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ON THE RELATIVE RANK OF THE SEMIGROUP OF ORIENTATION-PRESERVING TRANSFORMATIONS WITH RESTRICTED RANGE^{*}

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In this paper, we determine the relative rank of the semigroup $\mathcal{OP}(X, Y)$ of all orientation-preserving transformations on a finite chain X with restricted range $Y \subseteq X$ modulo the semigroup $\mathcal{O}(X, Y)$ of all order-preserving transformations on X with restricted range Y.

Let S be a semigroup. The rank of S (denoted by rank S) is defined to be the minimal number of elements of a generating set of S. The ranks of various well known semigroups have been calculated [4, 5, 6, 7]. For a set $A \subseteq S$, the relative rank of S modulo A, denoted by rank(S : A), is the minimal cardinality of a set $B \subseteq S$ such that $A \cup B$ generates S. The relative rank of a semigroup modulo a suitable set was first introduced by Ruškuc [10] in order to describe the generating sets of semigroups with infinite rank. But also if the rank is finite, the relative rank gives information about the generating sets. In the present paper, we will determine the relative rank for a particular class of transformation semigroups.

Let X be a finite chain, say $X = \{1 < 2 < \cdots < n\}$ and denote by $\mathcal{T}(X)$ the monoid (under composition) of all full transformations on X. A transformation $\alpha \in \mathcal{T}(X)$ is called *order-preserving* if $x \leq y$ implies $x\alpha \leq y\alpha$, for all $x, y \in X$. We denote by $\mathcal{O}(X)$ the submonoid of $\mathcal{T}(X)$ of all order-preserving full transformations on X. We say that a transformation $\alpha \in \mathcal{T}(X)$ is *orientation-preserving* if there are subsets $X_1, X_2 \subseteq X$ with $\emptyset \neq X_1 < X_2$, (i.e. $x_1 < x_2$ for $x_1 \in X_1$ and $x_2 \in X_2$), $X = X_1 \cup X_2$, and $x\alpha \leq y\alpha$, whenever either $(x, y) \in X_1^2 \cup X_2^2$ with x < y or $(x, y) \in X_2 \times X_1$. Note that $X_2 = \emptyset$ provides $\alpha \in \mathcal{O}(X)$. We denote by $\mathcal{OP}(X)$ the submonoid of $\mathcal{T}(X)$ of all orientationpreserving full transformations on X. An equivalent notion of an orientation-preserving transformation was first introduced by McAlister in [9] and, independently, by Catarino and Higgins in [1]. It is interesting to note that the relative rank of $\mathcal{OP}(X)$ modulo $\mathcal{O}(X)$ as well as the relative rank of $\mathcal{T}(X)$ modulo $\mathcal{OP}(X)$ is one (see [1, 8]).

Let $Y = \{a_1 < a_2 < \cdots < a_m\}$ be a nonempty subset of X, and denote by $\mathcal{T}(X, Y)$ the subsemigroup $\{\alpha \in \mathcal{T}(X) | X\alpha \subseteq Y\}$ of $\mathcal{T}(X)$ of all elements with range (image) restricted to Y. In 1975, Symons [11] introduced and studied the semigroup $\mathcal{T}(X, Y)$, which is called semigroup of transformations with restricted range. In [2], Fernandes,

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Honyam, Quinteiro, and Singha determine the rank of the order-preserving counterpart $\mathcal{O}(X,Y)$ of $\mathcal{T}(X,Y)$. Recently, the regularity, the Green's relations, and the rank of the semigroup $\mathcal{OP}(X,Y)$ of all orientation-preserving transformations in $\mathcal{T}(X,Y)$ were studied by the same authors in [3]. Recall, the rank of $\mathcal{T}(X,Y)$ is the Sterling number S(n,m) of second kind with |X| = n and |Y| = m. On the other hand, rank $\mathcal{O}(X,Y) = \binom{n-1}{m-1} + |Y^{\#}|$, where $Y^{\#}$ denotes the set of all $y \in Y$ with one of the following properties: (i) y has no successor in X; (ii) y is a no successor of any element in X; (iii) both the successor of Y and the element whose successor is y belong to Y. Moreover in [3], Fernandes et al. show that rank $\mathcal{OP}(X,Y) = \binom{n-1}{m-1} + a$, where $a \in \{0,1\}$ depending on

the set Y. In this paper, we determine the relative rank of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$.

Let $\alpha \in \mathcal{OP}(X, Y)$. The kernel of α is the equivalence relation ker α with $(x, y) \in \ker \alpha$ if $x\alpha = y\alpha$. It corresponds uniquely to a partition on X. This justifies to regard ker α as partition on X. We will call a block of this partition a ker α -class. In particular, $x\alpha^{-1} := \{y \in X : y\alpha = x\}$, for $x \in X\alpha$, are ker α -classes. We say that a partition Pis a subpartition of a partition Q of X if for all $p \in P$ there is a $q \in Q$ with $p \subseteq q$. A set $T \subseteq X$ with $|T \cap x\alpha^{-1}| = 1$ for all $x \in X\alpha$, is called a transversal of ker α . Let $A \subseteq X$. Then $\alpha|_A : A \to Y$ denotes the restriction of α to A and A will be called convex if x < y < z with $x, z \in A$ implies $y \in A$.

Let $l \in \{1, \ldots, m\}$. We denote by \mathcal{P}_l the set of all partitions $\{A_1, \ldots, A_l\}$ of X such that $A_2 < A_3 < \cdots < A_l$ are convex sets (if l > 1) and A_1 is the union of two convex sets with $1, n \in A_1$. For $P \in \mathcal{P}_m$ with the blocks $A_1, A_2 < \cdots < A_m$, let α_P be the transformation on X defined by

$$\alpha_P := a_i$$
, whenever $x \in A_i$ for $1 \le i \le m$

in the case $1 \notin Y$ or $n \notin Y$ and

$$x\alpha_P := \begin{cases} a_{i+1}, & \text{if } x \in A_i \text{ for } 1 \le i < m \\ a_1 & \text{if } x \in A_m \end{cases}$$

in the case $1, n \in Y$. Clearly, ker $\alpha_P = P$. With $X_1 := \{1, \ldots, \max A_m\}, X_2 := \{\max A_m + 1, \ldots, n\}$ and $X_1 = \{1, \ldots, \max A_{m-1}\}, X_2 = \{\max A_{m-1} + 1, \ldots, n\}$, respectively, we can easy verify that α_P is orientation-preserving. Further, let $\eta \in \mathcal{T}(X, Y)$ be defined by

$$x\eta := \begin{cases} a_{i+1} & \text{if } a_i \leq x < a_{i+1} \text{ for } 1 \leq i < m \\ a_1 & \text{if } x = a_m \\ a_{\Gamma} & \text{otherwise} \end{cases} \quad \text{with } \Gamma := \begin{cases} 1 & \text{if } 1 \notin Y \\ 2 & \text{otherwise,} \end{cases}$$

in the case $1 \notin Y$ or $n \notin Y$ and

$$x\eta := \begin{cases} a_{i+1} & \text{if} \quad a_i \le x < a_{i+1}, \quad 2 \le i < m \\ a_1 & \text{if} \quad x = a_m = n \\ a_2 & \text{if} \quad x < a_2 \end{cases}$$

in the case $1, n \in Y$. In fact, $\eta \in \mathcal{OP}(X, Y)$ and $\eta|_Y$ is a permutation on Y, namely $\eta|_Y = \begin{pmatrix} a_1 & \dots & a_{m-1} & a_m \\ a_2 & \dots & a_m & a_1 \end{pmatrix}$. We will show that $A := \{\alpha_P : P \in \mathcal{P}_m\} \cup \{\eta\}$ is a 110

relative generating set of $\mathcal{OP}(X, Y)$ modulo $\mathcal{O}(X, Y)$.

Lemma 1. For each $\alpha \in \mathcal{OP}(X, Y)$ with rank $\alpha = m$, there is an $\widehat{\alpha} \in \{\alpha_P : P \in \mathcal{P}_m\} \cup \mathcal{O}(X, Y)$ with ker $\alpha = \ker \widehat{\alpha}$.

Proof. Let $\alpha \in \mathcal{OP}(X, Y)$ and let $X_1, X_2 \subseteq X$ as in the definition of orientationpreserving transformation. If $X_2 = \emptyset$ then $\alpha \in \mathcal{O}(X, Y)$. Suppose now that $X_2 \neq \emptyset$ and let $X_1 \alpha = \{x_1 < \cdots < x_r\}$ and $X_2 \alpha = \{y_1 < \cdots < y_s\}$ for suitable natural numbers r and s. We observe that $X_1 \alpha$ and $X_2 \alpha$ have at most one joint element (only $x_1 = y_s$ could be possible). If $x_1 \neq y_s$ then ker $\alpha = \{x_1 \alpha^{-1} < \cdots < x_r \alpha^{-1} < y_1 \alpha^{-1} < \cdots < y_s \alpha^{-1}\} = \ker \widehat{\alpha}$ with $\widehat{\alpha} = \begin{pmatrix} x_1 \alpha^{-1} & \cdots & x_r \alpha^{-1} & y_1 \alpha^{-1} & \cdots & y_s \alpha^{-1} \\ a_1 & \cdots & a_r & a_{r+1} & \cdots & a_{r+s} \end{pmatrix} \in \mathcal{O}(X, Y)$. If $x_1 = y_s$ then $1, n \in x_1 \alpha^{-1} = y_s \alpha^{-1}$ and ker $\alpha = \ker \alpha_P$ with $P = \{x_1 \alpha^{-1}, x_2 \alpha^{-1} < \cdots < x_r \alpha^{-1} < y_1 \alpha^{-1} < \cdots < y_{s-1} \alpha^{-1}\} \in \mathcal{P}_m$. \Box

Lemma 2. $\mathcal{OP}(X, Y) = \langle \mathcal{O}(X, Y), A \rangle.$

Proof. Let $\beta \in \mathcal{OP}(X, Y)$ with rank $\beta = m$. Then there is $\theta \in \{\alpha_P : P \in \mathcal{P}_m\} \cup \mathcal{O}(X, Y)$ with ker $\beta = \ker \theta$ by Lemma 1. In particular, there is $r \in \{0, \ldots, m-1\}$ with $a_1\theta^{-1} = a_{r+1}\beta^{-1}$. Then it is easy to verify that $\beta = \theta\eta^r$, where $\eta^0 := \eta^m$.

Suppose now that $i := \operatorname{rank} \beta < m$ and that $\ker \beta \in \mathcal{P}_i$, say $\ker \beta = \{A_1, A_2 < \cdots < A_i\}$ with $1, n \in A_1$. Then there is a subpartition $P \in \mathcal{P}_m$ of $\ker \beta$. We put $\theta := \alpha_P$, $a := \min X\beta$, and let T be a transversal of $\ker \theta$. In particular, we have $Y = \{x(\theta|_T)\eta^k : x \in T\}$ for all $k \in \{1, \ldots, m\}$. Since both mappings $\theta|_T : T \to Y$ and $\eta|_Y : Y \to Y$ are bijections, there is $k \in \{1, \ldots, m\}$ with $a_1((\theta|_T)\eta^k)^{-1}\beta = a$ and $a_1((\theta|_T)\eta^{k+1})^{-1}\beta \neq a$. Moreover, since $(\theta|_T)\eta^k$ is a bijection from T to Y and both transformations $\theta\eta^k$ and β are orientation-preserving, it is easy to verify that $f^* := ((\theta|_T)\eta^k)^{-1}\beta$ can be extended to an orientation-preserving transformation f defined by

$$xf := \begin{cases} a_1 f^* & \text{if } x < a_1 \\ a_i f^* & \text{if } a_i \le x < a_{i+1}, \ 1 \le i < m \\ a_m f^* & \text{if } a_m \le x, \end{cases}$$

i.e. f and f^* coincide on Y. Moreover, $a_1 f = a_1 f^* = a_1 ((\theta|_T)\eta^k)^{-1}\beta = a$. In order to show that f is order-preserving, it left to verify that $nf \neq a$. Assume that nf = a, where $n \geq a_m$. Then $nf = a_m f^* = a_m f$, i.e. $(n, a_m) \in \ker f$ and $n\eta = a_m \eta = a_1$. So, there is $x^* \in T$ such that $x^*((\theta|_T)\eta^k) = a_m$, i.e. $x^* = a_m((\theta|_T)\eta^k)^{-1}$. Now, we have a = $nf = a_m f^* = a_m((\theta|_T)\eta^k)^{-1}\beta = a_m(\eta^k|_Y)^{-1}(\theta|_T)^{-1}\beta = a_1(\eta|_Y)^{-1}(\eta^k|_Y)^{-1}(\theta|_T)^{-1}\beta =$ $a_1((\theta|_T)\eta^{k+1})^{-1}\beta \neq a$, a contradiction.

Finally, we will verify that $\beta = \theta \eta^k f \in \langle \mathcal{O}(X, Y), A \rangle$. For this let $x \in X$. Then there is $\tilde{x} \in T$ such that $(x, \tilde{x}) \in \ker \beta$. So, we have $x \theta \eta^k f = x \theta \eta^k f^* = \tilde{x} \theta \eta^k ((\theta|_T) \eta^k)^{-1} \beta = \tilde{x} \beta = x \beta$.

Suppose now that ker $\beta \notin \mathcal{P}_i$ and let $X\beta = \{b_1, \dots, b_i\}$ such that $b_1\beta^{-1} < \dots < b_i\beta^{-1}$. Then we define a transformation φ by $x\varphi := a_j$ for all $x \in b_{j-1}\beta^{-1}$ and $2 \leq j \leq i+1$. Clearly, $\varphi \in \mathcal{O}(X, Y)$. Further, we define a transformation $\nu \in \mathcal{T}(X, Y)$ by

$$x\nu := \begin{cases} b_j & \text{if } a_j < x \le a_{j+1}, \ 2 \le j \le i \\ b_i & \text{otherwise.} \end{cases}$$

Since β is orientation-preserving, there is $k \in \{1, \ldots, i\}$ such that k = i or $b_1 < \cdots < b_{k-1} > b_k < \cdots < b_i$. Then $X_1 := \{a_1, \ldots, a_{k+1} - 1\}$ and $X_2 := \{a_{k+1}, \ldots, n\}$ give a partition of X providing that ν is orientation-preserving. Clearly, rank $\nu = i$ and 111

 $1\nu = n\nu = b_i$. Thus, it is easy to verify that ker $\nu \in \mathcal{P}_i$. Hence, $\nu \in \langle \mathcal{O}(X,Y), A \rangle$ by the previous case and it remains to show that $\beta = \varphi \nu \in \langle \mathcal{O}(X, Y), A \rangle$. For this let $x \in X$. Then $x \in b_j \beta^{-1}$ for some $j \in \{1, \ldots, i\}$, i.e. $x \varphi \nu = a_{j+1} \nu = b_j = x \beta$.

The previous lemma shows that A is a relative generating set for $\mathcal{OP}(X, Y)$ modulo $\mathcal{O}(X,Y)$. It remains to show that A is of minimal size.

Lemma 3. Let $B \subseteq \mathcal{OP}(X,Y)$ be a relative generating set of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$. Then $\mathcal{P}_m \subseteq \{\ker \alpha : \alpha \in B\}.$

Proof. Let $P \in \mathcal{P}_m$. Since $\alpha_P \in \mathcal{OP}(X, Y) = \langle \mathcal{O}(X, Y), B \rangle$, there are $\theta_1 \in \mathcal{O}(X, Y) \cup$ B and $\theta_2 \in \mathcal{OP}(X,Y)$ with $\alpha_P = \theta_1 \theta_2$. Because of rank $\alpha_P = m$, we obtain ker $\alpha_P =$ ker θ_1 . Since $1\alpha_P = n\alpha_P$, we conclude that $\theta_1 \notin \mathcal{O}(X,Y)$, i.e. $\theta_1 \in B$ with ker $\theta_1 =$ $\ker \alpha_P = P. \quad \Box$

In order to find a formula for the number of elements in \mathcal{P}_m , we have to compute the number of possible partitions of X into m + 1 convex sets. This number is $\binom{n-1}{m}$.

Remark 4. $|\mathcal{P}_m| = \binom{n-1}{m}.$

Now we are able to state the main result of the paper. The relative rank of $\mathcal{OP}(X, Y)$ modulo $\mathcal{O}(X, Y)$ depends of the fact whether both 1 and n belong to Y or not.

Theorem 5. If $1 \notin Y$ or $n \notin Y$ then $\operatorname{rank}(\mathcal{OP}(X,Y) : \mathcal{O}(X,Y)) = \binom{n-1}{m}$. **Proof.** It is easy to verify that $P := \ker \eta \in \mathcal{P}_m$ and $\eta = \alpha_P$. Thus, $A = \{\alpha_P : P \in \mathcal{P}_m\}$ is a generating set of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$ by Lemma 2, i.e. the relative rank of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$ is bounded by the cardinality of \mathcal{P}_m , which is $\binom{n-1}{m}$ by Remark 4. But this number cannot be reduced by Lemma 3. \Box **Theorem 6.** If $\{1, n\} \subseteq Y$ then rank $(\mathcal{OP}(X, Y) : \mathcal{O}(X, Y)) = 1 + \binom{n-1}{m}$.

Proof. Let $B \subseteq \mathcal{OP}(X, Y)$ be a relative generating set of $\mathcal{OP}(X, Y)$ modulo $\mathcal{O}(X, Y)$. By Lemma 3, we know that $\mathcal{P}_m \subseteq \{ \ker \alpha : \alpha \in B \}$. Assume that the equality holds. Note that $1\eta = a_1\eta > a_m\eta = n\eta$, i.e. ker $\eta \notin \mathcal{P}_m$ and η is not order-preserving. Hence, there are $\theta_1, \ldots, \theta_l \in \mathcal{O}(X, Y) \cup B$, for a suitable natural number l, such that $\eta = \theta_1 \cdots \theta_l$. From rank $\eta = m$, we obtain ker $\theta_1 = \ker \eta$ and rank $\theta_i = m$ for $i \in \{1, \ldots, l\}$ and thus, $\{1,n\} \subseteq Y$ implies $(1,n) \notin \ker \theta_i$ for $i \in \{2,\ldots,l\}$. This implies $\theta_2,\ldots,\theta_l \in \mathcal{O}(X,Y)$. Since ker $\theta_1 = \ker \eta \notin \mathcal{P}_m$, we get $\theta_1 \in \mathcal{O}(X, Y)$, and consequently, $\eta = \theta_1 \theta_2 \cdots \theta_l \in \mathcal{O}(X, Y)$ $\mathcal{O}(X,Y)$, a contradiction. So, we have verified that $|\mathcal{P}_m| < |B|$, i.e. the relative rank of $\mathcal{OP}(X,Y)$ modulo $\mathcal{O}(X,Y)$ is greater than $\binom{n-1}{m}$. But it is bounded by $1 + \binom{n-1}{m}$ due to Lemma 2. This proves the assertion.

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ВЪРХУ ОТНОСИТЕЛНИЯ РАНГ НА ПОЛУГРУПАТА ОТ ВСИЧКИ ЗАПАЗВАЩИ ОРИЕНТАЦИЯТА ПРЕОБРАЗОВАНИЯ С ОГРАНИЧЕНО МНОЖЕСТВО ОТ ОБРАЗИ

Илинка Димитрова, Йорг Копиц, Китисак Тинпун

Нека S е полугрупа и A е подмножество на S. Относителен ранг на полугрупата S по модул A се нарича най-малкото кардинално число на множество $B \subseteq S$, такова че $A \cup B$ поражда S. Означава се с rank(S : A). Нека X е крайна верига, например $X = \{1 < 2 < \cdots < n\}$. Моноида от всички пълни преобразования на множеството X относно операцията композиция на преобразования се означава с $\mathcal{T}(X)$. Едно преобразование $\alpha \in \mathcal{T}(X)$ се нарича запазващо наредбата, ако от $x \leq y$ следва, че $x\alpha \leq y\alpha$ за всяко $x, y \in X$. С $\mathcal{O}(X)$ се означава полугрупата от всички запазващи наредбата преобразования на X. Преобразованието $\alpha \in \mathcal{T}(X)$ се нарича запазващи ориентацията, ако съществуват подмножества $X_1, X_2 \subseteq X$ със свойствата $\emptyset \neq X_1 < X_2, X = X_1 \cup X_2$ и $x\alpha < y\alpha$ за всяко $(x, y) \in X_1^2 \cup X_2^2 \cup X_2 \times X_1$. Полугрупата от всички запазващи ориентацията преобразования на X се означава с $\mathcal{OP}(X)$.

Нека $Y = \{a_1 < a_2 < \cdots < a_m\}$ е непразно подмножество на X. С $\mathcal{T}(X,Y)$ се означава подполугрупата $\{\alpha \in \mathcal{T}(X) | X \alpha \subseteq Y\}$ на $\mathcal{T}(X)$ от всички пълни преобразования на X с множество от образи, съдържащо се в Y. Обект на разглеждане в настоящата работа е полугрупата $\mathcal{OP}(X,Y)$ от всички запазващи ориентацията преобразования на X с множество от образи, съдържащо се в Y. Намерен е относителният ранг на полугрупата $\mathcal{OP}(X,Y)$ по модул полугрупата $\mathcal{O}(X,Y)$ от всички запазващи наредбата преобразования на X с множество от образи, съдържащо се в Y.