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THE PROJECTIONS ON SOME TRIANGLES OF A SIMPLEX ARE DERIVATIONS*

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In this paper we construct some examples of derivations in finite endomorphism semirings.

1. Introduction and preliminaries. Rings with derivations has been studied by many authors in the last 60 years, especially the relationship between derivations and the structure of rings. But about the derivations in semirings we know only the definition, see [1], examples and properties of derivations in strings, see [3], and authors results, see [5] and [6] for derivations in a tetrahedron.

Here we investigate the derivations in finite endomorphism semiring, which can be represented as a simplex, see [4]. Using the notations and results of [4], [5] and [6] we show that projections on some triangles of arbitrary n–simplex are derivations. Similar authors' results can be found in [7]. Concerning background of combinatorics the reader is refer to [2].

Consider a finite chain $\mathrm{C}_n = (\{0, 1, \ldots, n-1\}, \vee)$ and denote the endomorphism semiring of this chain by $\widehat{\mathcal{E}}_{\mathrm{C}_n}$ For some elements $a_0, a_1, \ldots, a_{k-1} \in \mathrm{C}_n$, such that $a_0 < a_1 < \ldots < a_{k-1}$, where $k \leq n$, we denote $A = \{a_0, a_1, \ldots, a_{k-1}\}$. Now, consider endomorphisms $\alpha \in \widehat{\mathcal{E}}_{\mathrm{C}_n}$ with property $Im(\alpha) \subseteq A$. The set of all such endomorphisms α is a simplicial complex and especially it is a maximal simplex. We denote this simplex (or k–simplex) by $\sigma_k^{(n)}(A) = \sigma^{(n)}\{a_0, a_1, \ldots, a_{k-1}\}$. Any simplex $\sigma^{(n)}\{a_0, a_1, \ldots, a_{k-1}\}$ is a subsemiring of $\widehat{\mathcal{E}}_{\mathrm{C}_n}$.

The 2-simplices, which are faces of the simplex $\sigma^{(n)}\{a_0, a_1, \ldots, a_{k-1}\}$ are called triangles. We denote any triangle of $\sigma^{(n)}\{a_0, a_1, \ldots, a_{k-1}\}$ by $\triangle^{(n)}\{a, b, c\}$, where $a, b, c \in A$. In the present paper we consider only triangles such that $a = a_0$ and $b = a_1$.

The endomorphisms $\alpha \in \sigma^{(n)}\{a_0, a_1, \ldots, a_{k-1}\}$ such that

$$\alpha(0) = \cdots = \alpha(i_0 - 1) = a_0, \alpha(i_0) = \cdots = \alpha(i_0 + i_1 - 1) = a_1, \ \cdots$$

$$\alpha(i_0 + \cdots + i_{k-2}) = \cdots = \alpha(i_0 + \cdots + i_{k-1} - 1) = a_{k-1}$$

we denote by $\alpha = (a_0)_{i_0}(a_1)_{i_1} \ldots (a_{k-1})_{i_{k-1}}$, where $i_0 + i_1 + \cdots + i_{k-1} = n$. So, arbitrary endomorphism from the triangle $\triangle^{(n)}\{a_0, a_1, a_m\}$ has the form

$$(a_0)_{i_0}(a_1)_{i_1}(a_m)_{i_2}, \text{ where } i_0 + i_1 + i_2 = n.$$

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2. Projection on the triangles $\triangle^{(\mathbf{n})}\{\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_m}\}$. Let us consider the map

$$\partial_{1m} : \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\} \to \triangle^{(n)}\{a_0, a_1, a_m\},$$

where $2 \leq m \leq k-1$, such that for $\alpha = (a_0)_{i_0}(a_1)_{i_1}\dots(a_{k-1})_{i_{k-1}}$, $i_0+i_1+\dots+i_{k-1} = n$,

$$\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}.$$

Lemma 1. *For any endomorphisms $\alpha, \beta \in \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}$, it follows*

$$\partial_{1m}(\alpha + \beta) = \partial_{1m}(\alpha) + \partial_{1m}(\beta). \tag{1}$$

Proof. Let $\alpha = (a_0)_{i_0}(a_1)_{i_1}\dots(a_{k-1})_{i_{k-1}}$ and $\beta = (a_0)_{j_0}(a_1)_{j_1}\dots(a_{k-1})_{j_{k-1}}$, $i_0 + \dots + i_{k-1} = j_0 + \dots + j_{k-1} = n$. Without loss of generality assume $i_0 + i_1 \leq j_0 + j_1$.

Case 1. Let $i_0 \leq j_0$. Then $\alpha + \beta = (a_0)_{i_0}(a_1)_x(a_2)_{s_2}\dots(a_{k-1})_{s_{k-1}}$.

Since $i_2 + \dots + i_{k-1} \geq j_2 + \dots + j_{k-1}$, it follows that $s_2 + \dots + s_{k-1} = i_2 + \dots + i_{k-1}$ and then $x = n-i_0-(s_2+\dots s_{k-1}) = i_1$. So, $\alpha+\beta = (a_0)_{i_0}(a_1)_{i_1}(a_2)_{s_2}\dots(a_{k-1})_{s_{k-1}}$ and $\partial_{1m}(\alpha + \beta) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1} = \partial_{1m}(\alpha)$. Now $\partial_{1m}(\alpha) + \partial_{1m}(\beta) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}+(a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1} = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1} = \partial_{1m}(\alpha)$ and (1) holds.

Case 2. Let $i_0 > j_0$. Then $\alpha + \beta = (a_0)_{j_0}(a_1)_y(a_2)_{s_2}\dots(a_{k-1})_{s_{k-1}}$, where $s_2 + \dots + s_{k-1} = i_2 + \dots + i_{k-1}$. Hence, $y = n - j_0 - (s_2 + \dots s_{k-1}) = i_0 + i_1 - j_0$. So, $\alpha + \beta = (a_0)_{j_0}(a_1)_{i_0+i_1-j_0}(a_2)_{s_2}\dots(a_{k-1})_{s_{k-1}}$ and $\partial_{1m}(\alpha + \beta) = (a_0)_{j_0}(a_1)_{i_0+i_1-j_0}(a_m)_{n-i_0-i_1}$. On the other hand $\partial_{1m}(\alpha) + \partial_{1m}(\beta) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}+(a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1} = (a_0)_{j_0}(a_1)_{i_0+i_1-j_0}(a_m)_{n-i_0-i_1}$. Hence (1) holds and this completes the proof. □

Consider the set $S_{1m} = \{\alpha | \alpha \in \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}, \alpha(a_2) = \dots = \alpha(a_m) \leq a_1\}$.

For $\alpha, \beta \in S_{1m}$ we find $(\alpha + \beta)(a_s) = \alpha(a_s) + \beta(a_s) \leq a_1$ where $s = 2, \dots, m$, and $(\alpha\beta)(a_s) = \beta(\alpha(a_s)) \leq \beta(a_1) \leq \beta(a_2) \leq a_1$ where $s = 2, \dots, m$.

So, S_{1m} is a subsemiring of $\sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}$.

Let $R_{1m} = \{\alpha | \alpha \in \sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}, \alpha(a_2) \geq a_2, \alpha(a_m) \leq a_m\}$.

For $\alpha, \beta \in R_{1m}$ we find

$(\alpha + \beta)(a_2) = \alpha(a_2) + \beta(a_2) \geq a_2$, and $(\alpha + \beta)(a_m) = \alpha(a_m) + \beta(a_m) \leq a_m$,
$(\alpha\beta)(a_2) = \beta(\alpha(a_2)) \geq \beta(a_2) \geq a_2$, and $(\alpha\beta)(a_m) = \beta(\alpha(a_m)) \leq \beta(a_m) \leq a_m$.

So, R_{1m} is a subsemiring of $\sigma^{(n)}\{a_0, a_1, \dots, a_{k-1}\}$.

Now for $\alpha \in S_{1m}$ and $\beta \in R_{1m}$ we obtain:

- $(\alpha + \beta)(a_2) = \alpha(a_2) + \beta(a_2) \geq \alpha(a_2) + a_2 = a_2$, $(\alpha + \beta)(a_m) = \alpha(a_m) + \beta(a_m) \leq a_1 + a_m = a_m$. Hence $\alpha + \beta \in R_{1m}$.

Let $\alpha(a_2) = \dots = \alpha(a_m) = a_0$. Then we have:

- $(\alpha\beta)(a_s) = \beta(\alpha(a_s)) = \beta(a_0)$ for all $s = 2, \dots m$. If $\beta(a_0) \leq a_1$ it follows $\alpha\beta \in S_{1m}$. If $\beta(a_0) \geq a_2$, it follows $(\alpha\beta)(a_2) = \beta(a_0) \geq a_2$. Since $\beta(a_0) \leq \beta(a_m) \leq a_m$, then $(\alpha\beta)(a_m) = \beta(a_0) \leq a_m$. Hence $\alpha\beta \in R_{1m}$.
- $(\beta\alpha)(a_m) = \alpha(\beta(a_m)) \leq \alpha(a_m) = a_0$. Hence $(\beta\alpha)(a_2) = \dots = (\beta\alpha)(a_m) = a_0$ and $\beta\alpha \in S_{1m}$.

Let $\alpha(a_2) = \dots = \alpha(a_m) = a_1$. Then we have:

- $(\alpha\beta)(a_s) = \beta(\alpha(a_s)) = \beta(a_1)$ for all $s = 2, \dots m$. If $\beta(a_1) \leq a_1$ it follows $\alpha\beta \in S_{1m}$. If $\beta(a_1) \geq a_2$, it follows $(\alpha\beta)(a_2) = \beta(a_1) \geq a_2$. Since $\beta(a_1) \leq \beta(a_m) \leq a_m$, then $(\alpha\beta)(a_m) = \beta(a_1) \leq a_m$. Hence $\alpha\beta \in R_{1m}$.
- $(\beta\alpha)(a_m) = \alpha(\beta(a_m)) \leq \alpha(a_m) = a_1$. If $(\beta\alpha)(a_m) = a_0$, it follows $(\beta\alpha)(a_2) = \dots = (\beta\alpha)(a_m) = a_0$ and $\beta\alpha \in S_{1m}$. Let $(\beta\alpha)(a_m) = a_1$. Since $a_1 = (\beta\alpha)(a_m) \geq (\beta\alpha)(a_2) = \alpha(\beta(a_2)) \geq \alpha(a_2) = a_1$, it follows that $(\beta\alpha)(a_2) = \dots = (\beta\alpha)(a_m) = a_1$ and $\beta\alpha \in S_{1m}$.

Let us denote $\mathcal{D}_{\partial_{1m}} = S_{1m} \cup R_{1m}$. Thus we have proved

Lemma 2. *The set $\mathcal{D}_{\partial_{1m}}$, where $2 \le m \le k-1$, is a subsemiring of $\sigma^{(n)}\{a_0, a_1, \ldots, a_{k-1}\}$.*

Lemma 3. *For any $\alpha, \beta \in \mathcal{D}_{\partial_{1m}}$, it follows*

$$\partial_{1m}(\alpha\beta) = \partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta). \tag{2}$$

Proof. Let $\alpha = (a_0)_{i_0}(a_1)_{i_1} \ldots (a_{k-1})_{i_{k-1}}$, $\beta = (a_0)_{j_0}(a_1)_{j_1} \ldots (a_{k-1})_{j_{k-1}}$, where $i_0 + i_1 + \cdots + i_{k-1} = j_0 + j_1 + \cdots + j_{k-1} = n$.

Case 1. Let $\beta \in S_{1m}$.

Case 1-1. Let $\beta(a_2) = \cdots = \beta(a_m) = a_0$, $m \ge 2$ and p, where $m \le p \le k-1$, be the largest positive integer such that $\beta(a_p) = a_0$. Then $\beta(a_0) = \beta(a_1) = a_0$.

• If $a_1 \notin Im(\beta)$, $\alpha\beta = (a_0)_{i_0+\cdots+i_p}(a_2)_{s_2} \ldots (a_{k-1})_{s_{k-1}}$, $i_0 + \cdots + i_p + s_2 + \cdots + s_{k-1} = n$. Now $\partial_{1m}(\alpha\beta) = (a_0)_{i_0+\cdots+i_p}(a_m)_{n-(i_0+\cdots+i_p)}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = \overline{a_0}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_p \le j_0 - 1 \le j_0 + j_1 - 1 < a_{p+1}$, we have $\alpha\partial_{1m}(\beta) = (a_0)_{i_0+\cdots+i_p}(a_m)_{n-(i_0+\cdots+i_p)}$ and (2) holds.

• Let $\beta(a_{p+1}) = a_1$ and q, $p+1 \le q \le k-1$, be the largest positive integer such that $\beta(a_q) = a_1$. Then $\alpha\beta = (a_0)_{i_0+\cdots+i_p}(a_1)_{i_{p+1}+\cdots+i_q}(a_2)_{s_2} \ldots (a_{k-1})_{s_{k-1}}$, where $i_0 + \cdots + i_q + s_2 + \cdots + s_{k-1} = n$. Now $\partial_{1m}(\alpha\beta) = (a_0)_{i_0+\cdots+i_p}(a_1)_{i_{p+1}+\cdots+i_q}(a_m)_{n-(i_0+\cdots+i_q)}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = \overline{a_0}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_p \le j_0 - 1 < a_{p+1} \le a_q \le j_0 + j_1 - 1$, it follows that $\alpha\partial_{1m}(\beta) = (a_0)_{i_0+\cdots+i_p}(a_1)_{i_{p+1}+\cdots+i_q}(a_m)_{n-(i_0+\cdots+i_q)}$ and (2) holds.

Case 1-2. Let $\beta(a_2) = \cdots = \beta(a_m) = a_1$, $m \ge 2$.

• Let $\beta(a_0) = \beta(a_1) = a_0$. Then $\alpha\beta = (a_0)_{i_0+i_1}(a_1)_{i_2+\cdots+i_m}(a_2)_{s_2} \ldots (a_{k-1})_{s_{k-1}}$, $i_0 + \cdots + i_m + s_2 + \cdots + s_{k-1} = n$ and $\partial_{1m}(\alpha\beta) = (a_0)_{i_0+i_1}(a_1)_{i_2+\cdots+i_m}(a_m)_{n-(i_0+\cdots+i_m)}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_0)_{i_0+i_1}(a_1)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_1 \le j_0 - 1 < a_2 \le a_m \le j_0 + j_1 - 1$, it follows that $\alpha\partial_{1m}(\beta) = (a_0)_{i_0+i_1}(a_1)_{i_2+\cdots+i_m}(a_m)_{n-(i_0+\cdots+i_m)}$ and (2) holds.

• Let $\beta(a_0) = a_0$, $\beta(a_1) = a_1$. Then $\alpha\beta = (a_0)_{i_0}(a_1)_{i_1+\cdots+i_m}(a_2)_{s_2} \ldots (a_{k-1})_{s_{k-1}}$, where $i_0 + \cdots + i_m + s_2 + \cdots + s_{k-1} = n$ and it follows $\partial_{1m}(\alpha\beta) = (a_0)_{i_0}(a_1)_{i_1+\cdots+i_m}(a_m)_{n-(i_0+\cdots+i_m)}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$, so $\partial_{1m}(\alpha)\beta = (a_0)_{i_0}(a_1)_{n-i_0}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$, $a_0 \le j_0 - 1 < a_1 < a_m \le j_0 + j_1 - 1$, it follows that $\alpha\partial_{1m}(\beta) = (a_0)_{i_0}(a_1)_{i_1+\cdots+i_m}(a_m)_{n-(i_0+\cdots+i_m)}$ and (2) holds.

• Let $\beta(a_0) = \beta(a_1) = a_1$. Then $\alpha\beta = (a_1)_{i_0+\cdots+i_m}(a_2)_{s_2} \ldots (a_{k-1})_{s_{k-1}}$, where $i_0 + \cdots + i_m + s_2 + \cdots + s_{k-1} = n$ and it follows $\partial_{1m}(\alpha\beta) = (a_1)_{i_0+\cdots+i_m}(a_m)_{n-(i_0+\cdots+i_m)}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = \overline{a_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $\le j_0 - 1 < a_0 < a_m \le j_0 + j_1 - 1$, it follows that $\alpha\partial_{1m}(\beta) = (a_1)_{i_0+\cdots+i_m}(a_m)_{n-(i_0+\cdots+i_m)}$ and (2) holds.

Case 2. Let $\beta \in R_{1m}$, i.e. $\beta(a_2) \ge a_2$ and $\beta(a_m) \le a_m$. Let p, $2 \le p \le m$, be the largest positive integer such that $\beta(a_2) = a_p$ and q, $p \le q \le m$, be the least positive integer such that $\beta(a_m) = a_q$.

Case 2-1. Let $\beta(a_1) = a_0$. Then $\alpha\beta = (a_0)_{i_0+i_1}(a_p)_{s_p} \ldots (a_{k-1})_{s_{k-1}}$, where $i_0 + i_1 + s_p + \cdots + s_{k-1} = n$. Then $\partial_{1m}(\alpha\beta) = (a_0)_{i_0+i_1}(a_m)_{n-i_0-i_1}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_0)_{i_0+i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_1 \le j_0 - 1 \le j_0 + j_1 - 1 < a_2$, it follows that $\alpha\partial_{1m}(\beta) = (a_0)_{i_0+i_1}(a_m)_{n-i_0-i_1}$ and (2) holds.

Case 2-2. Let $\beta(a_0) = a_0$ and $\beta(a_1) = a_1$. Then $\alpha\beta = (a_0)_{i_0}(a_1)_{i_1}(a_p)_{s_p}\ldots(a_{k-1})_{s_{k-1}}$, where $i_0 + i_1 + s_p + \cdots + s_{k-1} = n$ and $\partial_{1m}(\alpha\beta) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and then $\partial_{1m}(\alpha)\beta = (a_0)_{i_0}(a_1)_{i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_0 \leq j_0 - 1 < a_1 \leq j_0 + j_1 - 1 < a_2$, it follows that $\alpha\partial_{1m}(\beta) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and (2) holds.

Case 2-3. Let $\beta(a_0) = \beta(a_1) = a_1$. Then $\alpha\beta = (a_1)_{i_0+i_1}(a_p)_{s_p}\ldots(a_{k-1})_{s_{k-1}}$, where $i_0 + i_1 + s_p + \cdots + s_{k-1} = n$ and $\partial_{1m}(\alpha\beta) = (a_1)_{i_0+i_1}(a_m)_{n-i_0-i_1}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_1)_{i_0+i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $j_0 - 1 < a_0 < a_1 \leq j_0 + j_1 - 1 < a_2$, it follows that $\alpha\partial_{1m}(\beta) = (a_1)_{i_0+i_1}(a_m)_{n-i_0-i_1}$ and (2) holds.

Case 2-4. Let $\beta(a_0) = a_0$ and $\beta(a_1) = a_r$, where $2 \leq r \leq p$. Then $\alpha\beta = (a_0)_{i_0}(a_r)_{i_1}(a_{r+1})_{s_{r+1}}\ldots(a_{k-1})_{s_{k-1}}$, where $i_0+i_1+s_{r+1}+\cdots+s_{k-1} = n$ and $\partial_{1m}(\alpha\beta) = (a_0)_{i_0}(a_m)_{n-i_0}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_0)_{i_0}(a_r)_{i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_0 \leq j_0 - 1 \leq j_0 + j_1 - 1 < a_1$, it follows that $\alpha\partial_{1m}(\beta) = (a_0)_{i_0}(a_m)_{n-i_0}$ and (2) holds.

Case 2-5. Let $\beta(a_0) = a_1$ and $\beta(a_1) = a_r$, where $2 \leq r \leq p$. Then $\alpha\beta = (a_1)_{i_0}(a_r)_{i_1}(a_{r+1})_{s_{r+1}}\ldots(a_{k-1})_{s_{k-1}}$, where $i_0+i_1+s_{r+1}+\cdots+s_{k-1} = n$ and $\partial_{1m}(\alpha\beta) = (a_1)_{i_0}(a_m)_{n-i_0}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_1)_{i_0}(a_r)_{i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $j_0 - 1 < a_0 \leq j_0 + j_1 - 1 < a_1$, it follows that $\alpha\partial_{1m}(\beta) = (a_1)_{i_0}(a_m)_{n-i_0}$ and (2) holds.

Case 2-6. Let $\beta(a_0) = a_\ell$, where $2 \leq \ell \leq p$ and $\beta(a_1) = a_r$, where $\ell \leq r \leq p$. Then $\alpha\beta = (a_\ell)_{i_0}(a_r)_{i_1}(a_{r+1})_{s_{r+1}}\ldots(a_{k-1})_{s_{k-1}}$, where $i_0 + i_1 + s_{r+1} + \cdots + s_{k-1} = n$ and $\partial_{1m}(\alpha\beta) = \overline{a_m}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and then $\partial_{1m}(\alpha)\beta = (a_\ell)_{i_0}(a_r)_{i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $j_0-1 \leq j_0+j_1-1 < a_0$, it follows that $\alpha\partial_{1m}(\beta) = \overline{a_m}$ and (2) holds. $\square$

Theorem 1. *The map $\partial_{1m} : \mathcal{D}_{\partial_{1m}} \to \triangle^{(n)}\{a_0, a_1, a_m\}$, where $1 \leq m \leq k-1$, is a derivation. The semiring $\mathcal{D}_{\partial_{1m}}$ is the maximal subsemiring of the simplex $\sigma^{(n)}\{a_0, a_1, \ldots, a_{k-1}\}$, such that $\partial_{1m} : \sigma^{(n)}\{a_0, a_1, \ldots, a_{k-1}\} \to \triangle^{(n)}\{a_0, a_1, a_m\}$ is a derivation.*

Proof. From Lemmas 1, 2 and 3 we find that $\partial_{1m} : \sigma^{(n)}\{a_0, a_1, \ldots, a_{k-1}\} \to \triangle^{(n)}\{a_0, a_1, a_m\}$ is a derivation.

Now we shall prove for arbitrary m, $2 \leq m \leq k-1$, and arbitrary endomorphism $\alpha \in \sigma^{(n)}\{a_0, a_1, \ldots, a_{k-1}\}$ that if the second multiplier $\beta \notin \mathcal{D}_{\partial_{1m}}$, then the equality (2) does not hold.

Case 1. Let ℓ, $2 \leq \ell < m$, be the largest positive integer such that $\beta(a_\ell) = a_0$ and $\beta(a_m) = a_1$. Then $\alpha\beta = (a_0)_{i_0+\cdots+i_\ell}(a_1)_{i_{\ell+1}+\cdots+i_m}(a_2)_{s_2}\ldots(a_{k-1})_{s_{k-1}}$, $i_0 + \cdots + i_m + s_2 + \cdots + s_{k-1} = n$. So, it follows $\partial_{1m}(\alpha\beta) = (a_0)_{i_0+\cdots+i_\ell}(a_1)_{i_{\ell+1}+\cdots+i_m}(a_m)_{n-(i_0+\cdots+i_m)}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_0)_{i_0+i_1}(a_1)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_\ell \leq j_0 - 1 < a_{\ell+1} < a_m \leq j_0 + j_1 - 1$, we have $\alpha\partial_{1m}(\beta) = (a_0)_{i_0+\cdots+i_\ell}(a_1)_{i_{\ell+1}+\cdots+i_m}(a_m)_{n-(i_0+\cdots+i_m)}$. Now, it follows $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_0)_{i_0+i_1}(a_1)_{n-i_0-i_1} + (a_0)_{i_0+\cdots+i_\ell}(a_1)_{i_{\ell+1}+\cdots+i_m}(a_m)_{n-(i_0+\cdots+i_m)} = (a_0)_{i_0+i_1}(a_1)_{i_2+\cdots+i_m}(a_m)_{n-(i_0+\cdots+i_m)} > \partial_{1m}(\alpha\beta)$. Hence, (2) does not hold.

Case 2. Let ℓ, where $2 \leq \ell < m$, be the largest positive integer such that $\beta(a_\ell) = a_0$, $\beta(a_{\ell+1}) = a_1$ and p, $\ell + 1 \leq p < m$, be the largest positive integer such that $\beta(a_p) = a_1$. Let q, where $2 \leq q \leq m$, be the least positive integer such that $\beta(a_m) = a_q$.

Now $\alpha\beta = (a_0)_{i_0+\cdots+i_\ell}(a_1)_{i_{\ell+1}+\cdots+i_p}(a_2)_{s_2}\ldots(a_{k-1})_{s_{k-1}}$, where $i_0 + \cdots + i_p + s_2 + \cdots + s_{k-1} = n$. So, it follows $\partial_{1m}(\alpha\beta) = (a_0)_{i_0+\cdots+i_\ell}(a_1)_{i_{\ell+1}+\cdots+i_p}(a_m)_{n-(i_0+\cdots+i_p)}$.

Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_0)_{i_0+i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_\ell \leq j_0 - 1 < a_{\ell+1} < a_p \leq j_0 + j_1 - 1$, it follows that $\alpha\partial_{1m}(\beta) = (a_0)_{i_0+\cdots+i_\ell}(a_1)_{i_{\ell+1}+\cdots+i_p}(a_m)_{n-(i_0+\cdots+i_p)}$. Now we obtain $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_0)_{i_0+i_1}(a_q)_{n-i_0-i_1} + (a_0)_{i_0+\cdots+i_\ell}(a_1)_{i_{\ell+1}+\cdots+i_p}(a_m)_{n-(i_0+\cdots+i_p)}$. Since $n = i_0+\cdots i_p+s_2+\cdots s_{k-1} \geq i_0+\cdots i_p$ it follows that $n-i_0-i_1 \geq i_2+\cdots i_p \geq i_{\ell+1}+\cdots+i_p$.

Hence, $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) \geq (a_0)_{n-(i_{\ell+1}+\cdots+i_p)}(a_q)_{i_{\ell+1}+\cdots+i_p}+$
$(a_0)_{i_0+\cdots+i_\ell}(a_1)_{i_{\ell+1}+\cdots+i_p}(a_m)_{n-(i_0+\cdots+i_p)} = (a_0)_{i_0+\cdots+i_\ell}(a_q)_{i_{\ell+1}+\cdots+i_p}(a_m)_{n-(i_0+\cdots+i_p)} > \partial_{1m}(\alpha\beta)$. Hence, (2) does not hold.

Case 3. Let ℓ, where $2 \leq \ell < m$, be the largest positive integer such that $\beta(a_\ell) = a_0$ and $\beta(a_{\ell+1}) = a_p$, where $2 \leq p \leq m$. Then $\alpha\beta = (a_0)_{i_0+\cdots+i_\ell}(a_p)_{s_p}\dots(a_{k-1})_{s_{k-1}}$, where $i_0+\cdots+i_m+s_2+\cdots+s_{k-1} = n$. Then $\partial_{1m}(\alpha\beta) = (a_0)_{i_0+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_0)_{i_0+i_1}(a_q)_{n-i_0-i_1}$, where $\beta(a_m) = a_q$, $p \leq q \leq m$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_\ell \leq j_0 - 1 \leq j_0 + j_1 - 1 < a_{\ell+1} < a_m$, it follows that $\alpha\partial_{1m}(\beta) = (a_0)_{i_0+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)}$. Now we obtain $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_0)_{i_0+i_1}(a_q)_{n-i_0-i_1} + (a_0)_{i_0+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)} = (a_0)_{i_0+i_1}(a_q)_{i_2+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)} > \partial_{1m}(\alpha\beta)$. Hence, (2) does not hold.

Case 4. Let $\beta(a_2) = a_1$ and ℓ, where $2 \leq \ell < m$, be the largest positive integer such that $\beta(a_\ell) = a_1$. Let q, $2 \leq q \leq m$, be the least positive integer such that $\beta(a_m) = a_q$.

Case 4-1. Let $\beta(a_1) = a_0$. Now $\alpha\beta = (a_0)_{i_0+i_1}(a_1)_{i_2+\cdots+i_\ell}(a_2)_{s_2}\dots(a_{k-1})_{s_{k-1}}$, where $i_0+\cdots+i_\ell+s_2+\cdots+s_{k-1} = n$. So, it follows $\partial_{1m}(\alpha\beta) = (a_0)_{i_0+i_1}(a_1)_{i_2+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_0)_{i_0+i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_1 \leq j_0 - 1 < a_2 \leq a_\ell \leq j_0 + j_1 - 1$, it follows that $\alpha\partial_{1m}(\beta) = (a_0)_{i_0+i_1}(a_1)_{i_2+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)}$. Now $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_0)_{i_0+i_1}(a_q)_{n-i_0-i_1} + (a_0)_{i_0+i_1}(a_1)_{i_2+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)} = (a_0)_{i_0+i_1}(a_q)_{i_2+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)} > \partial_{1m}(\alpha\beta)$. Hence, (2) does not hold.

Case 4-2. Let $\beta(a_0) = a_0$ and $\beta(a_1) = a_1$. It follows now $\alpha\beta = (a_0)_{i_0}(a_1)_{i_1+\cdots+i_\ell}(a_2)_{s_2}\dots(a_{k-1})_{s_{k-1}}$, where $i_0+\cdots+i_\ell+s_2+\cdots+s_{k-1} = n$. Then $\partial_{1m}(\alpha\beta) = (a_0)_{i_0}(a_1)_{i_1+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_0)_{i_0}(a_1)_{i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_0 \leq j_0 - 1 < a_1 \leq a_\ell \leq j_0 + j_1 - 1$, it follows that $\alpha\partial_{1m}(\beta) = (a_0)_{i_0}(a_1)_{i_1+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)}$. Thus, $\partial_{1m}(\alpha)\beta+\alpha\partial_{1m}(\beta) = (a_0)_{i_0}(a_1)_{i_1}(a_q)_{n-i_0-i_1}+(a_0)_{i_0}(a_1)_{i_1+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)} = (a_0)_{i_0}(a_1)_{i_1}(a_q)_{i_2+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)} > \partial_{1m}(\alpha\beta)$. Hence, (2) does not hold.

Case 4-3. Let $\beta(a_0) = a_1$. Now $\alpha\beta = (a_1)_{i_0+\cdots+i_\ell}(a_2)_{s_2}\dots(a_{k-1})_{s_{k-1}}$, where $i_0+\cdots+i_\ell+s_2+\cdots+s_{k-1} = n$ and $\partial_{1m}(\alpha\beta) = (a_1)_{i_0+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_1)_{i_0+i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $j_0 - 1 < a_0 < a_\ell \leq j_0 + j_1 - 1$, it follows that $\alpha\partial_{1m}(\beta) = (a_1)_{i_0+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)}$. So, $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_1)_{i_0+i_1}(a_q)_{n-i_0-i_1}+(a_1)_{i_0+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)} = (a_1)_{i_0+i_1}(a_q)_{i_2+\cdots+i_\ell}(a_m)_{n-(i_0+\cdots+i_\ell)} > \partial_{1m}(\alpha\beta)$. Hence, (2) does not hold.

Case 5. Let $\beta(a_2) = a_\ell$, where $2 \leq \ell$ and $\beta(a_m) = a_q$, where m ¡ q.

Case 5-1. Let $\beta(a_1) = a_0$. Now $\alpha\beta = (a_0)_{i_0+i_1}(a_\ell)_{s_\ell}\dots(a_{k-1})_{s_{k-1}}$, $i_0+i_1+s_\ell+\cdots+s_{k-1} = n$ and $\partial_{1m}(\alpha\beta) = (a_0)_{i_0+i_1}(a_m)_{n-i_0-i_1}$. But $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_0)_{i_0+i_1}(a_q)_{n-i_0-i_1} > \partial_{1m}(\alpha\beta)$, hence (2) does not hold.

Case 5-2. Let $\beta(a_0) = a_0$ and $\beta(a_1) = a_1$. Now $\alpha\beta = (a_0)_{i_0}(a_1)_{i_1}(a_\ell)_{s_\ell}\dots(a_{k-1})_{s_{k-1}}$, $i_0+i_1+s_\ell+\cdots+s_{k-1} = n$ and $\partial_{1m}(\alpha\beta) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$. But $\partial_{1m}(\alpha) =$

$(a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_0)_{i_0}(a_1)_{i_1}(a_q)_{n-i_0-i_1} > \partial_{1m}(\alpha\beta)$, hence (2) does not hold.

Case 5-3. Let $\beta(a_0) = \beta(a_1) = a_1$. Now $\alpha\beta = (a_1)_{i_0+i_1}(a_\ell)_{s_\ell}\dots(a_{k-1})_{s_{k-1}}$, $i_0 + i_1 + s_\ell + \cdots + s_{k-1} = n$, and $\partial_{1m}(\alpha\beta) = (a_1)_{i_0+i_1}(a_m)_{n-i_0-i_1}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_1)_{i_0+i_1}(a_q)_{n-i_0-i_1} > \partial_{1m}(\alpha\beta)$, hence (2) does not hold.

Case 5-4. Let $\beta(a_0) = a_0$ and $\beta(a_1) = a_{h_1}$, where $2 \leq h_1 \leq \ell$. Now $\alpha\beta = (a_0)_{i_0}(a_{h_1})_{i_1}(a_\ell)_{s_\ell}\dots(a_{k-1})_{s_{k-1}}$, where $i_0 + i_1 + s_\ell + \cdots + s_{k-1} = n$. Then $\partial_{1m}(\alpha\beta) = (a_0)_{i_0}(a_m)_{n-i_0}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_0)_{i_0}(a_{h_1})_{i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_0 \leq j_0 - 1 \leq j_0 + j_1 - 1 < a_1$, it follows that $\alpha\partial_{1m}(\beta) = (a_0)_{i_0}(a_m)_{n-i_0}$.

So, we have $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_0)_{i_0}(a_{h_1})_{i_1}(a_q)_{n-i_0-i_1} + (a_0)_{i_0}(a_m)_{n-i_0}$.

- If $h_1 \leq m$, it follows $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_0)_{i_0}(a_m)_{i_1}(a_q)_{n-i_0-i_1} > \partial_{1m}(\alpha\beta)$.
- If $h_1 > m$, it follows $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_0)_{i_0}(a_{h_1})_{i_1}(a_q)_{n-i_0-i_1} > \partial_{1m}(\alpha\beta)$.

Hence, (2) does not hold.

Case 5-5. Let $\beta(a_0) = a_1$, $\beta(a_1) = a_{h_1}$, $2 \leq h_1 \leq \ell$. Now $\alpha\beta = (a_1)_{i_0}(a_{h_1})_{i_1}(a_\ell)_{s_\ell}\dots(a_{k-1})_{s_{k-1}}$, $i_0 + i_1 + s_\ell + \cdots + s_{k-1} = n$ and $\partial_{1m}(\alpha\beta) = (a_1)_{i_0}(a_m)_{n-i_0}$. But $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_1)_{i_0}(a_{h_1})_{i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $a_1 \leq j_0 - 1$, $\alpha\partial_{1m}(\beta) = (a_1)_{i_0}(a_m)_{n-i_0}$. So, $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_1)_{i_0}(a_{h_1})_{i_1}(a_q)_{n-i_0-i_1} + (a_1)_{i_0}(a_m)_{n-i_0}$.

- If $h_1 \leq m$, it follows $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_1)_{i_0}(a_m)_{i_1}(a_q)_{n-i_0-i_1} > \partial_{1m}(\alpha\beta)$.
- If $h_1 > m$, it follows $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_1)_{i_0}(a_{h_1})_{i_1}(a_q)_{n-i_0-i_1} > \partial_{1m}(\alpha\beta)$.

Hence, (2) does not hold.

Case 5-6. Let $\beta(a_0) = a_{h_0}$ and $\beta(a_1) = a_{h_1}$, where $2 \leq h_0 \leq h_1 \leq \ell$. Now $\alpha\beta = (a_{h_0})_{i_0}(a_{h_1})_{i_1}(a_\ell)_{s_\ell}\dots(a_{k-1})_{s_{k-1}}$, where $i_0 + i_1 + s_\ell + \cdots + s_{k-1} = n$ and $\partial_{1m}(\alpha\beta) = \overline{a_m}$. Clearly $\partial_{1m}(\alpha) = (a_0)_{i_0}(a_1)_{i_1}(a_m)_{n-i_0-i_1}$ and so $\partial_{1m}(\alpha)\beta = (a_{h_0})_{i_0}(a_{h_1})_{i_1}(a_q)_{n-i_0-i_1}$. Since $\partial_{1m}(\beta) = (a_0)_{j_0}(a_1)_{j_1}(a_m)_{n-j_0-j_1}$ and $\leq j_0 + j_1 + j_2 - 1 < a_0$, it follows that $\alpha\partial_{1m}(\beta) = \overline{a_m}$. So, we obtain $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_{h_0})_{i_0}(a_{h_1})_{i_1}(a_q)_{n-i_0-i_1} + \overline{a_m}$.

- If $h_1 \leq m$, it follows $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_m)_{i_0+i_1}(a_q)_{n-i_0-i_1} > \partial_{1m}(\alpha\beta)$.
- If $h_0 \leq m < h_1$, we have $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_m)_{i_0}(a_{h_1})_{i_1}(a_q)_{n-i_0-i_1} > \partial_{1m}(\alpha\beta)$.
- If $m < h_0$, it follows $\partial_{1m}(\alpha)\beta + \alpha\partial_{1m}(\beta) = (a_{h_0})_{i_0}(a_{h_1})_{i_1}(a_q)_{n-i_0-i_1} > \partial_{1m}(\alpha\beta)$.

Hence, (2) does not hold again and this completes the proof. □

REFERENCES

[1] J. Golan. Semirings and Their Applications, Kluwer, Dordrecht, 1999.

[2] R. Stanley. Enumerative combinatorics, Vol. 2, Cambridge University Press, 1999.

[3] I. Trendafilov. Derivations in Some Finite Endomorphism Semirings. *Discussiones Mathematicae General Algebra and Applications*, **32** (2012), 77–100.

[4] I. Trendafilov. Simplices in the Endomorphism Semiring of a Finite Chain, Hindawi Publishing Corporation, Algebra, Vol. 2014, Article ID 263605

[5] D. Vladeva, I. Trendafilov. Derivations in a tetrahedron – I part. *Proc. Techn. Univ.-Sofia*, **65**, No 2 (2015), 117–126.

[6] D. Vladeva, I. Trendafilov. Derivations in a tetrahedron – II part. *Proc. Techn. Univ.-Sofia*, **65**, No 2 (2015), 127–136.

[7] D. Vladeva, I. Trendafilov. Derivations in a endomorphism semiring (to appear).

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ПРОЕКЦИИТЕ ВЪРХУ НЯКОИ ТРИЪГЪЛНИЦИ НА СИМПЛЕКС СА ДИФЕРЕНЦИРАНИЯ

Димитринка Иванова Владева, Иван Димитров Трендафилов

В тази статия построяваме примери на диференцирания в краен полупръстен от ендоморфизми.