ON BERGE'S MAXIMUM THEOREM WITH CONCAVE FUNCTION OF UTILITY IN THE SECOND VARIABLE*

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#### Abstract

In the present paper we show a new property of Berge's Maximum Theorem with concave utility functions. It is proven that the maximum multifunction is convexvalued and continuous when the action multifunction is compact-valued, convexvalued and continuous, and the utility function is continuous and concave in its second variable.


1. Introduction. The well-known Berge's Maximum Theorem has become one of the most useful and powerful theorems in optimization theory and its applications in the different fields in mathematics. In particular, in [5] and [6], Slavov and Evans discuss two applications of this theorem in multi-objective optimization. In [1], [4], [8] and [9], the authors consider many different applications of Berge's Maximum Theorem in game theory and mathematical economics.

The original variant of Berge's Maximum Theorem is essentially as follows:
Theorem 1 ([3], [7, Theorem 9.14]). Let $X \subset R^{n}$ and $Y \subset R^{m}, u: X \times Y \rightarrow R$ be a continuous function, and $D: X \Rightarrow Y$ be a compact-valued and continuous multifunction. Then, the function $h: X \rightarrow R$ defined by $h(x)=\max \{u(x, y) \mid y \in D(x)\}$ is continuous on $X$, and the multifunction $S: X \Rightarrow Y$ defined by $S(x)=\{y \in D(x) \mid u(x, y)=h(x)\}$ is compact-valued and upper semi-continuous on $X$.

According to Walker's terminology [9] we say that $X$ is a set of environments, $Y$ is a set of actions, $u$ is a utility function, $D$ is an action multifunction, $h$ is a marginal function, $S$ is a maximum multifunction, $D(x)$ is the set of feasible actions available in environment $x$, and $S(x)$ is the set of optimal actions in environment $x$.

It is important to note that Berge's Maximum Theorem gives conditions under which marginal function $h$ and maximum multifunction $S$ are continuous and upper semicontinuous, respectively.

Berge's Maximum Theorem is often used in a special situation when the action multifunction $D$ is convex-valued and the utility function $u$ is quasi-concave or strictly quasi-concave in its second variable in addition to the hypotheses of Theorem 1.

Now, we focus our attention on the classical variant of Barge's Maximum Theorem, also called the Maximum Theorem.

[^0]Theorem 2 ([2, Maximum Theorem] [5, Theorem 2] [7, Theorem 9.17 and Corollary 9.20]). Let $X \subset R^{n}$ and $Y \subset R^{m}, u: X \times Y \rightarrow R$ be a continuous function, and $D: X \Rightarrow Y$ be a compact-valued and continuous multifunction. Define $h$ and $S$ as in Theorem 1.
(a) Then $h$ is a continuous function on $X$, and $S$ is a compact-valued and upper semi-continuous multifunction on $X$.
(b) If $u(x, \cdot)$ is quasi-concave in $y$ for each $x \in X$, and $D$ is convex-valued, then $S$ is convex-valued.
(c) If $u(x, \cdot)$ is strictly quasi-concave in $y$ for each $x \in X$, and $D$ is convex-valued, then $S$ is a continuous function on $X$.
(d) If $u$ is concave on $X \times Y$, and $D$ has a convex graph, then $h$ is a concave function on $X$ and $S$ is a convex-valued multifunction on $X$.
(e) If $u$ is strictly concave on $X \times Y$, and $D$ has a convex graph, then $h$ is a strictly concave and continuous function on $X$, and $S$ is a continuous function on $X$.

The key goal of this work is to present a new property of the Maximum Theorem, that is, the maximum multifunction is continuous when the action multifunction is convexvalued and the utility function is concave in its second variable.
2. Preliminaries. For a better understanding of the paper, we recall some useful definitions and notations. Let $\|\cdot\|$ be the Euclidean norm on $R^{n}, d(x, y)=\|x-y\|$ be the Euclidean distance between $x, y \in R^{n}$ and $\tau$ be the Euclidean topology induced by $d$. It is known that the Euclidean norm is strictly convex.

Let us consider a multifunction $F: X \Rightarrow R^{m}$ where the feasible domain $X \subset R^{n}$ is nonempty. Note that the image of each point by a multifunction is nonempty. Recall some standard topological definitions and their equivalent statements for a continuous multifunction.
(1) $F$ is called upper semi-continuous (briefly usc) at a point $x \in X$ if and only if for each open set $V \subset R^{n}$ such that $F(x) \subset V$, these exists a set $U$ of $\tau$ containing $x$ such that $y \in U \cap X$ implies $F(y) \subset V$. This is equivalent to " $F$ is upper semi-continuous at a point $x \in X$ if and only if $\left\{x_{k}\right\}_{k=1}^{\infty} \subset X$ and $\left\{y_{k}\right\}_{k=1}^{\infty} \subset F(X)$ are a pair of sequences such that $\lim _{k \rightarrow \infty} x_{k}=x$ and $y_{k} \in F\left(x_{k}\right)$ for all $k \in N$, then there exists a convergent subsequence of $\left\{y_{k}\right\}_{k=1}^{\infty}$ whose limit belongs to $F(x)$ ". $F$ is usc on $X$ if and only if $F$ is usc at each point $x \in X$.
(2) $F$ is called lower semi-continuous (briefly lsc) at a point $x \in X$ if and only if for each open set $V \subset R^{n}$ such that $F(x) \cap V \neq \emptyset$, there exists a set $U$ of $\tau$ containing $x$ such that $y \in U \cap X$ implies $F(y) \cap V \neq \emptyset$. This is equivalent to " $F$ is lower semi-continuous at a point $x \in X$ if and only if $\left\{x_{k}\right\}_{k=1}^{\infty} \subset X$ is a sequence convergent to $x$ and $y \in F(x)$, then there exists a sequence $\left\{y_{k}\right\}_{k=1}^{\infty} \subset F(X)$ such that $y_{k} \in F\left(x_{k}\right)$ for all $k \in N$ and $\lim _{k \rightarrow \infty} y_{k}=y " . F$ is lsc on $X$ if and only if $F$ is lsc at each point $x \in X$.
(3) $F$ is called continuous at a point $x \in F$ if and only if $F$ is both usc and lsc at point $x \in X . F$ is continuous on $X$ if and only if $F$ is continuous at each point $x \in X$.

Note that a function being continuous is equivalent to it being upper or lower semicontinuous as a multifunction; therefore, if $X \subset R^{n}, s: X \rightarrow R^{m}$ is a function and $S: X \Rightarrow R^{m}$ is a multifunction defined by $S(x)=\{s(x)\}$, then the following statements are equivalent: (i) $s$ is continuous on $X$. (ii) $S$ is upper semi-continuous on $X$. (iii) $S$ is
lower semi-continuous on $X$. As a result we get that continuous functions are a special class of continuous multifunctions. In our case this means that if $|S(x)|=1$ for all $x \in X$, then $S$ is a continuous function on $X$, see Theorem 2(c) and 2(e).

Remark 1. Consider the original variant of the Maximum Theorem. The following important two facts hold [7, Examples 9.15 and 9.16]:
(1) $S$ is only upper semi-continuous, and not necessarily also lower semi-continuous.
(2) The continuity of $u$ on $X \times Y$ cannot be replaced with $u(\cdot, y)$ is continuous on $X$ for each fixed $y \in Y$ and $u(x, \cdot)$ is continuous on $Y$ for each fixed $x \in X$.

It is known that quasi-concavity and concavity of the functions play a special role in optimization theory and we use the Maximum Theorem under convexity as a mathematical tool in convex optimization. Recall the definitions of quasi-concave and concave functions in the usual sense.

Definition 1. A real function $g: X \rightarrow R$ on a convex subset $X \subset R^{n}$ is called:
(a) quasi-concave on $X$ if and only if for any $x, y \in X$ and $t \in[0,1]$, then

$$
g(t x+(1-t) y) \geq \min (g(x), g(y))
$$

(b) strictly quasi-concave on $X$ if and only if for any $x, y \in X, x \neq y$ and $t \in(0,1)$, then $g(t x+(1-t) y)>\min (g(x), g(y))$.
(c) concave on $X$ if and only if for any $x, y \in X$ and $t \in[0,1]$, then $g(t x+(1-t) y) \geq$ $t g(x)+(1-t) g(y)$.
(d) strictly concave on $X$ if and only if for any $x, y \in X$ and $t \in(0,1)$, then $g(t x+(1-t) y)>t g(x)+(1-t) g(y)$.

It is easy to show that concavity implies quasi-concavity and strict concavity implies strict quasi-concavity, but in general the converse does not hold.

Remark 2. It is known that for each $x \in X$ :
(1) a function $g$ being quasi-concave is equivalent to its upper contour set $\{y \in$ $X \mid g(y) \geq g(x)\}$ being convex.
(2) if a function $g$ is strictly quasi-concave, then its upper contour set $\{y \in X \mid g(y) \geq$ $g(x)\}$ is strictly convex, but in general the converse does not hold.

Let us define the closed linear segment $\{t y+(1-t) x \mid t \in[0,1]\}$ by $[x, y]$ for $x, y \in R^{n}$ and $x \neq y$, and define the open linear segment $] x, y]$, analogously, $] x, y]=\{t y+(1-t) x \mid t \in$ $(0,1)\}$. Note that the definition of quasi-concave or concave function allows for linear segments (closed or open) in the boundary of its upper contour set. But, the definition of strictly quasi-concave or strictly concave function does not allow for linear segments (closed or open) in the boundary of its upper contour set.

Remark 3. If $g$ is a quasi-concave or concave function, then the set of maximizers is convex, and if $g$ is strictly quasi-concave or strictly concave, then the maximizer is unique.

Remark 4. Consider a real continuous function $g$ on $X \subset R^{n}$. Let $g$ be a concave function on closed linear segment $[x, y] \subset X$ such that $g(x)<g(y)$. For $A(x)=\{p \in$ $[x, y] \mid g(p) \geq g(y)\}$ it is easy to show that there exists a unique point $z \in A(x)$ such that $d(x, z)=d(x, A(x))$ and $x \neq z$, because set $A(x)$ is compact and convex, and we use the Euclidian distance. Note that $d(x, z) \leq d(x, y)$ and $g(x)<g(z)=g(y)$. Now consider function $g$ on closed linear segment $[x, z]$ and a function $f$ on closed linear segment $[0,1]$ 180
such that $f(t)=g(x+t(z-x))$ for $t \in[0,1]$. From concavity of function $g$ on $[x, z]$ and $g(x)<g(y)$ it follows that function $f$ is strictly increasing on $[0,1]$.
3. Main result. In this section we present the basic theorem of this paper.

Theorem 3. Let $X \subset R^{n}$ and $Y \subset R^{m}, u: X \times Y \rightarrow R$ be a continuous function, and $D: X \Rightarrow Y$ be a compact-valued and continuous multifunction. Define $h$ and $S$ as in Theorem 1. If $D$ is convex-valued and $u(x, \cdot)$ is concave in $y$ for each $x \in X$, then $S$ is a compact-valued, convex-valued and continuous multifunction on $X$.

Proof. From Theorem 2(a) and 2(b) we have that $S$ is a compact-valued, convexvalued and upper semi-continuous multifunction on $X$

We want to prove that $S$ is lower semi-continuous on $X$. It will be shown that if $x_{0} \in X,\left\{x_{k}\right\}_{k=1}^{\infty} \subset X$ is a sequence convergent to point $x_{0} \in X$ and $y_{0} \in S\left(x_{0}\right)$, then there exists a sequence $\left\{y_{k}\right\}_{k=1}^{\infty} \subset D(X)$ such that $y_{k} \in S\left(x_{k}\right)$ for all $k \in N$ and $\lim _{k \rightarrow \infty} y_{k}=y_{0}$, i.e. multifunction $S$ is lower semi-continuous at point $x_{0} \in X$.

Now, fix an arbitrary point $x_{0} \in X$. Let $\left\{x_{k}\right\}_{k=1}^{\infty} \subset X$ be a sequence convergent to $x_{0}$ and $y_{0} \in S\left(x_{0}\right)$. From $D$ is lower semi-continuous on $X$ it immediately follows that there exists a sequence $\left\{z_{k}\right\}_{k=1}^{\infty} \subset D(X)$ such that $z_{k} \in D\left(x_{k}\right)$ for all $k \in N$ and $\lim _{k \rightarrow \infty} z_{k}=y_{0}$.

For each $k \in N$, if $z_{k} \in S\left(x_{k}\right)$, then let $y_{k}=z_{k}$ and if $z_{k} \notin S\left(x_{k}\right)$, then it can be easily seen that there exists a unique point $y_{k} \in S\left(x_{k}\right)$ such that $d\left(z_{k}, y_{k}\right)=d\left(z_{k}, S\left(x_{k}\right)\right)$. Note that $S\left(x_{k}\right) \subset D\left(x_{k}\right)$ is a nonempty, compact and convex set, and we use the Euclidian distance.

Consider points $\left(x_{0}, y_{0}\right) \in X \times Y$ and $\left(x_{k}, z_{k}\right) \in X \times Y$ for $k \in N$. From $\lim _{k \rightarrow \infty} x_{k}=x_{0}$ it follows that $\lim _{k \rightarrow \infty} z_{k}=y_{0}, \lim _{k \rightarrow \infty}\left(x_{k}, z_{k}\right)=\left(x_{0}, y_{0}\right)$ and $\lim _{k \rightarrow \infty} u\left(x_{k}, z_{k}\right)=u\left(x_{0}, y_{0}\right)=$ $h\left(x_{0}\right)$. For each $k \in N$ we know that $h\left(x_{k}\right)=u\left(x_{k}, y_{k}\right) \geq u\left(x_{k}, z_{k}\right)$ and $\lim _{k \rightarrow \infty} h\left(x_{k}\right)=$ $h\left(x_{0}\right)$. This means that $u\left(x_{k}, y_{k}\right)-u\left(x_{k}, z_{k}\right)=h\left(x_{k}\right)-u\left(x_{k}, z_{k}\right)$ and we conclude that $\lim _{k \rightarrow \infty}\left(u\left(x_{k}, y_{k}\right)-u\left(x_{k}, z_{k}\right)\right)=\lim _{k \rightarrow \infty}\left(h\left(x_{k}\right)-u\left(x_{k}, z_{k}\right)\right)=h\left(x_{0}\right)-h\left(x_{0}\right)=0$, i.e. $\lim _{k \rightarrow \infty} d\left(u\left(x_{k}, y_{k}\right), u\left(x_{k}, z_{k}\right)\right)=0$.

Now we will prove that $\lim _{k \rightarrow \infty} y_{k}=y_{0}$.
Let us denote $A=\left\{k \in N \mid z_{k} \in S\left(x_{k}\right)\right\}$ and $B=\left\{k \in N \mid z_{k} \notin S\left(x_{k}\right)\right\}$.
It is interesting to point that $A \cap B=\emptyset, A \cup B=N, \lim _{k \rightarrow \infty} z_{k}=y_{0}, k \in A$ is equivalent to $z_{k}=y_{k}$, and $k \in B$ is equivalent to $z_{k} \neq y_{k}$.

Actually, there are the following possible cases: $A$ is infinite or $k \in A$ is infinite, or both.

Case 1. Let us assume that $A$ be infinite and let $k \in A$. Trivially, we have that $\lim _{k \rightarrow \infty} y_{k}=y_{0}$.

Case 2. Let us assume that $B$ be infinite and let $k \in B$. In this case, $u\left(x_{k}, y_{k}\right)>$ $u\left(x_{k}, z_{k}\right)$ and we will consider the function $u$ on closed linear segment $\left[\left(x_{k}, z_{k}\right),\left(x_{k}, y_{k}\right)\right]$. Obviously, $u\left(x_{k}, y_{k}\right)>u\left(x_{k}, p\right)>u\left(x_{k}, z_{k}\right)$ for all $p \in\left(z_{k}, y_{k}\right)$ because $u(x, \cdot)$ is concave on $\left[\left(x_{k}, z_{k}\right),\left(x_{k}, y_{k}\right)\right]$ and $\left\{y_{k}\right\}=S\left(x_{k}\right) \cap\left[z_{k}, y_{k}\right]$. This allows us to define a function $b:\left[z_{k}, y_{k}\right] \rightarrow\left[u\left(x_{k}, z_{k}\right), u\left(x_{k}, y_{k}\right)\right]$ by $b(p)=u\left(x_{k}, p\right)$ for $p \in\left[z_{k}, y_{k}\right]$, see also Remark 4 . Since $b\left(z_{k}\right)=u\left(x_{k}, z_{k}\right), b\left(y_{k}\right)=u\left(x_{k}, y_{k}\right)$ and continuity of $b$ on segment $\left[z_{k}, y_{k}\right]$ imply
$b\left(\left[z_{k}, y_{k}\right]\right)=\left[u\left(x_{k}, z_{k}\right), u\left(x_{k}, y_{k}\right)\right]$. For this function we also obtain $b\left(y_{k}\right)>b(p)>b\left(z_{k}\right)$ for all $p \in\left(z_{k}, y_{k}\right)$.

Let us fix an arbitrary point $p \in\left[z_{k}, y_{k}\right)$. It is obvious that if $q \in\left(p, y_{k}\right)$, then $u\left(x_{k}, y_{k}\right)=b\left(y_{k}\right)>u\left(x_{k}, q\right)=b(q)>u\left(x_{k}, p\right)=b(p)$ because $u(x, \cdot)$ is concave on segment $\left[\left(x_{k}, p\right),\left(x_{k}, y_{k}\right)\right]$ and $\left\{y_{k}\right\}=S\left(x_{k}\right) \cap\left[z_{k}, y_{k}\right]$, see Remark 4 .

Let us assume that $p, q \in\left[z_{k}, y_{k}\right]$ and $p \neq q$. There are two cases: either $q \in\left(p, y_{k}\right)$ or $p \in\left(q, y_{k}\right)$. So, if $q \in\left(p, y_{k}\right)$, then $b\left(y_{k}\right)>b(q)>b(p)$, but if $p \in\left(q, y_{k}\right)$, then $b\left(y_{k}\right)>b(p)>b(q)$. As a result we obtain $b(p) \neq b(q)$, i.e. $b$ is bijective.

Function $b$ is continuous and bijective on compact set $\left[z_{k}, y_{k}\right]$; therefore, $b^{-1}$ is continuous too. As a result we find that $\lim _{k \rightarrow \infty} d\left(u\left(x_{k}, z_{k}\right), u\left(x_{k}, y_{k}\right)\right)=0$ is equivalent to

$$
\begin{aligned}
& \lim _{k \rightarrow \infty} d\left(b^{-1}\left(u\left(x_{k}, z_{k}\right)\right), b^{-1}\left(u\left(x_{k}, y_{k}\right)\right)\right)=0, \\
& \lim _{k \rightarrow \infty} d\left(b^{-1}\left(u\left(x_{k}, z_{k}\right)\right), b^{-1}\left(u\left(x_{k}, y_{k}\right)\right)\right)=0
\end{aligned}
$$

is equivalent to $\lim _{k \rightarrow \infty} d\left(\left(x_{k}, z_{k}\right),\left(x_{k}, y_{k}\right)\right)=0$ and $\lim _{k \rightarrow \infty} d\left(\left(x_{k}, z_{k}\right),\left(x_{k}, y_{k}\right)\right)=0$ is equivalent to $\lim _{k \rightarrow \infty} d\left(z_{k}, y_{k}\right)=0$.

At the end of this case, since $0 \leq d\left(y_{k}, y_{0}\right) \leq d\left(y_{k}, z_{k}\right)+d\left(z_{k}, y_{0}\right), \lim _{k \rightarrow \infty} z_{k}=y_{0}$ and $\lim _{k \rightarrow \infty} d\left(z_{k}, y_{k}\right)=0$, we also conclude that $\lim _{k \rightarrow \infty} y_{k}=y_{0}$.

Case 3. Let us assume that $A$ and $B$ are both infinite. Using cases 1 and 2 , we conclude that $\lim _{k \rightarrow \infty} y_{k}=y_{0}$ for $k \in N$.

In conclusion, we obtain $\lim _{k \rightarrow \infty} y_{k}=y_{0}$ for $k \in N$, i.e. the multifunction $S$ is lower semi-continuous at point $x_{0} \in X$. Hence, we get that $S$ is continuous on $X$.

Continuing with this analysis, we also see that in Theorem 2 the maximum multifunction $S$ is lower semi-continuous on $X$, see Remark 1 .

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## ВЪРХУ ТЕОРЕМАТА ЗА МАКСИМУМА НА БЕРГ С ВДЛЪБНАТА ФУНКЦИЯ НА ПОЛЕЗНОСТ ВЪВ ВТОРАТА СИ ПРОМЕНЛИВА

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#### Abstract

В настоящата статия представяме едно ново свойство на Теоремата за максимума на Берг с вдлъбната функция на полезност. Доказано е, че максималната мултифункция е непрекъсната и с изпъкнали стойности, когато активната мултифункция е непрекъсната и с компактни и изпъкнали стойности, а функцията на полезност е непрекъсната и вдлъбната във втората си променлива.


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