')^2 + (x_2')^2 = 1 + (r')^2$ and $x_1'x_1'' + x_2'x_2'' = r'r''$. We consider the orthonormal tangent frame field $X = z_u$; $Y = \dfrac{z_v}{r}$, and the normal frame field $\{n_1, n_2\}$, defined by

$$\begin{aligned} n_1 &= \frac{1}{\sqrt{1+(r')^2}}\,(-x_2', x_1', 0, 0);\\ n_2 &= \frac{1}{\sqrt{1+(r')^2}}\,\big(r'x_1', r'x_2', (1+(r')^2)\cos v, (1+(r')^2)\sin v\big). \end{aligned}\tag{4}$$

Note that $\langle X, X\rangle = 1$; $\langle X, Y\rangle = 0$; $\langle Y, Y\rangle = -1$; $\langle n_1, n_1\rangle = 1$; $\langle n_1, n_2\rangle = 0$; $\langle n_2, n_2\rangle = -1$.

Calculating the second partial derivatives of $z(u, v)$ and the components of the second fundamental form, we obtain the formulas:

$$\begin{aligned} \sigma(X,X) &= \frac{x_1'x_2'' - x_1''x_2'}{\sqrt{1+(r')^2}}\,n_1 + \frac{r''}{\sqrt{1+(r')^2}}\,n_2,\\ \sigma(X,Y) &= 0,\\ \sigma(Y,Y) &= \qquad\qquad\qquad -\frac{\sqrt{1+(r')^2}}{r}\,n_2, \end{aligned}$$

which imply that the normal mean curvature vector field $H$ of $\mathcal{M}'$ is expressed as follows

$$H = \frac{1}{2r\sqrt{1+(r')^2}}\,\big(r(x_1'x_2'' - x_1''x_2')\,n_1 + (rr'' + (r')^2 + 1)\,n_2\big)\,. \tag{5}$$

Hence, $\langle H, H\rangle = \dfrac{r^2(x_1'x_2'' - x_1''x_2')^2 - (rr'' + (r')^2 + 1)^2}{4r^2(1+(r')^2)}$. In the present paper we are interested in rotational surfaces with non-zero constant mean curvature. So, we assume that $r^2(x_1'x_2'' - x_1''x_2')^2 - (rr'' + (r')^2 + 1)^2 \neq 0$.

It follows from (4) that

$$\begin{aligned} \nabla'_X n_1 &= -\frac{x_1'x_2'' - x_1''x_2'}{\sqrt{1+(r')^2}}\,X + \frac{r'}{1+(r')^2}(x_1'x_2'' - x_1''x_2')\,n_2,\\ \nabla'_Y n_1 &= 0,\\ \nabla'_X n_2 &= \frac{r''}{\sqrt{1+(r')^2}}\,X + \frac{r'}{1+(r')^2}(x_1'x_2'' - x_1''x_2')\,n_1,\\ \nabla'_Y n_2 &= \frac{\sqrt{1+(r')^2}}{r}\,Y. \end{aligned}\tag{6}$$

If $x_1'x_2'' - x_1''x_2' = 0,\ rr'' + (r')^2 + 1 \neq 0$, then from (6) we get $\nabla'_X n_1 = \nabla'_Y n_1 = 0$, which imply that the normal vector field $n_1$ is constant. Hence, the surface $\mathcal{M}'$ lies in the hyperplane $\mathbb{E}^3_2 = \mathrm{span}\{X, Y, n_2\}$.

So, further we consider rotational surfaces of elliptic type satisfying $x_1'x_2'' - x_1''x_2' \neq 0$ in an open interval $I \subset J$.

In the next theorem we give a local description of all constant mean curvature rotational surfaces of elliptic type.

**Theorem 3.1.** *Given a smooth positive function $r(u) : I \subset \mathbb{R} \to \mathbb{R}$, define the functions*

$$\varphi(u) = \eta \int \frac{\sqrt{(rr'' + (r')^2 + 1)^2 \pm 4C^2 r^2 (1 + (r')^2)}}{r(1 + (r')^2)} \, du, \quad \eta = \pm 1, \; C = const \neq 0$$

*and*

$$x_1(u) = \int \sqrt{1 + (r')^2} \, \cos \varphi(u) \, du,$$
$$x_2(u) = \int \sqrt{1 + (r')^2} \, \sin \varphi(u) \, du.$$

*Then the spacelike curve $c : \widetilde{z}(u) = (x_1(u), x_2(u), r(u), 0)$ is a generating curve of a constant mean curvature rotational surface of elliptic type.*

*Conversely, any constant mean curvature rotational surface of elliptic type is locally constructed as above.*

*Proof:* Let $\mathcal{M}'$ be a general rotational surface of elliptic type generated by a spacelike curve $c : \widetilde{z}(u) = (x_1(u), x_2(u), r(u), 0)\, ;\ u \in J$. We assume that $c$ is parameterized by the arc-length and $x_1' x_2'' - x_1'' x_2' \neq 0$ for $u \in I \subset J$. Using (5) we get that $\mathcal{M}'$ is of constant mean curvature if and only if

$$\frac{r^2 (x_1' x_2'' - x_1'' x_2')^2 - (rr'' + (r')^2 + 1)^2}{4r^2 (1 + (r')^2)} = \varepsilon C^2, \; \varepsilon = sign\langle H, H\rangle, \; C = const \neq 0. \tag{7}$$

Since the curve $c$ is parameterized by the arc-length, we have $(x_1')^2 + (x_2')^2 = 1 + (r')^2$, which implies that there exists a smooth function $\varphi = \varphi(u)$ such that

$$\begin{aligned} x_1'(u) &= \sqrt{1 + (r')^2} \, \cos \varphi(u), \\ x_2'(u) &= \sqrt{1 + (r')^2} \, \sin \varphi(u). \end{aligned} \tag{8}$$

It follows from (8) that $x_1' x_2'' - x_1'' x_2' = (1 + (r')^2)\varphi'$. Hence, condition (7) is written in terms of $r(u)$ and $\varphi(u)$ as follows:

$$\varphi'(u) = \eta \frac{\sqrt{(rr'' + (r')^2 + 1)^2 \pm 4C^2 r^2 (1 + (r')^2)}}{r(1 + (r')^2)}, \quad \eta = \pm 1. \tag{9}$$

Formula (9) allows us to recover the function $\varphi(u)$ from $r(u)$, up to integration constant. Using formulas (8), we can recover $x_1(u)$ and $x_2(u)$ from the functions $\varphi(u)$ and $r(u)$, up to integration constants. Consequently, the constant mean curvature rotational surface of elliptic type $\mathcal{M}'$ is constructed as described in the theorem.

Conversely, given a smooth function $r(u) > 0$, we can define the function

$$\varphi(u) = \eta \int \frac{\sqrt{(rr'' + (r')^2 + 1)^2 \pm 4C^2 r^2 (1 + (r')^2)}}{r(1 + (r')^2)} \, du,$$

where $\eta = \pm 1$, and consider the functions

$$x_1(u) = \int \sqrt{1 + (r')^2} \, \cos \varphi(u) \, du,$$
$$x_2(u) = \int \sqrt{1 + (r')^2} \, \sin \varphi(u) \, du.$$

A straightforward computation shows that the curve $c : \widetilde{z}(u) = (x_1(u), x_2(u), r(u), 0)$ is a spacelike curve generating a constant mean curvature rotational surface of elliptic type according to formula (1).

$\square$

*Remark*: In the special case when $rr'' + (r')^2 + 1 = 0$, i.e $r(u) = \pm\sqrt{-u^2 + 2au + b},\ a = const \neq 0,\ b = const$, the function $\varphi(u)$ is expressed as

$$\varphi(u) = \frac{2C}{\sqrt{a^2+b}}\left(\frac{u-a}{2}\sqrt{-u^2+2au+b} + \frac{a^2+b}{2}\arcsin\frac{u-a}{\sqrt{a^2+b}} + d\right),\ d = const.$$

***3.2. Constant mean curvature rotational surfaces of hyperbolic type.*** Now, we shall consider the rotational surface of hyperbolic type $\mathcal{M}''$, defined by (2). The tangent space of $\mathcal{M}''$ is spanned by the vector fields $z_u = (r' \cosh v, x_2', r' \sinh v, x_4')$; $z_v = (r \sinh v, 0, r \cosh v, 0)$, and the coefficients of the first fundamental form of $\mathcal{M}''$ are $E = 1;\ F = 0;\ G = -r^2(u)$.

The generating curve $c$ is a spacelike curve parameterized by the arc-length, i.e. $(r')^2 + (x_2')^2 - (x_4')^2 = 1$, and hence $(x_4')^2 - (x_2')^2 = (r')^2 - 1$. We assume that $(r')^2 \neq 1$, otherwise the surface lies in a 2-dimensional plane. Denote by $\varepsilon$ the sign of $(r')^2 - 1$.

We use the orthonormal tangent frame field $X = z_u;\ Y = \dfrac{z_v}{r}$ and the normal frame field $\{n_1, n_2\}$, defined by

$$\begin{aligned} n_1 &= \frac{1}{\sqrt{\varepsilon((r')^2-1)}}\,(0, x_4', 0, x_2'); \\ n_2 &= \frac{1}{\sqrt{\varepsilon((r')^2-1)}}\left((1-(r')^2)\cosh v, -r'x_2', (1-(r')^2)\sinh v, -r'x_4'\right). \end{aligned} \tag{10}$$

The frame field $\{X, Y, n_1, n_2\}$ satisfies $\langle X, X\rangle = 1;\ \langle X, Y\rangle = 0;\ \langle Y, Y\rangle = -1;\ \langle n_1, n_1\rangle = \varepsilon;\ \langle n_1, n_2\rangle = 0;\ \langle n_2, n_2\rangle = -\varepsilon$.

The mean curvature vector field $H$ of $\mathcal{M}''$ is expressed by the following formula

$$H = \frac{\varepsilon}{2r\sqrt{\varepsilon((r')^2-1)}}\left(r(x_4'x_2'' - x_4''x_2')\,n_1 - (rr'' + (r')^2 - 1)\,n_2\right).$$

If $x_2'x_4'' - x_2''x_4' = 0,\ rr'' + (r')^2 - 1 \neq 0$, $\mathcal{M}''$ lies in the hyperplane span$\{X, Y, n_2\}$. So, we assume that $x_2'x_4'' - x_2''x_4' \neq 0$ in an open interval $I \subset J$.

The local classification of constant mean curvature rotational surfaces of hyperbolic type is given by the following theorem:

**Theorem 3.2.** Case (A). *Given a smooth positive function $r(u) : I \subset \mathbb{R} \to \mathbb{R}$, such that $(r')^2 > 1$, define the functions*

$$\varphi(u) = \eta\int\frac{\sqrt{(rr'' + (r')^2 - 1)^2 \pm 4C^2r^2((r')^2-1)}}{r((r')^2-1)}\,du, \quad \eta = \pm 1,\ C = const \neq 0,$$

*and*

$$\begin{aligned} x_2(u) &= \int\sqrt{(r')^2-1}\,\sinh\varphi(u)\,du, \\ x_4(u) &= \int\sqrt{(r')^2-1}\,\cosh\varphi(u)\,du. \end{aligned}$$

*Then the spacelike curve $c : \widetilde{z}(u) = (r(u), x_2(u), 0, x_4(u))$ is a generating curve of a constant mean curvature rotational surface of hyperbolic type.*

Case (B). *Given a smooth positive function $r(u): I \subset \mathbb{R} \to \mathbb{R}$, such that $(r')^2 < 1$, define the functions*

$$\varphi(u) = \eta \int \frac{\sqrt{(rr'' + (r')^2 - 1)^2 \pm 4C^2 r^2((r')^2 - 1)}}{r((r')^2 - 1)}\, du, \quad \eta = \pm 1,\ C = const \neq 0,$$

*and*

$$x_2(u) = \int \sqrt{1 - (r')^2}\, \cosh \varphi(u)\, du,$$

$$x_4(u) = \int \sqrt{1 - (r')^2}\, \sinh \varphi(u)\, du.$$

*Then the spacelike curve $c : \widetilde{z}(u) = (r(u), x_2(u), 0, x_4(u))$ is a generating curve of a constant mean curvature rotational surface of hyperbolic type.*

*Conversely, any constant mean curvature rotational surface of hyperbolic type is locally described by one of the cases given above.*

The proof of the theorem is similar to the proof of Theorem 3.1.

In the case $rr'' + (r')^2 - 1 = 0$, i.e. $r(u) = \pm\sqrt{u^2 + 2au + b}$, $a = const \neq 0$, $b = const$, the function $\varphi(u)$ is expressed by the formula

$$\varphi(u) = \frac{2\eta C}{\sqrt{\varepsilon(a^2 - b)}} \left( \frac{u + a}{2} \sqrt{u^2 + 2au + b} - \frac{\varepsilon(a^2 - b)}{2} \ln \left| u + a + \sqrt{u^2 + 2au + b} \right| + d \right).$$

***3.3. Constant mean curvature rotational surfaces of parabolic type.*** Now we shall consider the rotational surface of parabolic type $\mathcal{M}'''$, defined by formula (3). The length of the mean curvature vector field of $\mathcal{M}'''$ is given by

$$\langle H, H \rangle = \frac{1}{4f^2 f'^2} \left( f^2 (x_1'' f' - x_1' f'')^2 - (f f'' + (f')^2)^2 \right).$$

In the case $x_1'' f' - x_1' f'' = 0$, $f f'' + (f')^2 \neq 0$ the surface $\mathcal{M}'''$ lies in a hyperplane of $\mathbb{E}^4_2$. So, we assume that $x_1'' f' - x_1' f'' \neq 0$ in an open interval $I \subset J$.

In the following theorem we give a local description of constant mean curvature rotational surfaces of parabolic type.

**Theorem 3.3.** *Given a smooth function $f(u): I \subset \mathbb{R} \to \mathbb{R}$, define the functions*

$$\varphi(u) = f' \left( A \pm \int \frac{1}{f'} \sqrt{\left((\ln |f f'|)'\right)^2 \pm 4C^2}\, du \right), \qquad C = const \neq 0,\ A = const,$$

*and*

$$x_1(u) = \int \varphi(u) du; \qquad g(u) = \int \frac{\varphi^2(u) - 1}{2f'(u)}\, du.$$

*Then the curve $c : \widetilde{z}(u) = x_1(u)\, e_1 + f(u)\, \xi_1 + g(u)\, \xi_2$ is a spacelike curve generating a constant mean curvature rotational surface of parabolic type.*

*Conversely, any constant mean curvature rotational surface of parabolic type is locally constructed as described above.*

In the special case $f f'' + (f')^2 = 0$, i.e. $f(u) = \pm\sqrt{2au + b}$, $a = const \neq 0, b = const$, we obtain the following expression for the function $\varphi(u)$:

$$\varphi(u) = \frac{1}{\sqrt{2au + b}} \left( A \pm \frac{2CB}{3a} \left( \sqrt{2au + b} \right)^3 \right), \ A = const,\ B = const \neq 0.$$

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## РОТАЦИОННИ ПОВЪРХНИНИ С ПОСТОЯННА СРЕДНА КРИВИНА В ЧЕТИРИМЕРНО ПСЕВДО-ЕВКЛИДОВО ПРОСТРАНСТВО С НЕУТРАЛНА МЕТРИКА


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В четиримерно псевдо-Евклидово пространство с неутрална метрика съществуват три типа ротационни повърхнини с двумерна ос – това са ротационни повърхнини от елиптичен, хиперболичен и параболичен тип. В настоящата статия е дадена класификация на ротационните повърхнини от елиптичен, хиперболичен и параболичен тип с постоянна средна кривина.