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HETEROCLINIC SOLUTIONS FOR A SECOND-ORDER DIFFERENCE EQUATION RELATED TO FISHER–KOLMOGOROV'S EQUATION*

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We study the existence of heteroclinic solutions of semilinear second-order difference equations related to the Fisher-Kolmogorov's equation $\Delta^2 y(t-1) + k\Delta y(t-1) + f(y(t)) = 0$ for $k \in (1,2)$. Analogous equation is considered in [5] and this paper continues the considerations there. The proves of the present results are based on monotonicity and continuity arguments.

1. Introduction. In the present paper we study the existence of heteroclinic solutions of the second-order difference equation

$$\Delta^2 y(t-1) + k\Delta y(t-1) + f(y(t)) = 0, t \in \mathbb{Z} \quad for \quad k \in (1,2) \tag{1}$$

under suitable assumptions, given in Section 1. This paper extends the consideration for the case $k \in (0,1)$.The proof of the presented results are based on monotonicity and continuity arguments. Equation (1) is related to Fisher-Kolmogorov's equation $u_t = u_{xx} + g(u)$, which was introduced in the papers of Fisher [7] and Kolmogorov [8] and it is originally motivated by models in population dynamic. Looking for traveling waves $u(x,t) = U(x - Ct)$, with speed C, one obtain the second-order ODE

$$U'' + CU' + g(U) = 0. \tag{2}$$

We note that a similar equation

$$\Delta^2 y(t-1) + c\Delta y(t) + f(y(t)) = 0, t \in \mathbb{Z} \tag{3}$$

is considered in [5]. It is easy to see, that (3) is equivalent to Equation (1) with $k = \dfrac{c}{c+1}$.

In [5] is considered the case, when $c > 0$, i.e. $k \in (0,1)$. In the present paper is considered the case $k \in (1,2)$ under some additional conditions for the function $f(.)$, given in Section 1. Here we derive our main results using simple monotonicity and continuity arguments. As it is described in [1], Equations (1) and (3) appear after a discretization and rescaling of Eq. (2). Fast and heteroclinic solutions of Eq. (2) are studied in the paper of Arias [2]. Several methods of considerations of various classes of difference equations can be found for example in [3, 4, 6, 9] etc.

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2. Basic assumptions and the behavior of solutions for $t \geq 0$. We consider the equation (1), where $\Delta y(t) = y(t+1) - y(t)$, $\Delta^2 y(t) = y(t+2) - 2y(t+1) + y(t)$, $\mathbb{Z}$ is the set of integers and the constant number $k \in (1,2)$. For the function $f(.)$ we suppose that the following conditions are fulfilled:

C1. For any two numbers $x \in [0,1]$ and $y \in [0,1]$ and $x \neq y$,

$$|f(x) - f(y)| < (2-k)|x-y|; \tag{4}$$

C2. $f : [0,1] \to \mathbb{R}_+$, where $f(0) = f(1) = 0$ and $f(y) > 0$ for $y \in (0,1)$;

C3. $f(x) < (k-1)(1-x)$ and f is strictly monotonous decreasing function in a small left neighbourhood of 1.

Obviously, we can define the function $f(y)$ for $y \in \mathbb{R}$, as $f(y) = 0$ for $y \notin [0,1]$. So the function $f(.)$ such defined, satisfies Lipshich condition for any two numbers $x \in \mathbb{R}$ and $y \in \mathbb{R}$. In particular thus defined function $f(y)$ is continuous $\forall y \in \mathbb{R}$.

Lemma 1. *Let l be an arbitrary positive integer and z is an arbitrary real number, such that $z \in (0,1)$. Then, there exists a real number $z_0 \in [z,1)$ such that there exists a solution $y(t)$ of (1), satisfying the conditions: $y(t_0) = z_0$, $y(t_0 + l) = z$, (t_0 is an arbitrary integer), $\{y(t)\}$ is a monotonous decreasing function for $t \geq t_0$, i.e. $y(t+1) < y(t)$ for $t \geq t_0$, $t \in \mathbb{Z}$, $y(t) > 0$ for $t \geq t_0$, $t \in \mathbb{Z}$ and $\lim y(t) = 0$ for $t \to +\infty$.*

We give only a sketch of the proof. We choose the solution $\{y(t)\}$ of (1) such that

$$y(t_0) = 1 - \varepsilon_1, \quad y(t_0+1) = 1 - \varepsilon_1 - \varepsilon_2, \tag{5}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are sufficiently small numbers. We define the function $\varepsilon_1(\varepsilon_2)$ so that the number $\varepsilon_1(\varepsilon_2)$ is the smallest positive root of $f(1 - \varepsilon_1(\varepsilon_2) - \varepsilon_2) = (k-1)\varepsilon_2$. We can prove that the function $\varepsilon_1(\varepsilon_2)$ is continuous function for $\varepsilon_2 \in \left(0, \varepsilon_2^0\right]$. We can construct the solution $\{y(t)\}$ of (1) such that it satisfies (5) with $\varepsilon_1 = \varepsilon_1(\varepsilon_2)$ for suitable ε_2, $\{y(t)\}$ is strictly monotonous decreasing function for $t \geq t_0$, $t \in \mathbb{Z}$, $\lim y(t) = 0$ for $t \to +\infty$ and $\{y(t)\}$ satisfies the other conditions of Lemma 1.

3. Heteroclinic solutions of (1). If $y(t)$ is a solution of (1), then

$$y(t) - y(t-1) = -\frac{1}{k-1}(y(t+1) - y(t)) - \frac{1}{k-1} f(y(t)) \tag{6}$$

and $y(t-1) = \dfrac{k-2}{k-1} y(t) + \dfrac{1}{k-1} y(t+1) + \dfrac{1}{k-1} f(y(t))$, $k \in (1,2)$. If $z(t)$ is another solution of (1), then we obtain:

$$\begin{aligned} y(t-1) - z(t-1) &= -\frac{1}{k-1}\left[(2-k)(y(t) - z(t)) - (f(y(t)) - f(z(t)))\right] \\ &\quad + \frac{1}{k-1}(y(t+1) - z(t+1)). \end{aligned} \tag{7}$$

From C1 the sign of $(2-k)(y(t) - z(t)) - (f(y(t)) - f(z(t)))$ coincides with the sign of $y(t) - z(t)$ for $k \in (1,2)$ and we conclude that: If $y(t+1) - z(t+1) \geq 0$ (≤ 0) and $y(t) - z(t) \leq 0$ (≥ 0), where at least one of these inequalities is strong, then $y(t-1) - z(t-1) > 0$ (< 0). In particular, if $\{y(t)\}$ is a solution of (1), then $\{y(t+1)\}$ is also a solution of (1) and for $z(t) = y(t+1)$, we can do the previous conclusion. If we have one of these situations, it follows that the sign of $y(t) - z(t)$ (or $y(t) - y(t+1)$) changes (oscillate) when $t \to -\infty$, i.e. for $t = t_0, t_0 - 1, t_0 - 2, \ldots$. And in each of these cases from (7) it follows that

$$|y(t-1)-z(t-1)| \geq \frac{1}{k-1}|y(t+1)-z(t+1)| \tag{8}$$

or

$$|y(t-1)-y(t)| \geq \frac{1}{k-1}|y(t+1)-y(t+2)|. \tag{9}$$

From (9) it follows that if $\{y(t)\}$ is heteroclinic solution, then for any $t \in \mathbb{Z}$, $t < -N$, N sufficiently large, it is not possible that $y(t+1)-y(t+2) \geq 0$ and $y(t)-y(t+1) \leq 0$, where at least one of these inequalities is strong and the converse case ($y(t+1)-y(t+2) \leq 0$ and $y(t)-y(t+1) \geq 0$, where at least one of these inequalities is strong) is not possible, because from (9) since $\frac{1}{k-1} > 1$ for $k \in (1,2)$, it follows that $\lim |y(t+1)-y(t)| = \infty$ for $t \to -\infty$ and since the signs of $y(t-1)-y(t)$ change alternatively when $t \to -\infty$, then $\lim_{t\to-\infty} y(t)$ does not exists. Thus if $\{y(t)\}$ is heteroclinic solution of (1), then $\{y(t)\}$ has to be monotonous. But if $y(t+1) > y(t)$ for some $t \in \mathbb{Z}$, then from (6) $y(t-1)-y(t) > 0$, i.e. $y(t-1) > y(t) < y(t+1)$ and hence the sequence $\{y(t)\}$ cannot be monotonous when $\forall t \in \mathbb{Z}$. By analogy, if $y(t+1) = y(t) \in (0,1)$ for some $t \in \mathbb{Z}$, then from (6) $y(t-1)-y(t) > 0$ and we obtain

$$\begin{aligned} y(t-2)-y(t-1) &= -\frac{1}{k-1}\left[(2-k)(y(t-1)-y(t)) - (f(y(t-1)) - f(y(t)))\right] \\ &\quad + \frac{1}{k-1}(y(t)-y(t+1)). \end{aligned}$$

Since $\frac{1}{k-1}(y(t)-y(t+1)) = 0$, then from C1,

$$y(t-2)-y(t-1) = -\frac{1}{k-1}\left[(2-k)(y(t-1)-y(t)) - (f(y(t-1)) - f(y(t)))\right] < 0,$$

i.e. $y(t-2) < y(t-1) > y(t)$ and hence again $\{y(t)\}$ cannot be monotonous when $t \in \mathbb{Z}$. Thus if $\{y(t)\}$ is heteroclinic solution of (1) then

$$y(t) > y(t+1), \ t \in \mathbb{Z} \tag{10}$$

and $\lim_{t\to\infty} y(t) = 0$. We prove now that

$$y(t) \in (0,1), \ t \in \mathbb{Z}. \tag{11}$$

Suppose the contrary, i.e. that $y(t_0) \in (0,1)$, but $y(t_0-1) \geq 1$. Then, putting $z(t) \equiv 1$, $t \in \mathbb{Z}$ we obtain that $y(t_0-2) < 1$, $y(t_0-3) > 1$, …, i.e. $y(t_0) < y(t_0-1) > y(t_0-2)$, which contradicts with (10). Thus we prove that (11) holds. If we take limit in (6) for $t \to -\infty$, then we obtain that $f(l_-) = 0$, where $l_- = \lim_{t\to-\infty} y(t)$ and from (11) $l_- \in (0,1]$. Since $f(u) > 0$ for $u \in (0,1)$, we conclude that the only possibility is $l_- = 1$. Thus any heteroclinic solution of (1) satisfies conditions (10) and (11) and $\lim_{t\to\infty} y(t) = 0$, $\lim_{t\to-\infty} y(t) = 1$.

Lemma 2. *For any $y_1 \in (0,1)$, there exists at most one heteroclinic solution $\{y(t)\}$ of (1) with the property $y(1) = y_1$.*

Proof. Let $\{y(t)\}$ and $\{z(t)\}$ are two heteroclinic solutions of (1) for which $y(1) = z(1) = y_1$. If $y(0) = z(0)$, then obviously $y(t) \equiv z(t)$ $\forall t \in \mathbb{Z}$. Let us suppose for example that $y(0) > z(0)$. Since $y(1) = z(1)$, then from (7) $y(-1) < z(-1)$, $y(-2) > z(-2)$, …

and $|y(t-2)-z(t-2)| \geq \dfrac{1}{k-1}|y(t)-z(t)|\ \forall t \in \mathbb{Z}$. Since $\dfrac{1}{k-1} > 1$ for $k \in (1,2)$, then it follows that $\lim\limits_{t\to-\infty}|y(t)-z(t)| = \infty$ and then both $y(t)$ and $z(t)$ cannot be heteroclinic solutions of (1). Lemma 2 is proved. □

Our aim now is to prove the existence of heteroclinic solution $\{y(t)\}$ of (1) for which $y(1) = y_1$. Our main result is

Theorem 3. *For any real number $y_1 \in (0,1)$, there exists an unique heteroclinic solution of (1) $\{y(t)\}$, satisfying the conditions: $y(1) = y_1$, $y(t) > y(t+1)$, $\forall t \in \mathbb{Z}$, $y(t) \in (0,1)$, $t \in \mathbb{Z}$ and $\lim\limits_{t\to+\infty} y(t) = 0$ and $\lim\limits_{t\to-\infty} y(t) = 1$.*

Proof. Let $\{y(t)\}$ be a solution of (1) for which $y(1) = y_1 \in (0,1)$. Let $y(0) = y_0$. If we denote $y(2) = y_2$, then from (6) $y_2 - y_1 = -(k-1)(y_1-y_0) - f(y_1) = (k-1)(y_0-y_1) - f(y_1) < 0$ iff

$$y_1 < y_0 < y_1 + \frac{f(y_1)}{k-1}, \tag{12}$$

i.e. (12) holds if and only if $y_2 < y_1 < y_0$. From (7) it is easy to obtain that

If $y(t-1) - z(t-1) \geq 0$ (>0) and $y(t) - z(t) \geq 0$ (>0), then

$$y(t+1) - z(t+1) \geq 0\ (>0). \tag{13}$$

So (12) is necessary and sufficient condition for $y(t) > y(t+1)$, $t = 0, 1, \ldots$. Thus we obtained that $\{y(t)\}$ is monotonously decreasing solution of (1) for $t \geq 0$ if and only if (12) holds. Further for any fixed real number y_0 satisfying (12), we can obtain the solution $\{y(t)\}$ of (1) for $t \in \mathbb{Z}$. Our aim is to prove that there exists y_0 satisfying (12) such that the obtained solution $y(t) = y(t, y_1, y_0)$ of (1) satisfies the conditions (10) and (11). Let now for $\forall n \in \mathbb{N}$ with $\{y_n(t)\}$ denote the solution of (1), satisfying Lemma 1, i.e. $y_n(1) = y_1$ and $y_n(t) > y_n(t+1)$ $\forall t \geq -n$, $t \in \mathbb{Z}$, i.e. $y_n(t) \equiv y(t, y_1, y_n(0))$ and obviously $y_n(0)$ satisfies (12). Then the obtained sequence $\{y_n(0)\}_{n=1}^{\infty}$ is bounded and one can choose a convergent subsequence $\{y_{n_k}(0)\}_{k=1}^{\infty}$ for which $n_k \to \infty$ when $k \to \infty$ and $\lim\limits_{k\to\infty} y_{n_k}(0) = \overline{y}_0 \in \left[y_1, y_1 + \dfrac{f(y_1)}{k-1}\right]$, see (12). That means the solutions of (1) $\{y(t, y_1, y_{n_k}(0))\}_{k=1}^{\infty}$ have the properties $y(1, y_1, y_{n_k}(0)) = y_1$, $y(0, y_1, y_{n_k}(0)) = y_{n_k}(0)$ and

$$y(t, y_1, y_{n_k}(0)) > y(t+1, y_1, y_{n_k}(0))\ \ \forall t \geq -n_k, t \in \mathbb{Z}. \tag{14}$$

We prove that $y(t) = y(t, y_1, \overline{y}_0)$ is the sought heteroclinic solution. We assume at first that there exists $s_0 \in \mathbb{N}$ such that $y(-s_0) > y(-s_0+1)$, but $y(-s_0) > y(-s_0-1)$, i.e. $y(-s_0, y_1, \overline{y}_0) > y(-s_0+1, y_1, \overline{y}_0)$ and $y(-s_0, y_1, \overline{y}_0) > y(-s_0-1, y_1, \overline{y}_0)$. Since $s_0 \in \mathbb{N}$ is a fixed number, then $y(-s_0, y_1, y_0)$, $y(-s_0+1, y_1, y_0)$, $y(-s_0-1, y_1, y_0)$ are continuous functions with respect to y_0. Hence for sufficiently large numbers $n_k \in \mathbb{N}$,

$$\begin{aligned} y(-s_0, y_1, y_{n_k}(0)) &> y(-s_0+1, y_1, y_{n_k}(0)) \\ y(-s_0, y_1, y_{n_k}(0)) &> y(-s_0-1, y_1, y_{n_k}(0)). \end{aligned} \tag{15}$$

But (15) contradicts (14) for sufficiently large numbers k, such that $n_k >> s_0$. Hence the above assumption is not true. This fact, the fact that $\overline{y}_0 \in \left[y_1, y_1 + \dfrac{f(y_1)}{k-1}\right]$ and (13) show that the solution of (1) $y(t) = y(t, y_1, \overline{y}_0)$ thus defined is a monotonously nonincreasing sequence, i.e.

$$y(t) \geq y(t+1)\,,\ t \in \mathbb{Z}. \tag{16}$$

We assume now that for some $t_0 \in \mathbb{Z}$, $y(t_0) = y(t_0+1)$. Then from (6) we obtain $y(t_0-1) - y(t_0) = \dfrac{1}{k-1} f(y(t_0)) > 0$ and from (7) and C1, $y(t_0-2) - y(t_0-1) = -\dfrac{1}{k-1}\left[(2-k)(y(t_0-1) - y(t_0)) - (f(y(t_0-1)) - f(y(t_0)))\right] < 0$, (because the last expression can be equal to 0 iff $y(t_0-2) = y(t_0-1) = y(t_0) = y(t_0+1)$, i.e. $y(t) \equiv 0$ or $y(t) \equiv 1$, $t \in \mathbb{Z}$, which is impossible). Thus we obtain that $y(t_0-2) < y(t_0-1) > y(t_0)$, which contradicts (16). Thus (16) be in force, must all inequalities in (16) to be strong, i.e. for the our solution $y(t) = y(t, y_1, \overline{y}_0)$, (10) holds, i.e. $y(t) > y(t+1)$, $\forall t \in \mathbb{Z}$. Also if $y_1 \in (0,1)$, then $y_0 > y_1 > 0$ and it follows that $y(t) > 0$, $t \in \mathbb{Z}$. But we proved that if $\{y(t)\}$ satisfies (10), then (11) also holds. Thus we proved that the solution $y(t) = y(t, y_1, \overline{y}_0)$ of (1) satisfies (10) and (11) and as we shown above,

$$\lim_{t\to+\infty} y(t) = 0 \quad \text{and} \quad \lim_{t\to-\infty} y(t) = 1 \tag{17}$$

i.e. $y(t)$ is the sought heteroclinic solution. $\square$

Corollary 4. *For any real number $y_0 \in (0,1)$ and arbitrary $t_0 \in \mathbb{Z}$, there exists an unique heteroclinic solution of* (1) $\{y(t)\}$ *satisfying the conditions:* $y(t_0) = y_0$, $y(t) > y(t+1)$, $t \in \mathbb{Z}$, $y(t) \in (0,1)$, $\forall t \in \mathbb{Z}$ *and* $\lim_{t\to+\infty} y(t) = 0$ *and* $\lim_{t\to-\infty} y(t) = 1$.

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ХЕТЕРОКЛИНИЧНИ РЕШЕНИЯ НА ЕДНО ДИФЕРЕНЧНО УРАВНЕНИЕ ОТ ВТОРИ РЕД СВЪРЗАНО С УРАВНЕНИЕТО НА ФИШЕР–КОЛМОГОРОВ

Дико Моис Суружон

В предлаганата статия се третира проблема за съществуване на хетероклинични решения за диференчно уравнение от втори ред, свързано с уравнението на Фишер–Колмогоров $\Delta^2 y(t-1) + k\Delta y(t-1) + f(y(t)) = 0$ за $k \in (1,2)$. Аналогично уравнение се разглежда в [5] и тази статия е продължение на разглежданията в [5] като доказателствата на представените резултати са базирани на съображения за монотонност и непрекъснатост.