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(2, 3)-GENERATION OF THE SPECIAL LINEAR GROUPS OF DIMENSION 11*

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In the present paper we prove that the group $PSL_{11}(q)$ is $(2,3)$-generated for any q. Actually, we give explicit generators x and y of respective orders 2 and 3, for the linear group $SL_{11}(q)$.

1. Introduction. $(2,3)$-generated groups are those groups which can be generated by an involution and an element of order 3 or, equivalently, they appear to be homomorphic images of the famous modular group $PSL_2(\mathbb{Z})$. It is known that many series of finite simple groups are $(2,3)$-generated. Most powerful result of Liebeck-Shalev and Lübeck-Malle (see [14]) states that, except for the infinite families $PSp_4(2^m)$, $PSp_4(3^m)$ and the Suzuki groups $Sz(2^{2m+1})$, all finite simple groups are $(2,3)$-generated, up to a finite number of exceptions. We have especially focused our attention to the projective special linear groups defined over finite fields. Many authors have been investigated the groups $PSL_n(q)$ with respect to that generation property. $(2,3)$-generation has been proved in the cases $n = 2$, $q \neq 9$ [9], $n = 3$, $q \neq 4$ [5],[2], $n = 4$, $q \neq 2$ [16], [15], [10], [12], $n = 5$, any q [19], [11], $n = 6$, any q [18], $n = 7$, any q [17], $n = 8$, any q [6], $n \geq 5$, odd $q \neq 9$ [3], [4], and $n \geq 13$, any q [13]. In this way the only cases that still remain open are those for $9 \leq n \leq 12$, even q or $q = 9$ (it is a well-known fact that the groups $PSL_2(9) \cong A_6$, $PSL_3(4)$ and $PSL_4(2) \cong A_8$ are not (2,3)-generated). In the forthcoming papers ([7], [8]) we prove that the groups $PSL_9(q)$ and $PSL_{10}(q)$ are $(2,3)$-generated for all q. In the present work we continue our investigation by proving the following:

Theorem. *The groups $SL_{11}(q)$ and $PSL_{11}(q)$ are $(2,3)$-generated for all q.*

2. Proof of the Theorem. Let $G = SL_{11}(q)$ and $\overline{G} = G/Z(G) = PSL_{11}(q)$, where $q = p^e$ and p is a prime. Set $d = (11, q-1)$ and $Q = (q^{11}-1)/(q-1)$. It is easily seen that here $(6, Q) = 1$. The group G acts (naturally) on an eleven-dimensional vector space $V = F^{11}$ over the field $F = GF(q)$.

To prove the theorem we make use of the known list of maximal subgroups of G given in [1]. In Aschbaher's notation any maximal subgroup of G belongs to one of the following families $C_1, C_2, C_3, C_5, C_6, C_8$, and S. Roughly speaking, they are:

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- C_1: stabilizers of subspaces of V,
- C_2: stabilizers of direct sum decompositions of V,
- C_3: stabilizers of extension fields of F of prime degree,
- C_5: stabilizers of subfields of F of prime index,
- C_6: normalizers of extraspecial groups in absolutely irreducible representations,
- C_8: classical groups on V contained in G,
- S: almost simple groups, absolutely irreducible on V, and the representation of their (simple) *socles* on V can not be realized over proper subfields of F; not continued in members of C_8.

In [1] the representatives of the conjugacy classes of maximal subgroups of G are specified in Tables 8.70 and 8.71. For the reader's convenience we provide the exact list of maximal subgroups of G together with their orders. The notation used here for group structures is standard group-theoretic notation as in [1]. Especially, $A \times B$ is the direct product of groups A and B, and we write $A : B$ or $A.B$ to denote a split extension of A by B or an extension of A by B of unspecified type, respectively; the cyclic group of order n is simple denoted by n, and E_{q^k} stands for an elementary abelian group of order q^k.
If M is a maximal subgroup of G then one of the following holds.

1. $M \cong E_{q^{10}} : GL_{10}(q)$ of order $q^{55}(q-1)(q^2-1)(q^3-1)(q^4-1)(q^5-1)(q^6-1)(q^7-1)(q^8-1)(q^9-1)(q^{10}-1)$.
2. $M \cong E_{q^{18}} : (SL_9(q) \times SL_2(q)) : (q-1)$ of order $q^{55}(q-1)(q^2-1)^2(q^3-1)(q^4-1)(q^5-1)(q^6-1)(q^7-1)(q^8-1)(q^9-1)$.
3. $M \cong E_{q^{24}} : (SL_8(q) \times SL_3(q)) : (q-1)$ of order $q^{55}(q-1)(q^2-1)^2(q^3-1)^2(q^4-1)(q^5-1)(q^6-1)(q^7-1)(q^8-1)$.
4. $M \cong E_{q^{28}} : (SL_7(q) \times SL_4(q)) : (q-1)$ of order $q^{55}(q-1)(q^2-1)^2(q^3-1)^2(q^4-1)^2(q^5-1)(q^6-1)(q^7-1)$.
5. $M \cong E_{q^{30}} : (SL_6(q) \times SL_5(q)) : (q-1)$ of order $q^{55}(q-1)(q^2-1)^2(q^3-1)^2(q^4-1)^2(q^5-1)^2(q^6-1)$.
6. $M \cong (q-1)^{10} : S_{11}$ (if $q \geq 5$) of order $2^8.3^4.5^2.7.11.(q-1)^{10}$.
7. $M \cong \dfrac{q^{11}-1}{q-1} : 11$ of order $11.\dfrac{q^{11}-1}{q-1}$.
8. $M \cong SL_{11}(q_0).(11, \dfrac{q-1}{q_0-1})$ (if $q = q_0^r$, r prime) of order $q_0^{55}(q_0^2-1)(q_0^3-1)(q_0^4-1)(q_0^5-1)(q_0^6-1)(q_0^7-1)(q_0^8-1)(q_0^9-1)(q_0^{10}-1)(q_0^{11}-1).(11, \dfrac{q-1}{q_0-1})$.

9. $M \cong 11_+^{1+2} : Sp_2(11)$ (if $q = p \equiv 1 \pmod{11}$ or $q = p^5$ and $p \equiv 3, 4, 5, 9 \pmod{11}$) of order $2^3.3.5.11^4$ (here 11_+^{1+2} stands for an extraspecial group of order 11^3 and exponent 11).

10. $M \cong d \times SO_{11}(q)$ (if q is odd) of order $d.q^{25}(q^2-1)(q^4-1)(q^6-1)(q^8-1)(q^{10}-1)$.

11. $M \cong (11, q_0 - 1) \times SU_{11}(q_0)$ (if $q = q_0^2$) of order $q_0^{55}(q_0^2 - 1)(q_0^3 + 1)(q_0^4 - 1)(q_0^5 + 1)(q_0^6 - 1)(q_0^7 + 1)(q_0^8 - 1)(q_0^9 + 1)(q_0^{10} - 1)(q_0^{11} + 1).(11, q_0 - 1)$.

12. $M \cong d \times L_2(23)$ (if $q = p \equiv 1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18 \pmod{23}$, $q \neq 2$) of order $2^3.3.11.23.d$.

13. $M \cong d \times U_5(2)$ (if $q = p \equiv 1 \pmod 3$) of order $2^{10}.3^5.5.11.d$.

14. $M \cong M_{24}$ (if $q = 2$) of order $2^{10}.3^3.5.7.11.23$.

We proceed now as follows. First we prove that there is only one type of maximal subgroups of G whose order is a multiple of Q; actually these are the groups (in Aschbaher's class C_3) in case 7 above, of order $11.\dfrac{q^{11}-1}{q-1}$. Into the second step we find out two elements x and y of respective orders 2 and 3 in G such that their product has got order Q. Finally, we deduce that the group G is generated by these two elements. Then the projective images of these elements will generate the group $\overline{G}$.

Let us start with the first step in our strategy. In order to prove the above mentioned arithmetic fact we use the well-known Zsigmondy's theorem, and take a primitive prime divisor of $p^{11e} - 1$, i.e., a prime r which divides $p^{11e} - 1$ but does not divide $p^i - 1$ for $0 < i < 11e$. Obviously $r \geq 23$ (as $r - 1$ is a multiple of $11e$) and also r divides Q. It is easy now to be seen that the only maximal subgroups of orders divisible by r are those in cases 11, and 12 or 14 with $r = 23$. In case 11 if $Q = \dfrac{q_0^{22}-1}{q_0^2-1}$ divides the order of M, then $\dfrac{q_0^{11}-1}{q_0-1}$ should be a factor of the integer $q_0^{55}(q_0+1)(q_0^2-1)(q_0^3+1)(q_0^4-1)(q_0^5+1)(q_0^6-1)(q_0^7+1)(q_0^8-1)(q_0^9+1)(q_0^{10}-1).(11, q_0-1)$, which is impossibile, by the same Zsigmondy's theorem. As for the groups in case 12, we have $Q = \dfrac{p^{11}-1}{p-1} \geq \dfrac{3^{11}-1}{3-1} > 2^3.3.11^2.23 \geq |M|$. Lastly, in case 14 $Q = 2^{11} - 1 = 23.89$ does not divide the order of M_{24}.

Further, let us choose for x the matrix

$$x = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

and y to be in the form

$$y = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_2 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_4 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & \delta_5 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \delta_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & \delta_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \delta_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & \delta_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \delta_{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then x is an involution of G and y is an element of order 3 in G for any δ_1, δ_2, δ_3, δ_4, δ_5, δ_6, δ_7, δ_8, δ_9, $\delta_{10} \in GF(q)$, also

$$z = xy = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \delta_{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & \delta_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \delta_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & \delta_7 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -\delta_6 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & \delta_5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_4 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_2 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_1 \end{bmatrix}.$$

The characteristic polynomial of z is

$$f_z(t) = t^{11} - \delta_1 t^{10} + (\delta_{10} - 1)t^9 + (2\delta_1 + \delta_3 + 1)t^8 - (\delta_1 + \delta_8 + \delta_9 + 2\delta_{10} + 1)t^7 - (\delta_1 - \delta_2 + \delta_3 + \delta_5 - \delta_{10})t^6 + (\delta_1 + \delta_3 - \delta_6 + \delta_7 + \delta_8 + \delta_9 + \delta_{10} + 1)t^5 + (\delta_1 - \delta_2 - \delta_4 - \delta_7 - \delta_8 - \delta_9 - \delta_{10})t^4 - (\delta_1 - \delta_2 - \delta_4 + \delta_9 + \delta_{10} + 2)t^3 + (\delta_2 + \delta_9 + \delta_{10} + 2)t^2 - (\delta_2 - 1)t - 1$$

Let take now an element ω of order Q in the multiplicative group of the field $GF(q^{11})$ and put

$$l(t) = (t - \omega)(t - \omega^q)(t - \omega^{q^2})(t - \omega^{q^3})(t - \omega^{q^4})(t - \omega^{q^5})(t - \omega^{q^6})(t - \omega^{q^7})(t - \omega^{q^8})(t - \omega^{q^9})(t - \omega^{q^{10}}) = t^{11} - at^{10} + bt^9 - ct^8 + dt^7 - et^6 + ft^5 - gt^4 + ht^3 - kt^2 + mt - 1.$$

The last polynomial has all its coefficients in the field $GF(q)$ and the roots of $l(t)$ are pairwise distinct (in fact, the polynomial $l(t)$ is irreducible over $GF(q)$ which is not necessary for our considerations). The polynomials $f_z(t)$ and $l(t)$ are identically equal if

$$\delta_1 = a,\ \delta_2 = -m + 1,\ \delta_3 = -2a - c - 1,\ \delta_4 = a + 2m - 2 - k + h,$$
$$\delta_5 = a - m + 3 + c + b + e,\ \delta_6 = -a - c + 1 + g - m + k - h - f,$$
$$\delta_7 = 3 - m + k - h + g + a + b + d,\ \delta_8 = -a + k - m + 1 - b - d,\ \delta_9 = m - 4 - b - k,$$
$$\delta_{10} = b + 1$$

For these values of $\delta_i (i = 1, \ldots, 10)$ $f_z(t) = l(t)$ and then, in $GL_{11}(q^{11})$, z is conjugate to diag $(\omega, \omega^q, \omega^{q^2}, \omega^{q^3}, \omega^{q^4}, \omega^{q^5}, \omega^{q^6}, \omega^{q^7}, \omega^{q^8}, \omega^{q^9}, \omega^{q^{10}})$ and hence z is an element of G of order Q.

Then, $H = \langle x, y \rangle$ is a subgroup of G of order divisible by $6Q$. We have already proved above that the only maximal subgroup of G whose order is a multiple of Q is that in Aschbaher's class C_3, of order $11Q$, which means that H can not be contained in any maximal subgroup of G. Thus $H = G$ and $G = \langle x, y \rangle$ is a $(2, 3)$-generated group; $\overline{G} = \langle \overline{x}, \overline{y} \rangle$ is a $(2, 3)$-generated group too. The theorem is proved. □

REFERENCES

[1] J. N. Bray, D.F. Holt, C. M. Roney-Dougal. The Maximal Subgroups of the Low-Dimensional Finite Classical Groups. London Math. Soc. Lecture Note Series **407**, Cambridge University Press, 2013.

[2] J. Cohen. On non-Hurwitz groups and noncongruence of the modular group. *Glasgow Math. J.*, **22** (1981), 1–7.

[3] L. Di Martino, N. A. Vavilov. $(2, 3)$-generation of $SL_n(q)$. I. Cases $n = 5, 6, 7$. *Comm. Alg.*, **22**, No 4 (1994), 1321–1347.

[4] L. Di Martino, N. A. Vavilov. $(2, 3)$-generation of $SL_n(q)$. II. Cases $n \geq 8$. *Comm. Alg.*, **24**, No 2 (1996), 487–515.

[5] D. Garbe. Über eine Klasse von arithmetisch definierbaren Normalteilern der Modulgruppe. *Math. Ann.*, **235**, No 3 (1978), 195–215.

[6] Ts. Genchev, E. Gencheva. $(2, 3)$-generation of the special linear groups of dimension 8. *Math. and Education in Math.*, **44** (2015), 167–173.

[7] Ts. Genchev, E. Gencheva, K. Tabakov. $(2, 3)$-generation of the special linear groups of dimension 9. *arXiv*, 2016 (in preparation).

[8] Ts. Genchev, E. Gencheva, K. Tabakov. $(2, 3)$-generation of the special linear groups of dimension 10. *arXiv*, 2016 (in preparation).

[9] A. M. Macbeath. Generators of the linear fractional group. *Proc. Symp. Pure Math.*, **12** (1969), 14–32.

[10] P. Manolov, K. Tchakerian. $(2, 3)$-generation of the groups $PSL_4(2^m)$. *Ann. Univ. Sofia, Fac. Math. Inf.*, **96** (2004), 101–104.

[11] M. A. Pellegrini, M. C. Tamburini Bellani. The simple classical groups of dimension less than 6 which are $(2, 3)$-generated. *arXiv* 1405.3149v2, 2015.

[12] M. A. Pellegrini, M. C. Tamburini Bellani, M. A. Vsemirnov. Uniform $(2, k)$-generation of the 4-dimensional classical groups. *J. Algebra*, **369** (2012), 322–350.

[13] P. Sanchini, M. C. Tamburini. Constructive $(2, 3)$-generation: a permutational approach. *Rend. Sem. Mat. Fis. Milano*, **64** (1994), 141–158.

[14] A. Shalev. Asymptotic group theory. *Notices Amer. Math. Soc.*, **48**, No 4 (2001), 383–389.

[15] M. C. Tamburini, S. Vassallo. $(2, 3)$-generazione di gruppi lineari. Scritti in onore di Giovanni Melzi. *Sci. Mat.*, **11** (1994), 391–399.

[16] M. C. Tamburini, S. Vassallo. $(2, 3)$-generazione di $SL_4(q)$ in caratteristica dispari e problemi collegati. *Boll. Un. Mat. Ital. B(7)*, **8** (1994), 121–134.

[17] K. Tabakov. $(2, 3)$-generation of the groups $PSL_7(q)$. *Math. and Education in Math.*, **42** (2013), 260–264.

[18] K. Tabakov, K. Tchakerian. $(2, 3)$-generation of the groups $PSL_6(q)$. *Serdica Math. J.*, **37**, No 4 (2011), 365–370.

[19] K. Tchakerian. (2, 3)-generation of the groups $PSL_5(q)$. *Ann. Univ. Sofia, Fac. Math. Inf.*, **97** (2005), 105–108.

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(2, 3)-ПОРОДЕНОСТ НА СПЕЦИАЛНИТЕ ЛИНЕЙНИ ГРУПИ ОТ РАЗМЕРНОСТ 11

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В настоящата работа разглеждаме специалните линейни групи от размерност 11 над крайни полета и доказваме, че те са епиморфни образи на добре известната модулярна група $PSL_2(\mathbb{Z})$. Последното означава, че всяка от разглежданите от нас групи $SL_{11}(q)$ и $PSL_{11}(q)$ се поражда от един свой елемент от ред 2 (инволюция) и още един елемент от ред 3. Предложеното доказателство е в сила за произволно крайно поле $GF(q)$, над което са дефинирани тези групи. Всъщност ние посочваме в явен вид две матрици, от ред две и три съответно, които пораждат групата $SL_{11}(q)$.