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ABOUT REMAINDERS OF RECTIFIABLE SPACES*

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The remainders of rectifiable and dissentive spaces are studied. Conditions are determined under which a remainder of a given dissentive or rectifiable space is paracompact.

1. Introduction. By a space we understand a completely regular topological space. We use the terminology from [11]. The Souslin number, or cellularity, of the space X is the smallest cardinal number $c(X)$ such that any family of pairwise disjoint non-empty open subsets has cardinality $\leq c(X)$ (see [11]).

A *remainder* of a Tychonov space X is the subspace $Y \setminus X$ of a compactification Y of X. It is well-known that a space X has a Hausdorff compactification if and only if X is Tychonov (see [11]).

In this article we consider what kind of remainders a space and a paracompact p-space can have. Distinct properties of remainders were studied in [2, 3, 4, 5, 6, 7, 8, 9, 12]. The present notes are motivated by the works [2, 5, 6] of A. V. Arhangel'skii.

A completely regular space X is a p-space if for any (or some) compactification Y of X there exists a countable family $\{\gamma_n : n \in \mathbb{N}\}$ of families of open subsets of Y such that $x \in \cap\{St(\gamma_n, x) : n \in \mathbb{N}\} \subseteq X$ for each $x \in X$, where $St(\gamma_n, x) = \cup\{U \in \gamma_n : x \in U\}$ (see [1]). It is well-known that paracompact p-spaces are preimages of metrizable spaces under perfect mappings (see [1]). Recall that a space X is of a countable type if every compact subspace F of X is contained in a compact subspace $\Phi \subseteq X$ which has a countable base of open neighbourhoods in X. Every p-space is of a countable type and every metrizable space is a p-space (see [1]).

2. On charming and dissentive spaces. Let $\mathcal{P}$ be a class of spaces with the following properties:

– any space $X \in \mathcal{P}$ is Lindelöf;

– if $\{X_n \in \mathcal{P} : n \in \mathbb{N}\}$ is a disjoint countable family of spaces, then the discrete sum $\oplus\{X_n : n \in \mathbb{N}\} \in \mathcal{P}$;

– if Y is a closed subspace of a space $X \in \mathcal{P}$, then $Y \in \mathcal{P}$;

– if $X, Y \in \mathcal{P}$, then $X \times Y \in \mathcal{P}$;

– if a space Y is a continuous image or a perfect inverse image of a space $X \in \mathcal{P}$, then $Y \in \mathcal{P}$.

Obviously, the class $\mathcal{P}$ is σ-additive.

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A space X is called a $\mathcal{P}$-charming space if there exists a subspace Y of X such that $Y \in \mathcal{P}$ and for any open neighbourhood U of Y in X we have $X \setminus U \in \mathcal{P}$ (see [5, 6]). The subspace Y is called a $\mathcal{P}$-kernel of the $\mathcal{P}$-charming space X. Obviously, any $\mathcal{P}$-charming space is Lindelöf and the class of $\mathcal{P}$-charming spaces is invariant under perfect mappings, is preserved by the continuous mappings, is σ-additive and hereditary with respect to closed subspaces.

K. Nagami has defined the important class of Σ-spaces [15] which contains the class of metrizable spaces and the class of spaces with a σ-discrete network. Lindelöf Σ-spaces can be characterized as continuous images of Lindelöf p-spaces. In particular, every Lindelöf p-space is in this class. Every space with a countable network, and every σ-compact space is in this class too.

If $\mathcal{P}$ is the class of Lindelöf Σ-spaces, then any $\mathcal{P}$-charming space is called a charming space [5, 6].

Example 2.1. Let K be an uncountable metrizable compact space without isolated points, $Z_0 \subseteq K$, $Z_1 = K \setminus Z_0$ be a dense countable subspace of K, S be an uncountable dense subspace of K, $Z_1 \subseteq S$, $\mathcal{T}$ be the topology on K. We put $A(K) = K \times \{0,1\}$, $A_0 = K \times \{0\}$, $A_1 = K \times \{1\}$, $Z = Z_0 \times \{0\}$, $Y = A_0 \cup (S \times \{1\})$ and $X = (Z_1 \times \{0\}) \cup (S \times \{1\})$. On Y consider the topology generated by the open base $\mathcal{B} = \{(U \times \{0,1\}) \setminus F : U \in \mathcal{T}, F$ *is a finite subset of* $A_1\} \cup \{\{y\} : y \in A_1\}$. The space $A(K)$ is the Alexandroff duplicate of the space K (see [11], Example 3.1.26). It is well-known that $A(Y)$ is a non-metrizable hereditarily paracompact space which is a compactification of the discrete space A_1 and the remainder $A_0 = A(K) \setminus A_1$ is a metrizable compact space homeomorphic to K. The spaces X and Y as the subspaces of $A(K)$ are spaces with the following properties:

– Y is a compact non-metrizable space;

– X is a a non-metrizable hereditarily paracompact space with a σ-disjoint base;

– X is not a p-space;

– Y is a compactification of X and the remainder $Z = Y \setminus X$ is a complete metrizable separable space;

– if S is a Lusin subspace (is an uncountable topological space without isolated points in which every nowhere dense subset is countable [14, 13]), then X is a Lindelöf space. In this case X is a charming space and a $\mathcal{P}_\omega$-charming space relatively to the class $\mathcal{P}_\omega$ of countable spaces.

A Mal'cev operation on a space X is a continuous mapping $\mu : X^3 \longrightarrow X$ such that $\mu(x,x,z) = z$ and $\mu(x,y,y) = x$ for all $x,y,z \in X$. A space is called a Mal'cev space if it admits a Mal'cev operation.

A dissentive operation (a dissentor) on a space G is a continuous mapping $\mu : G^3 \longrightarrow G$ satisfying the following conditions:

– $\mu(x,x,y) = y$ for all $x,y \in G$;

– for every open set U of G and all $b,c \in G$ the set $\mu(U,b,c) = \{\mu(x,b,c) : x \in U\}$ is open in G. A space is dissentive if it admits a dissentive operation (see [9]).

A dissentive space G with a dissenter μ is hereditary dissentive if for every non-empty subspace A of G the mapping μ is a dissentor on A provided $\mu(A,A,A) \subseteq A$ (in this case we say that A is a subdissentive subspace of the dissentive space G).

A rectification on a space X is a homeomorphism $g : X \times X \longrightarrow X \times X$ with the following two properties:

– $g(x \times X) = x \times X$ for every $x \in X$;

– there exists $e \in X$ such that $g(x, x) = (x, e)$ for every point $x \in X$.

A space with a rectification is called rectifiable. A non-empty space G is rectifiable if and only if there exist a pair of binary continuous operations $p, q : G \times G \longrightarrow G$ and a point $c \in G$ such that $p(x, x) = c$, $p(x, q(x, y)) = y$, $q(x, p(x, y)) = y$ for all $x, y \in G$ (see [10]). In this case the ternary operation $\mu(x, y, z) = p(x, q(y, z))$ is a Mal'cev dissentive operation. Hence every rectifiable space is hereditary dissentive and a Mal'cev space.

For a non-empty subset X of a dissentive space with the dissenter μ we put $a_0(X, G, \mu) = X$ and $a_n(X, G, \mu) = \{\mu(x, y, z) : x, y, z \in a_{n-1}(X, G, \mu)\}$ for each $n \in \mathbb{N}$. Let $a(X, G, \mu) = \cup\{a_n(X, G, \mu) : n \in \mathbb{N}\}$. Then $\mu(a(X, G, \mu),\ a(X, G, \mu),\ a(X, G, \mu)) = a(X, G, \mu)$. If $G = a(X, G, \mu)$, then we say that G is generated by the set X. If $a(X, G, \mu)$ is dense in G, then we say that G is topologically generated by the set X.

Theorem 2.1. *Let $\mathcal{K}$ be a class of hereditary dissentive spaces such that if $G \in \mathcal{K}$ and A is a subdissentive subspace of G, then $A \in \mathcal{K}$. The following assertions are equivalent for a class $\mathcal{P}$:*

1. *If $G \in \mathcal{K}$ and $G \in \mathcal{P}$, then $c(G) = \aleph_0$.*
2. *If $A, G \in \mathcal{K}$, A is a dense subspace of G and $A \in \mathcal{P}$, then $c(G) = \aleph_0$.*
3. *If $X \subseteq G \in \mathcal{K}$, $X \in \mathcal{P}$ and G is topologically generated by the set X, then $c(G) = \aleph_0$.*
4. *If $A, G \in \mathcal{K}$, A is a dense subspace of G and A is a $\mathcal{P}$-charming space, then $c(G) = \aleph_0$.*

Proof. The implications $4 \to 2$ and $3 \to 2 \to 1$ are obvious.

Claim 1. If $X \subseteq G \in \mathcal{K}$, $X \in \mathcal{P}$ and G is generated by the set X, then $G \in \mathcal{P}$.

We have $\mu(Y, Y, Y) = \mu(Y \times Y \times Y) = \{\mu(x, y, z) : x, y, z \in Y\}$ for every subset Y of G. If $X \in \mathcal{P}$, then $a_n(X, G, \mu) \in \mathcal{P}$ for each $n \in \mathbb{N}$. Hence $\cup\{a_n(X, G, \mu) : n \in \mathbb{N}\} \in \mathcal{P}$. Since $G = a(X, G, \mu) = \cup\{a_n(X, G, \mu) : n \in \mathbb{N}\}$, Claim 1 is proved.

Implication $1 \to 3$ follows from Claim 1.

Claim 2. If $A \in \mathcal{K}$ and A is a $\mathcal{P}$-charming space, then there exists a dense subdissentive subspace B of A such that $B \in \mathcal{P}$.

Assume that $X \subseteq A$ is the $\mathcal{P}$-kernel of the space A. We have two cases.

Case 1. The set X is dense in A.

In this case, by virtue of Claim 1, $B = a(X, G, \mu) \in \mathcal{P}$ and B is dense in A.

Case 2. The set X is not dense in A.

In this case there exists a point $b \in A$ and an open subset U of A such that $b \in U \subseteq cl_A U \subseteq A \setminus cl_A X$. By construction, $F = cl_A X \in \mathcal{P}$. Let $\mu : A^3 \longrightarrow A$ be the dissentor on A. For each $c \in A$ consider the mapping $f_c : A \longrightarrow A$, where $f_c(x) = \mu(x, b, c)$ for each $x \in A$. The set $U_c = f_c(U)$ is open in A and $c = f_c(b) \in U_c$. Let $F_c = f_c(F)$. Then $F_c \in \mathcal{P}$. Since A is a Lindelöf space and $\{U_c : c \in A\}$ is an open cover of A, there exists a countable subset $C \subseteq A$ such that $A = \cup\{U_c : c \in C\}$. Since $A = \cup\{C_c : c \in C\}$, we have $A \in \mathcal{P}$. In this case $B = A$. Claim 2 is proved.

Implication $2 \to 4$ follows from Claim 2. The proof is complete. □

Remark 2.1. Implications $4 \to 3 \to 2 \to 1 \to 3$ are true for topological algebras with a countable set of continuous operations. In particular, these implications are true for Mal'cev spaces.

Corollary 2.2. *Theorem 3.1 is true for the class of rectifiable spaces.*

For topological groups the following assertions are proved in [5, 6].

Corollary 2.3. *Let G be a Mal'cev hereditary dissentive space. If G is a charming space, then G contains a dense Mal'cev hereditary dissentive Σ-subspace and $c(G) = \aleph_0$.*

Proof. In [16, 17] V. V. Uspenskii proved that $c(G) = \aleph_0$ for any Σ-Lindelöf-Mal'cev space G. Theorem 2.1 completes the proof.

Corollary 2.4. *Let G be a rectifiable space. If G is a charming space, then G contains a dense rectifiable Σ-subspace and $c(G) = \aleph_0$.*

Corollary 2.5. *Let G be a dissentive space. The following conditions are equivalent:*

1. *Some remainder of G is paracompact.*
2. *All remainders of G are Lindelöf.*
3. *G is a space of countable type.*

Proof. Implication $2 \to 1$ is obvious. Implications $3 \to 2 \to 3$ follow from the famous classical result of M. Henriksen and J. Isbell [12] which affirms that a Tychonov space X is of countable type if and only if the remainder in any (or some) Hausdorff compactification of X is Lindelöf.

Let Y be a compactification of G and $Z = Y \setminus G$ be a paracompact space. In ([9], Corollary 4.8) it was proved that any remainder of G is either pseudocompact or Lindelöf. Since a pseudocompact paracompact space is compact, the space Z is compact or Lindelöf. By virtue of the mentioned theorem of M. Henriksen and J. Isbell [12], G is a space of countable type. The proof is complete. $\square$

Corollary 2.6. *Let G be a paracompact rectifiable space. The following conditions are equivalent:*

1. *Some remainder of G is paracompact.*
2. *Some remainder of G is Lindelöf.*
3. *All remainders of G are charming spaces.*
4. *G is a paracompact p-space.*

Proof. In ([9], Corollary 2.8) was proved that a rectifiable space of countable type is a p-space. In ([5], Corollary 3.5) it was proved that for a paracompact p-space X every remainder is a charming space. The proof is complete. $\square$

Theorem 2.7. *Suppose that G is a rectifiable space and that Y is a compactification of G. Then the remainder $Z = Y \setminus G$ is a p-space if and only if at least one of the following conditions holds:*

(*a*) *G is a Lindelöf p-space.*

(*b*) *G is σ-compact;*

(*c*) *G is a locally compact space.*

Proof. If G is a locally compact space, then the remainder Z is a compact space and a p-space.

Assume that Z is a p-space and G is a non-locally compact rectifiable space. In this case Z is dense in Y and Y is a compactification of Z too. By virtue of ([9], Corollary 4.8) the remainder Z of G is either pseudocompact or Lindelöf.

Case 1. Z is a Lindelöf space.

In this case Z is a Lindelöf p-space. By virtue of ([5], Theorem 1.3), if X is a Lindelöf p-space, then any remainder of X is a Lindelöf p-space. Hence G is a Lindelöf p-space.

Case 2. Z is a pseudocompact space.

It is well-known that a pseudocompact p-space is Čech-complete. In this case G is σ-compact. The proof is complete. $\square$

Remark 2.8. It is not known if a rectifiable space of countable type is paracompact. If the response of this question is positive, then the requirement that G is a paracompact space in the conditions of Corollary 2.6 can be omitted. In this aspect we mention that a topological group of countable type is paracompact.

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ОТНОСНО КОМПАКТИФИЦИРАЩИТЕ МНОЖЕСТВА НА РЕКТИФИЦИРУЕМИ ПРОСТРАНСТВА

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Изследвани са компактифициращите множества на ректифицируеми топологични пространства. Намерени са условия, при които компактифициращите множества на ректифицируеми пространства са паракомпактни пространства.