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SOME GEOMETRIC PROBLEMS SOLVED THROUGH VECTORS*

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The use of vectors for reasoning and for proving results in elementary geometry is illustrated on several examples.

Introduction. In previous publications [1, 3, 2] we drew attention to the utility of employing vectors for doing calculations and solving problems in plane geometry. Here we give a further evidence of this utility. The examples demonstrate that the vector approach provides direct and simple proofs that may be difficult or impossible to obtain using other methods.

Along with linear operations and scalar product $(\cdot)$, we make use of the 'perp' operation $({}^{\perp})$ and the area product of planar vectors. The perp rotates a planar vector at a right angle counterclockwise. The area product can be defined as $\mathbf{u}\times\mathbf{v} = \mathbf{u}^{\perp}\cdot\mathbf{v}$ and is in fact the signed area of the parallelogram, formed by (any representatives of) the vectors $\mathbf{u}$ and $\mathbf{v}$, taken with the same origin – including the degenerate case of collinear vectors. Both the perp and the area product were introduced, along with their basic algebraic properties, in the above mentioned publications.

Throughout the text, bold letters denote exclusively vectors. Specifically, a single capital letter designates the position vector of a point with the same name (all such vectors being related to a single reference origin), and a pair of capital letters, such as $\mathbf{PQ}$, designates the vector $\mathbf{Q}-\mathbf{P}$.

For triangles, the standard notation A, B, C, a, b, c is used, including $\mathbf{a}=\mathbf{BC}$, $\mathbf{b}=\mathbf{CA}$, $\mathbf{c}=\mathbf{AB}$ $(\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0})$.

Proofs of propositions are enclosed in ⌜ and ⌟.

Heron's formula. Heron's formula is usually proved by using the Pythagorean theorem or trigonometric functions, or by exploiting the dissection of a triangle induced by its incircle. Here is a fairly straightforward proof that makes use of vectors.

⌜ From

$$S^2_{ABC} = \left(\frac{\mathbf{a}\times\mathbf{b}}{2}\right)^2 = \frac{a^2\,b^2-(\mathbf{a}\cdot\mathbf{b})^2}{4} = \frac{(a\,b-\mathbf{a}\cdot\mathbf{b})(a\,b+\mathbf{a}\cdot\mathbf{b})}{4},$$

$$a\,b-\mathbf{a}\cdot\mathbf{b} = \frac{(a+b)^2-(\mathbf{a}+\mathbf{b})^2}{2} = \frac{(a+b)^2-c^2}{2} = \frac{(a+b+c)(a+b-c)}{2}, \quad \text{and}$$

$$a\,b+\mathbf{a}\cdot\mathbf{b} = \frac{(\mathbf{a}+\mathbf{b})^2-(a-b)^2}{2} = \frac{c^2-(a-b)^2}{2} = \frac{(a-b+c)(-a+b+c)}{2}$$

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we obtain

$$S_{ABC}^2 = \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{16}$$

and therefore

$$S_{ABC} = \sqrt{s(s-a)(s-b)(s-c)}\,, \qquad \text{where} \quad s = (a+b+c)/2\,. \;\lrcorner$$

A triangle identity. If P, A, B, and C are any points in the plane, and A_1, B_1, C_1 are the orthogonal projections of P on BC, CA, and AB, then

$$AB\,.\,\overline{AC_1} + BC\,.\,\overline{BA_1} + CA\,.\,\overline{CB_1}$$

does not depend on P and equals $\dfrac{a^2+b^2+c^2}{2}$.

The lengths of AC_1, BA_1, and CB_1 are taken with positive or negative signs according to these segments having the same or opposite directions as AB, BC, and CA, respectively. Their values, or, better still, the ratios $\overline{AC_1} : AB$, $\overline{BA_1} : BC$, and $\overline{CB_1} : CA$, can be considered triangular coordinates of P with respect to $\triangle ABC$, in a similar sense as cevian and barycentric coordinates are.

$$\ulcorner\; \begin{aligned} AB\,.\,\overline{AC_1} + BC\,.\,\overline{BA_1} + CA\,.\,\overline{CB_1} &= \mathbf{AB}\cdot\mathbf{AC_1} + \mathbf{BC}\cdot\mathbf{BA_1} + \mathbf{CA}\cdot\mathbf{CB_1} \\ &= \mathbf{c}\cdot\mathbf{AP} + \mathbf{a}\cdot\mathbf{BP} + \mathbf{b}\cdot\mathbf{CP} \\ &= \mathbf{c}\cdot(\mathbf{P}-\mathbf{A}) + \mathbf{a}\cdot(\mathbf{P}-\mathbf{B}) + \mathbf{b}\cdot(\mathbf{P}-\mathbf{C}) \\ &= (\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{P} - (\mathbf{a}\cdot\mathbf{B} + \mathbf{b}\cdot\mathbf{C} + \mathbf{c}\cdot\mathbf{A}) \\ &= -(\mathbf{a}\cdot\mathbf{B} + \mathbf{b}\cdot\mathbf{C} + \mathbf{c}\cdot\mathbf{A})\,. \end{aligned}$$

As $\mathbf{A}\cdot\mathbf{a} + \mathbf{B}\cdot\mathbf{b} + \mathbf{C}\cdot\mathbf{c} = 0$ (which follows immediately from $\mathbf{a} = \mathbf{C}-\mathbf{B}$ etc):

$$\begin{aligned} -(\mathbf{a}\cdot\mathbf{B} + \mathbf{b}\cdot\mathbf{C} + \mathbf{c}\cdot\mathbf{A}) &= -(\mathbf{a}\cdot\mathbf{B} + \mathbf{b}\cdot\mathbf{C} + \mathbf{c}\cdot\mathbf{A}) + (\mathbf{A}\cdot\mathbf{a} + \mathbf{B}\cdot\mathbf{b} + \mathbf{C}\cdot\mathbf{c}) \\ &= -(\mathbf{a}\cdot(\mathbf{B}-\mathbf{A}) + \mathbf{b}\cdot(\mathbf{C}-\mathbf{B}) + \mathbf{c}\cdot(\mathbf{A}-\mathbf{C})) \\ &= -(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})\,. \end{aligned}$$

Finally, from $(\mathbf{a}+\mathbf{b}+\mathbf{c})^2 = 0$, there follows $-(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = \dfrac{a^2+b^2+c^2}{2}$, which completes the proof. $\lrcorner$

It appears to be not very difficult to prove the above proposition without using vectors when P is inside $\triangle ABC$, but apparently several other cases need to be considered. Vector algebra has the great advantage of being a language in which no such separate cases emerge: right from the outset the problem is dealt with in full generality.

Van Aubel's theorem of quadrilaterals. This theorem is usually formulated as follows.

Given an arbitrary planar quadrilateral $ABCD$, and squares built outwardly on each side, the lines connecting the centres of the opposite squares are perpendicular to each other and of equal length.

$\ulcorner$ Let $\mathbf{a} = \mathbf{AB}$, $\mathbf{b} = \mathbf{BC}$, $\mathbf{c} = \mathbf{CD}$, and $\mathbf{d} = \mathbf{DA}$. Then the centres of the squares are

$$\mathbf{A} + 1/2\,(\mathbf{a} - \mathbf{a}^\perp), \quad \mathbf{B} + 1/2\,(\mathbf{b} - \mathbf{b}^\perp), \quad \mathbf{C} + 1/2\,(\mathbf{c} - \mathbf{c}^\perp), \quad \mathbf{D} + 1/2\,(\mathbf{d} - \mathbf{d}^\perp),$$

so the lines connecting the first to the third and the second to the fourth centre are represented as vectors thus:

$$\mathbf{u} \;=\; \mathbf{AC} + \frac{1}{2}(\mathbf{c}-\mathbf{a}-\mathbf{c}^\perp+\mathbf{a}^\perp) \;=\; \frac{1}{2}(2\,\mathbf{b}+\mathbf{a}+\mathbf{c}+\mathbf{a}^\perp-\mathbf{c}^\perp)\,,$$

$$\mathbf{v} \;=\; \mathbf{BD} + \frac{1}{2}(\mathbf{d}-\mathbf{b}-\mathbf{d}^\perp+\mathbf{b}^\perp) \;=\; \frac{1}{2}(2\,\mathbf{c}+\mathbf{b}+\mathbf{d}+\mathbf{b}^\perp-\mathbf{d}^\perp)\,,$$

In order to prove the theorem, it remains to establish $\mathbf{u}^\perp - \mathbf{v} = \mathbf{0}$. This easily follows from $(\mathbf{x}^\perp)^\perp = -\mathbf{x}$ for any $\mathbf{x}$, $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=\mathbf{0}$, and the distributivity of $^\perp$ over vector addition. ⌟

The proof demonstrates that the theorem actually holds for any points A, B, C, and D, not necessarily forming a simple quadrilateral and even not necessarily different. It suffices to build the squares uniformly on the same side (left or right) of AB, BC, CD, and DA (and if some of these segments reduces to a point, that point also represents the respective square and its centre).

Precedence-conforming turns. For any non-zero, non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ in the plane, we say that $\mathbf{u}$ precedes $\mathbf{v}$ ($\mathbf{u} \prec \mathbf{v}$) if $\mathbf{v}$ is in the left half-plane with respect to $\mathbf{u}$ when (any representatives of) the two vectors share the same origin.

The area product $\mathbf{u}\times\mathbf{v}$ is either positive or negative according to which of $\mathbf{u}\prec\mathbf{v}$ and $\mathbf{v}\prec\mathbf{u}$ (if any) holds. Similarly, for $\mathbf{u}\neq\mathbf{0}$, the direction of $\mathbf{u}^\perp$ is such that $\mathbf{u}\prec\mathbf{u}^\perp$. Thus the definitions of both $\times$ and $^\perp$ depend on the $\prec$ relation, and in fact each of $\times$ and $^\perp$ can be derived from the other.

Of course, the $\prec$ relation is itself based on the informal, non-geometric notion of 'left'. However, for two pairs of vectors $\mathbf{u}\times\mathbf{v}\neq 0$ and $\mathbf{u}'\times\mathbf{v}'\neq 0$, the notion of whether they have the same or different precedence — i. e., whether they have *conforming precedence* — is entirely geometric.

If $\mathbf{u}\times\mathbf{v}\neq 0$, any vector $\mathbf{p}$ in the plane can be uniquely decomposed with respect to $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{p} = \frac{(\mathbf{p}\times\mathbf{v})\,\mathbf{u} + (\mathbf{u}\times\mathbf{p})\,\mathbf{v}}{\mathbf{u}\times\mathbf{v}}\,.$$

Substituting $\mathbf{p}^\perp$ for $\mathbf{p}$ in the above equality yields

$$\mathbf{p}^\perp = \frac{(\mathbf{p}^\perp\times\mathbf{v})\,\mathbf{u} + (\mathbf{u}\times\mathbf{p}^\perp)\,\mathbf{v}}{\mathbf{u}\times\mathbf{v}} = \frac{(\mathbf{u}\cdot\mathbf{p})\,\mathbf{v} - (\mathbf{v}\cdot\mathbf{p})\,\mathbf{u}}{\mathbf{u}\times\mathbf{v}}\,,$$

or, equivalently,

$$(\mathbf{u}\times\mathbf{v})\,\mathbf{p}^\perp = (\mathbf{u}\cdot\mathbf{p})\,\mathbf{v} - (\mathbf{v}\cdot\mathbf{p})\,\mathbf{u}\,.$$

This and the definition of $\times$ and $^\perp$ assert the validity of the following

Proposition. *Let* $\mathbf{q} = (\mathbf{u}\cdot\mathbf{p})\,\mathbf{v} - (\mathbf{v}\cdot\mathbf{p})\,\mathbf{u}$. *Then*

(1) $$\mathbf{q}\perp\mathbf{p}, \quad |\mathbf{q}| = |\mathbf{u}\times\mathbf{v}||\mathbf{p}|, \quad \textit{and} \quad \mathbf{p}\prec\mathbf{q} \Leftrightarrow \mathbf{u}\prec\mathbf{v}.$$

Unlike $\mathbf{p}^\perp$, whose direction depends on our conventional understanding of 'left', $(\mathbf{u}\cdot\mathbf{p})\,\mathbf{v} - (\mathbf{v}\cdot\mathbf{p})\,\mathbf{u}$, divided by $|\mathbf{u}\times\mathbf{v}|$ if one wants the same magnitude as $\mathbf{p}$, is perpendicular to $\mathbf{p}$ in a way conforming to $\mathbf{u}\prec\mathbf{v}$.

The above proposition also works conversely: if a vector, $\mathbf{q}$, is known to satisfy (1), then it is none other than $(\mathbf{u}\cdot\mathbf{p})\,\mathbf{v} - (\mathbf{v}\cdot\mathbf{p})\,\mathbf{u}$. For, let $\mathbf{q}$ be such a vector. If $\mathbf{q}' = (\mathbf{u}\cdot\mathbf{p})\,\mathbf{v} - (\mathbf{v}\cdot\mathbf{p})\,\mathbf{u}$, then, according to the proposition, (1) holds for $\mathbf{q}'$ as well. It follows that $\mathbf{q}$ and $\mathbf{q}'$ have the same attitude (perpendicular to $\mathbf{p}$), magnitude ($|\mathbf{u}\times\mathbf{v}||\mathbf{p}|$), and sense of direction ($\mathbf{p}\prec\mathbf{q}\Leftrightarrow\mathbf{p}\prec\mathbf{q}'$), whence $\mathbf{q}=\mathbf{q}'$.

The proposition is also valid if we consider $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{p}$ to be spatial vectors, and $\mathbf{u}\times\mathbf{v}$ the vector product of $\mathbf{u}$ and $\mathbf{v}$. Indeed, $|\mathbf{u}\times\mathbf{v}|$ – the length of $\mathbf{u}\times\mathbf{v}$ – then still equals the area of the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$, just as for planar vectors and their area product. Scalar products in space also have the same geometric meaning as in the plane, and the precedence relation retains its meaning in any oriented plane. Specifically, in space, there are two 'opposite' planes spanning $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u}\prec\mathbf{v}$ is true in one of them, $\mathbf{v}\prec\mathbf{u}$ in the other. Correspondingly, if $\mathbf{p}$ is coplanar with $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{p}$ precedes $(\mathbf{u}\cdot\mathbf{p})\,\mathbf{v}-(\mathbf{v}\cdot\mathbf{p})\,\mathbf{u}$ in the former plane, and the reverse precedence takes place in the latter.

The vector triple product. The above observation enables a rather straightforward derivation of a known formula for the vector triple product $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ in space.

If $\mathbf{a}\times\mathbf{b}\neq\mathbf{0}$ and $\mathbf{c}=\mathbf{c}'+\mathbf{c}''$, so that $\mathbf{c}'\perp\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}''\parallel\mathbf{a}\times\mathbf{b}$, then

(2) $$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\times\mathbf{b})\times\mathbf{c}'\,,$$

so whatever $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is equal to, it depends only on $\mathbf{c}'$ — the part of $\mathbf{c}$ which is in the $(\mathbf{a},\mathbf{b})$ plane. The vector $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}'$ is in the same plane, is perpendicular to $\mathbf{c}'$ — hence $|(\mathbf{a}\times\mathbf{b})\times\mathbf{c}'|=|\mathbf{a}\times\mathbf{b}||\mathbf{c}'|$ — and is preceded by $\mathbf{c}'$ in the oriented plane $(\mathbf{a},\mathbf{b})$ (the one in which $\mathbf{a}$ precedes $\mathbf{b}$). This is exactly where (1) applies, therefore

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}'=(\mathbf{a}\cdot\mathbf{c}')\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{c}')\,\mathbf{a}\,.$$

Taking into account $\mathbf{c}''\cdot\mathbf{a}=\mathbf{c}''\cdot\mathbf{b}=0$, we also obtain

$$(\mathbf{a}\cdot\mathbf{c}')\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{c}')\,\mathbf{a}=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}\,,$$

from which and (2) follows

(3) $$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}\,.$$

Of course, (3) also holds when $\mathbf{a}\times\mathbf{b}=\mathbf{0}$, as in this case both sides equal $\mathbf{0}$.

There is a large number of published proofs of (3). Ours is, as far as we know, the only one that is both expressed solely in terms of vectors (no trigonometry or coordinates involved) and explicitly based on planar vector algebra, which we believe reflects the essence of this identity.

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НЯКОИ ГЕОМЕТРИЧНИ ЗАДАЧИ, РЕШЕНИ ЧРЕЗ ВЕКТОРИ

Бойко Бл. Банчев

Дават се няколко примера за използването на вектори при разсъждения и доказване на твърдения от елементарната геометрия.