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MINIMAL SUBBASES OF DISCRETE SPACES*

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The aim of this paper is to determine for any natural number n the least natural number k, such that the discrete space of cardinality n has a subbase of cardinality k.

1. Introduction. A *base* of a topological space X is a collection of open subsets of X, such that every open subset of X can be represented as a union of some elements of this collection. A *subbase* of a topological space X is a collection of open subsets of X, such that the finite intersections of this collection (including the empty intersection) form a base of X.

In this paper we are interested in discrete spaces X only.

We fix a natural number $n \geq 2$ and a discrete space X with n elements. Let $a : \{1, 2, \dots, n\} \to X$ be a bijection, so that $X = \{a_1, a_2, \dots, a_n\}$ and $a_i \neq a_j$ for $i \neq j$. Every subset of X is open.

We are interested in bases and subbases of X of minimum cardinality.

2. The problem for bases. The problem for bases is rather easy. We have the following

Theorem 1. *A collection of subsets of the discrete space X is a base for X if and only if it contains all singletons of X.*

Proof. Let $x \in X$. The singleton $\{x\}$ has exactly 2 subsets – $\emptyset, \{x\}$. So if $\{x\}$ is the union of a collection of sets, then at least one set in the collection must be $\{x\}$ and all the others must be $\emptyset$ or $\{x\}$. Therefore, any base of X contains the singletons of X. The converse is trivial, because any set is the union of its singletons. $\square$

It follows that we have a least base of X with respect to inclusion, which has cardinality n. It is the family $\{\{x\} | x \in X\}$ of the singletons of X. Of course, the same argument is valid for infinite cardinals n.

3. The problem for subbases. We fix a natural number k.

An *antichain* of $\{1, 2, \dots, k\}$ is a finite sequence of subsets of $\{1, 2, \dots, k\}$, such that any two members of the sequence are incomparable with respect to inclusion. Of course, every antichain has different members. In set theory antichains are usually regarded as sets, but for our purposes we need sequences.

A *subbase sequence* of X is a finite sequence of subsets of X, such that the set of all members of the sequence is a subbase of X.

We show that there exists a bijective correspondence between the subbase sequences of X of length k and the antichains of $\{1, 2, \dots, k\}$ of length n.

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Let $S = (S_1, S_2, \ldots, S_k)$ be a subbase sequence of X of length k. We form the sequence $\mathbb{A}(S) = (A_1, A_2, \ldots, A_n)$ of subsets of $\{1, 2, \ldots, k\}$ by defining

$$(1) \qquad A_1 = \{p | a_1 \in S_p\}, \;\; A_2 = \{p | a_2 \in S_p\}, \;\; \ldots, \;\; A_n = \{p | a_n \in S_p\}.$$

We claim that $\mathbb{A}(S)$ is an antichain of $\{1, 2, \ldots, k\}$ of length n. Indeed, suppose that $A_i \subseteq A_j$ for some i, j, such that $i \neq j$. We have for all p

$$p \in A_i \Longrightarrow p \in A_j,$$

that is for all p

$$(2) \qquad a_i \in S_p \Longrightarrow a_j \in S_p.$$

Since S is a subbase sequence, there exist $S_{p_1}, S_{p_2}, \ldots, S_{p_r}$, such that

$$\{a_i\} = S_{p_1} \cap S_{p_2} \cap \ldots \cap S_{p_r}.$$

We have that $a_i \in S_{p_1}, \ldots, a_i \in S_{p_r}$, therefore, from (2) $a_j \in S_{p_1}, \ldots, a_j \in S_{p_r}$, that is

$$a_j \in S_{p_1} \cap S_{p_2} \cap \ldots \cap S_{p_r} = \{a_i\}.$$

So $a_j = a_i$, which is a contradiction.

Conversely, let $A = (A_1, A_2, \ldots, A_n)$ be an antichain of $\{1, 2, \ldots, k\}$ of length n. We define the sequence $\mathbb{S}(A) = (S_1, S_2, \ldots, S_k)$ of subsets of X by

$$(3) \qquad S_1 = \{a_i | 1 \in A_i\}, \;\; S_2 = \{a_i | 2 \in A_i\}, \;\; \ldots, \;\; S_k = \{a_i | k \in A_i\}.$$

We show that $\mathbb{S}(A)$ is a subbase sequence of X. In order to do this, we prove that

$$(4) \qquad \{a_i\} = \bigcap \{S_p | p \in \{1, \ldots, k\} \text{ and } a_i \in S_p\}$$

for all $i \in \{1, 2, \ldots, n\}$. The inclusion from left to right is obvious. For the reverse inclusion let $j \in \{1, 2, \ldots, n\}$ be a number, such that a_j is in the intersection in the right-hand side of (4), that is a_j belongs to all S_p, such that $a_i \in S_p$. We claim that $A_i \subseteq A_j$. Indeed, we have for all p

$$p \in A_i \Longrightarrow a_i \in S_p \Longrightarrow a_j \in S_p \Longrightarrow p \in A_j.$$

But A is an antichain so it follows $i = j$ and $a_i = a_j$. This proves the inclusion from right to left. The equality (4) shows that every singleton of X is a finite intersection of members of $\mathbb{S}(A)$ and since X is discrete, it follows that $\mathbb{S}(A)$ is a subbase sequence of X.

Theorem 2. *For all antichains A of $\{1, 2, \ldots, k\}$ of length n we have $A = \mathbb{A}(\mathbb{S}(A))$. For all subbase sequences S of X of length k we have $S = \mathbb{S}(\mathbb{A}(S))$.*

Proof. For the first part we start with an antichain A of $\{1, 2, \ldots, k\}$ of length n, $A = (A_1, A_2, \ldots, A_n)$. Let $S = \mathbb{S}(A) = (S_1, S_2, \ldots, S_k)$ be the corresponding subbase sequence. From (3) we have $S_p = \{a_i | p \in A_i\}$. Let $A' = \mathbb{A}(\mathbb{S}(A))$, $A' = (A'_1, A'_2, \ldots, A'_n)$, where from (1) $A'_i = \{p | a_i \in S_p\}$. For all $i \in \{1, 2, \ldots, n\}$ we have

$$p \in A'_i \Longleftrightarrow a_i \in S_p \Longleftrightarrow p \in A_i,$$

so $A'_i = A_i$. It follows that $A = A' = \mathbb{A}(\mathbb{S}(A))$. For the second part we start with a subbase sequence S of length k, $S = (S_1, S_2, \ldots, S_k)$. Let $A = \mathbb{A}(S) = (A_1, A_2, \ldots, A_n)$ be the corresponding antichain, $A_i = \{p | a_i \in S_p\}$. Let $S' = \mathbb{S}(\mathbb{A}(S)) = (S'_1, S'_2, \ldots, S'_k)$, where from (3) $S'_p = \{a_i | p \in A_i\}$. For all $p \in \{1, 2, \ldots, k\}$ we have

$$a_i \in S'_p \Longleftrightarrow p \in A_i \Longleftrightarrow a_i \in S_p,$$

so $S'_p = S_p$. It follows that $S = S' = \mathbb{S}(\mathbb{A}(S))$. $\square$

As a corollary the correspondence defined above is bijective.

Usually a subbase is thought of as a collection, not a sequence. Therefore, it is interesting to characterize the antichains A, such that the subbase sequence $\mathbb{S}(A)$ has different members and therefore can be regarded as a set.

Let A be an antichain of $\{1, 2, \ldots, k\}$. The numbers $p, q \in \{1, 2, \ldots, k\}$ are said to be *indiscernible* by A, if for all i we have $p \in A_i \iff q \in A_i$. The antichain A is *discernible* if no pair of different numbers is indiscernible by A. Let $S = \mathbb{S}(A)$, $S = (S_1, S_2, \ldots, S_k)$. Two numbers $p, q \in \{1, 2, \ldots, k\}$ are indiscernible by A if and only if $S_p = S_q$. So we obtain

Proposition 1. *For an antichain A of $\{1, 2, \ldots, k\}$ we have: the subbase sequence $\mathbb{S}(A)$ consists of different members if and only if the antichain A is discernible.*

It is obvious that excluding the empty set from a subbase produces again a subbase. Therefore, it is interesting to characterize the antichains A, which correspond to subbase sequences not containing the empty set.

An antichain A of $\{1, 2, \ldots, k\}$ of length n is *total*, if for all $p \in \{1, 2, \ldots, k\}$ there exists $i \in \{1, 2, \ldots, n\}$ such that $p \in A_i$, that is the union $A_1 \cup A_2 \cup \ldots \cup A_n$ is equal to $\{1, 2, \ldots, k\}$.

Proposition 2. *For an antichain A of $\{1, 2, \ldots, k\}$ we have: the subbase sequence $\mathbb{S}(A)$ consists of non-empty sets if and only if the antichain A is total.*

Proof. We have $S_p = \{a_i | p \in A_i\} = \emptyset$ if and only if p does not belong to any A_i. There exists p with $S_p = \emptyset$ if and only if there exists p which does not belong to the union $A_1 \cup A_2 \cup \ldots \cup A_n$. It remains to take the negation of both sides of this equivalence. $\square$

According to a theorem of Sperner in [1], the maximum length of an antichain of $\{1, 2, \ldots, k\}$ is the central binomial coefficient $c(k) = \binom{k}{\lfloor k/2 \rfloor}$. It can be achieved by taking all subsets of $\{1, 2, \ldots, k\}$ having cardinality exactly $\lfloor k/2 \rfloor$. Since a subsequence of an antichain is again an antichain, it follows that for all $l \le c(k)$ there is an antichain of $\{1, 2, \ldots, k\}$ of length l.

Lemma 1. *If there exists a subbase of X of cardinality k, then $n \le c(k)$.*

Proof. Suppose there exists a subbase of X of cardinality k. Then we can choose a subbase sequence S of length k. Let $A = \mathbb{A}(S)$ be the corresponding antichain of $\{1, 2, \ldots, k\}$ of length n. Since $c(k)$ is the maximum length of an antichain of $\{1, 2, \ldots, k\}$, we conclude that $n \le c(k)$. $\square$

Theorem 3. *The unique natural number k, satisfying the inequalities*

$$c(k-1) < n \le c(k),$$

is the minimum possible cardinality for a subbase of X.

Proof. We show that there exists a subbase of X of cardinality exactly k. Since $n \le c(k)$ we can choose an antichain A of $\{1, 2, \ldots, k\}$ of length n. Let $S = \mathbb{S}(A)$ be the corresponding subbase sequence of length k. Let $l \le k$ be the cardinality of the subbase of X, consisting of the members of S. Applying the above lemma, we obtain $n \le c(l)$. If $l \le k - 1$, then $n \le c(l) \le c(k-1)$, which contradicts the above assumption. It follows that $l = k$, so there exists a subbase of X of cardinality k. The same argument shows that k is the least possible. $\square$

The number k from the theorem can also be defined as the least natural number, satisfying the inequality $n \le c(k)$.

The problem for subbases for infinite cardinals n is a lot easier than its finite counterpart. For infinite cardinals n, the minimum cardinality of a subbase of the space X of cardinality n is n itself. It cannot be less than n, because otherwise there wouldn't exist enough finite intersections to obtain all singletons of X.

4. Examples. In this last section we give some examples.

The following table gives some values for n and the corresponding values for k – the minimum cardinality of a subbase for the discrete space X with n elements.

n	k
10	5
100	9
1000	13
10000	16
100000	20
1000000	23
10000000	26
100000000	30
1000000000	33
10000000000	37
100000000000	40

One subbase of minimum cardinality $k = 6$ of the discrete space X with $n = 20$ elements consists of the sets:

$$S_1 = \{a_1, a_2, a_3, a_5, a_6, a_8, a_{11}, a_{12}, a_{14}, a_{17}\},$$
$$S_2 = \{a_1, a_2, a_4, a_5, a_7, a_9, a_{11}, a_{13}, a_{15}, a_{18}\},$$
$$S_3 = \{a_1, a_3, a_4, a_6, a_7, a_{10}, a_{12}, a_{13}, a_{16}, a_{19}\},$$
$$S_4 = \{a_2, a_3, a_4, a_8, a_9, a_{10}, a_{14}, a_{15}, a_{16}, a_{20}\},$$
$$S_5 = \{a_5, a_6, a_7, a_8, a_9, a_{10}, a_{17}, a_{18}, a_{19}, a_{20}\},$$
$$S_6 = \{a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, a_{18}, a_{19}, a_{20}\}.$$

It corresponds to an antichain of $\{1, 2, 3, 4, 5, 6\}$ in which every set has exactly 3 members.

REFERENCES

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МИНИМАЛНИ ПРЕДБАЗИ НА ДИСКРЕТНИ ПРОСТРАНСТВА

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Целта на тази статия е да се определи за всяко естествено число n най-малкото естествено число k, такова че дискретното пространство с кардиналност n има предбаза с кардиналност k.