МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2019
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2019
Proceedings of the Forty-eighth Spring Conference of the Union of Bulgarian Mathematicians
Borovetz, April 1–5, 2019

EXISTENCE OF INFINITELY MANY SOLUTIONS OF PROBLEMS FOR p-LAPLACIAN DIFFERENTIAL EQUATIONS VIA VARIATIONAL METHOD*

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We study the multiplicity of weak solutions for second-order one-dimensional p-Laplacian differential equations. Some historical notes on critical point theory, variational method and Clark's theorem are given.

1. Introduction. In this paper we present some historical notes on the critical point theory and variational method for solvability of problems for differential equations. First, in Section 2, we consider preliminary notes on the cathegory of Lusternik and Schnirelman [7] and the genus of Krasnoselskii [6]. Then, we introduce the Palais-Smale (PS)-condition and formulate the critical point theorem of D. Clark [2] and its generalization due to Heinz [5]. We formulate also its recent extension due to Liu and Wang [8]. In Section 3 we consider an application of extended Clark's theorem for the existence of infinitely many weak solutions for the Dirichlet's problem (P) for second-order one-dimensional p-Laplacian equation

$$(\varphi_p\left(u'(x)\right))' - a(x)\varphi_p\left(u(x)\right) + f(x,u(x)) = 0\,,\ x \in [0,T] \tag{1.1}$$

with the boundary conditions

$$u(0) = u(T) = 0. \tag{1.2}$$

Here $\varphi_p\left(t\right) = |t|^{p-2}t$, $t \in \mathbb{R}$ for $p > 1$ and $f(x,u)$ is a nonlinear term which satisfies some growth conditions. The partial case $p = 2$ is known as stationary Fisher-Kolmogorov equation (see [4], [10], [11] and references therein).

Let the function $F(x,u)$ be the potential of the function $f(x,u)$

$$F(x,u) = \int_0^u f(x,s)ds.$$

Let $L^p(0,T)$ be the usual Lebesque's space with norm $\|u\|_p = \left(\int_0^T |u(x)|^p dx\right)^{1/p}$.

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* **2010 Mathematics Subject Classification:** 39A10, 34B15, 34C37, 35A15.

Key words: Second-order differential equation, Dirichlet's problem, Clark's theorem, variational method.

This research is supported by the Bulgarian National Science Fund under Project DN 12/4 Advanced analytical and numerical methods for nonlinear differential equations with applications in finance and environmental pollution, 2017.

For the function a, we suppose

(A) $a \in C([0,T]\times\mathbb{R})$ and there exist positive constants a_1 and a_2 such that $a_1 \le a(x) \le a_2$.

Next, we assume the growth conditions for nonlinear term $f(x,u)$:

(F_1) $f \in C((0,T)\times\mathbb{R})$ and there exist constants q, $1<q<p$ and functions $b \in L^r(0,T)$, where $r = \dfrac{p}{p-q}$, such that

$$|f(x,u)| \le b(x)|u|^{q-1},$$

for all $(x,u) \in (0,T)\times\mathbb{R}$.

(F_2) There exists an open bounded interval $J \subset (0,T)$ and q positive constants c and such that

$$F(x,u) \ge c|u|^q,$$

for all $(x,u) \in J \times \mathbb{R}$.

(F_3) $f(x,u) = -f(x,-u)$, for all $(x,u) \in (0,T)\times\mathbb{R}$.

To treat the solvability of problem (P) we introduce the following Banach space

$$X := W_0^{1,p}(0,T) = \{u \in W^{1,p}(0,T) : u(0) = u(T) = 0\}$$

where $W^{1,p}(0,T)$ is the Sobolev space

$$W^{1,p} = \{u \in L^p(0,T) : u' \in L^p(0,T)\}$$

with the norm

$$\|u\|_{W^{1,p}} = \left(\int_0^T (|u'(x)|^p + |u(x)|^p)dx\right)^{1/p}.$$

By assumption (A) the last norm is equivalent to the norm

$$\|u\| = \left(\int_0^T (|u'(x)|^p + a(x)|u(x)|^p)dx\right)^{1/p}.$$

We define the functional $I : X \to \mathbb{R}$

$$I(u) := \frac{1}{p}\int_0^T (|u'(x)|^p + a(x)|u(x)|^p)dx - \int_0^T F(x,u(x))dx,$$

The functional I is C^1 differentiable under assumptions $(F_1)-(F_2)$ and (A) and

$$\langle I'(u), v\rangle = \int_0^T (\varphi_p(u'(x))v'(x) + a(x)\varphi_p(u(x)v(x) - f(x,u(x))v(x))dx,$$

where $u, v \in X$ and $\langle\cdot,\cdot\rangle$ denotes the duality brackets between the spaces X^* and X. The critical points of I, i.e. the points u for which $\langle I'(u), v\rangle = 0$ for all $v \in X$ are the weak solutions of the problem (P). Our main result is as follows:

Theorem 1.1. *Let $p>1$ and the functions f and a satisfy assumptions $(F_1),(F_2)$ and (A). Then the problem (P) has at least one nontrivial weak solution. If in addition the assumption (F_3) holds, then the problem (P) has infinitely many pairs of weak solutions $(u_m, -u_m)$ such that $u_m \neq 0$ and $\lim_{m\to\infty} \max_{x\in[0,T]} |u_m(x)| = 0$.*

Remark 1.2. We show an example of a function f which satisfies the assumptions

$(F_1)-(F_3)$. Let $p=\frac{5}{2}, q=\frac{3}{2}$, $T=\pi$, $J=(0,1)$ and

$$f(x,u)=\frac{|\cos x|}{x^{1/3}}|u|^{-1/2}u.$$

Then $r=\frac{5}{2}$, and $b(x)=\frac{|\cos x|}{x^{1/3}}\in L^{5/2}(0,\pi)$ by $b(x)\leq\frac{1}{x^{1/3}}$ and $\int_0^\pi(\frac{1}{x^{1/3}})^{5/2}dx=\int_0^\pi\frac{1}{x^{5/6}}dx=6\sqrt[6]{\pi}$. We have

$$F(x,u)=\frac{2|\cos x|}{3x^{1/3}}|u|^{3/2}$$

and for $(x,u)\in J\times\mathbb{R}$

$$F(x,u)\geq\frac{2}{3}\cos 1^{1/3}|u|^{3/2},$$

which satisfies (F_2) with $c=\frac{2}{3}\cos 1^{1/3}$.

The paper is organized as follows. In Section 2 we give notes on above mentioned critical point theorems and variational formulation of the problem. In Section 3 we present an existence result for the problem (P) and give it's proof.

2. Preliminaries. Critical point theory and variational method deal with problems for which there exists a smooth functional whose critical points are the solutions of considered problems. The minimax method characterizes a critical value c of a functional J on a Banach space E as a mini max value over a certain class of sets $\mathcal{A}$ of E

$$c=\inf_{A\in\mathcal{A}}\max_{u\in A}J(u).$$

Perhaps one of first examples using a mini max method due to E. Fisher (1905) for characterization of eigenvalues of a real symmetric $n\times n$ matrix M

$$\lambda_k=\inf_{X_{k-1}}\max_{x\perp X_{k-1},|x|=1}\langle Mx,x\rangle,$$

where $E=R^n, X_j$ is a subspace of E with dimension j.

A theory of mini max was elaborated in early 1930 by L. Lusternik and L. Schnirelaman [7] in the study of manifolds. They introduce the Lusternik-Schnirelaman (LS) cathegory $\mathrm{Cat}(Y)$ of a topological space Y, as the least number n, such that there is a cover of $n+1$ open subsets of Y, each of them contractible to a point of Y.

For instance $\mathrm{Cat}(\mathbb{S}^n)=1, \mathrm{Cat}(\mathbb{T})=2$ where $\mathbb{S}^n$ is the sphere in $\mathbb{R}^{n+1}$ and $\mathbb{T}=\mathbb{S}\times\mathbb{S}$ is the torus in $\mathbb{R}^3$.

A simpler notion the LS-cathegory is that of *genus* due to M. Krasnoselskii [6] and the equivalent definition of Coffman [3]. Let E be a real Banach space and Ξ denotes the family of sets $A\subset E$ such that A is closed and symmetric with respect to the origin 0, i.e. $x\in A$ iff $-x\in A$. For $A\in\Xi$, the genus $\gamma(A)$ is defined as the smallest integer n, such that there exists a continuous map $\varphi:A\to\mathbb{R}^n\setminus 0$.

One can see that for $n\geq 1$ and A homemorphic to $\mathbb{S}^n$, $\gamma(A)>1$ by intermediate value theorem. Moreover $\gamma(\mathbb{S}^n)=n+1$ follows by Borsuk-Ulam theorem (see [9]).

After the work of Lusternik and Schnirelman [7] many authors studied the existence of multiple critical points of functionals on manifolds and infinite dimensional Banach spaces using compactness conditions known as Palais-Smale (PS) or (C) conditions. Let

E be a Banach space and $J \in C^1(E, \mathbb{R})$.

The functional J satisfies (PS)-condition if any sequence $u_n \subset E$, such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ contains a strongly convergent subsequence (whose limit as a consequence is a critical point of J.)

Clark [2] in 1972 used the (PS)-condition in a localized form as (A)-condition:

Any sequence $u_n \subset E$, such that $J(u_n) < 0$, $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ has a strongly convergent subsequence.

Brezis and Nirenberg [1] in 1991 used a variant of (PS)-conditions known as $(PS)_c$ and $(C)_c$ conditions:

Any sequence $u_n \subset E$, such that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$ contains a convergent subsequence.

J satisfies $(C)_c$ condition if whenever $u_n \subset E$ is a sequence, such that $J(u_n) \to c$ and $J'(u_n) \to 0$, then c is a critical value of J.

(PS)-conditions is equivalent to $(PS)_c$ condition for any $c \in \mathbb{R}$. $(PS)_c$ condition implies $(C)_c$ condition but not converse. For instance the function $f(x) = \cos(x), x \in \mathbb{R}$ satisfies $(C)_c$ but not $(PS)_1$ condition.

D. Clark [2] in 1972 used (A) condition, Krasnoselskii genus in a multiple critical point theorem, known recently as Clark's theorem. We present it's variant proved in the paper of. H. Heinz [5], 1986. Let Σ be the class of closed subsets A of $E \setminus \{0\}$ which are symmetric with respect to origin, i.e. $A = -A$. The Lusternik-Schnirelman levels c_j of $J \in C^1(E, \mathbb{R})$ are defined by

$$c_j = \inf_{A \in \Sigma} \sup_{u \in A} J(u), j \geq 1.$$

Theorem 2.1 (Clark's theorem, [2], [5])**.** *Let E be a Banach space, $J \in C^1(E, \mathbb{R})$. Suppose that J satisfies the assumptions:*

(i) J is bounded below on E, J is even and $J(0) = 0$,

(ii) J satisfies $(PS)-$ condition,

$(iii)_m$ For some integer m, there exists a set $K \in \Sigma$ such that $\gamma(K) = m$ and $\sup_{u \in K} J(u) < 0$

Then, there exist antipodal pairs $(u_j, -u_j)$ of critical points of J such that $J(u_j) = c_j$ for $1 \leq j \leq m$.

As a consequence we have

Proposition 2.2. *If J satisfies hypotheses (i), (ii) and $(iii)_m$ for every $m \in \mathbb{N}$, then we have $\lim_{j \to \infty} c_j = 0$.*

Rabinowitz [12], p.53 gave a simpler formulation as follows

Theorem 2.3 ([12])**.** *Let E be a Banach space, $J \in C^1(E, \mathbb{R})$ with J is even, bounded below and satisfying (PS). Suppose $J(0) = 0$, there is a set $K \subset E$ such that K is homeomorphic to $\mathbb{S}^{j-1}$ by an odd map, and $\sup_K J < 0$. Then J possesses at least j distinct pairs of critical points.*

Theorem 2.3 was applied in [11] for the study of extended Fisher-Kolmogorov equations.

Theorem 2.1 was recently extended by Z. Liu and Z. Wang [8] in 2015. They consider the question, when the critical points u_j of Proposition 2.2 satisfy the condition $u_j \to 0$ as $j \to \infty$. Their basic result is formulated as follows:

Theorem 2.4 (Generalized Clark's theorem, [5], [8]). *Let E be a Banach space, $J \in C^1(E,\mathbb{R})$. Assume that J satisfies the (PS) condition, it is even, bounded from below and $J(0)=0$. If for any $k \in \mathbb{N}$, there exists a $k-$dimensional subespace E^k of E and $\rho_k > 0$ such that $\sup\limits_{E^k \cap S_{\rho_k}} J < 0$, where $S_\rho = \{u \in E\,,\ \|u\|_E = \rho\}$, then at least one of the following conclusions holds:*

1. *There exists a sequence of critical points $\{u_k\}$ satisfying $J(u_k) < 0$ for all k and $\lim\limits_{k\to\infty} \|u_k\|_E = 0$.*
2. *There exists $r > 0$ such that for any $0 < \alpha < r$ there exists a critical point u such that $\|u\|_E = \alpha$ and $J(u) = 0$.*

Realize that Theorem 2.4 implies the existence of many pairs of critical points $(u_m, -u_m)$, $u_m \neq 0$ such that $J(u_m) \leq 0$, $\lim\limits_{m\to+\infty} J(u_m) = 0$ and $\lim\limits_{m\to+\infty} \|u_m\|_E = 0$.

We present also the minimization theorem which will be used in the proof of the main result Theorem1.1 (see [4], [9],[10]).

Theorem 2.5 (Minimization theorem). *Let $J\colon E \to \mathbb{R}$ be a weakly sequentially lower semi continuous functional on a reflexive Banach space E and let J has a minimum on E, i.e., there exists $u_0 \in E$ such that $J(u_0) = \inf\limits_{u\in E} J(u)$. If J is differentiable, then u_0 is a critical point of J.*

We have the usual Wirtinger and Sobolev inequalities and compact embedding $X \hookrightarrow C([0,T])$, (see [9], [10]).

Lemma 2.6. *If $u \in X$, then*

(i)

$$\int_0^T |u(x)|^p dx \leq T^p \int_0^T |u'(x)|^p dx,$$

(ii)

$$\|u\|_\infty := \max|u(x)| \leq T^{\frac{p-1}{p}} \left(\int_0^T |u'(x)|^p dx\right)^{\frac{1}{p}}.$$

3. Proof of the main result. Before the proof of Theorem 1.1 we formulate lemma as follows.

Lemma 3.1. *Assume that assumptions (A) and (F_1) hold. Then the functional $I : X \to \mathbf{R}$ is bounded from below and satisfies (PS) condition.*

Proof of Lemma 3.1. We have by (F_1)

$$|F(x,u)| \leq \frac{1}{q} b(x)|u|^q.$$

Then, for Hölder inequality and assumption (A)

$$\int_0^T F(x,u(x))dx \leq C_1 \|b\|_{\frac{p}{p-q}} \|u\|^q, \tag{3.1}$$

where C_1 is a positive constant. By Lemma 2.6 we have

$$I(u) \geq \frac{1}{p}\|u\|^p - C_1\|b\|_{\frac{p}{p-q}} \|u\|^q$$

which implies by $1 < q < p$, that I is a coercive functional, i.e. $I(u) \to \infty$ as $\|u\| \to \infty$ and bounded from below. The statement that the functional I satisfies the PS-condition is following the arguments in [10] and we omit the details. $\square$

Proof of Theorem 1.1. The conditions of minimization Theorem 2.5 are satisfied and the functional $I : X \to \mathbf{R}$ has a minimum point u_0 which is a weak solution of the problem (P). It can be shown by (F_2) that u_0 is a nonzero solution. Let $v \in W_0^{1,p}(J) \cap X$ be such that $v(x) = 0$, $x \in [0, T] \backslash J$ be a nonzero function and $\|v\|_\infty \le 1$. Then for $t > 0$ by (F_2)

$$\begin{aligned} I(tv) &= \frac{t^p}{p}\|v\|^p - \int_0^T F(x, tv(x))dx = \frac{t^p}{p}\|v\|^p - \int_J F(x, tv(x))dx \\ &\le \frac{t^p}{p}\|v\|^p - ct^q \int_J |v(x)|^q dx. \end{aligned}$$

By the last inequality it follows that for sufficiently small $t, I(tv) < 0$ because $1 < q < p$. Then $I(u_0) = \min\{I(u) : u \in X\} < 0$ and u_0 is a nonzero weak solution because $I(0) = 0$.

Next, we show that under conditions $(F_1) - (F_3)$ and (A), the functional I satisfies conditions of Theorem 2.4. Let $J = (a, b) \subset (0, T)$ and for $n \in \mathbf{N}$ take n disjoint open intervals $J_k = (x_{k-1}, x_k), k = 1, \ldots, n$ where $x_k = a + \dfrac{k}{n}(b - a)$, $k = 0, 1, \ldots, n$. We have $\cup\{J_k : k = 1, \ldots, n\} \subset J$. Next, we choose functions $v_k \in C_0^\infty(J_k)$ such that $\|v_k\|_\infty < \infty$ and $\|v_k\| = 1$. Let X_n be the $n-$dimensional subspace of X

$$X_n := span\{v_1, v_2, \ldots, v_n\}$$

and

$$S_n := \{u \in X_n : \|u\| = 1\}.$$

For $u = \displaystyle\sum_{k=1}^n \lambda_k v_k \in X_n$ we have

$$\begin{aligned} \|u\|^p &= \int_0^T (|u'(x)|^p + a(x)|u(x)|^p)dx \\ &= \sum_{k=1}^n |\lambda_k|^p \int_{J_k} (|v_k'(x)|^p + a(x)|v_k(x)|^p)dx \\ &= \sum_{k=1}^n |\lambda_k|^p \end{aligned}$$

and

$$\|u\|_q^q = \sum_{k=1}^n |\lambda_k|^q \int_{J_k} |v_k(x)|^q dx. \tag{3.2}$$

Since the norms $\|\cdot\|$ and $\|\cdot\|_q$ are equivalent on the finite dimensional space X_n there are positive constants d_{1n} and d_{2n} such that for $u \in X_n$:

$$d_{1n}\|u\| \le \|u\|_q \le d_{2n}\|u\|. \tag{3.3}$$

Then, for $u \in S_n$ by (3.2) and (3.3) we have:

$$\begin{aligned} I(tu) &= \frac{t^p}{p}\|u\|^p - \int_0^T F(x, tu(x))dx = \frac{t^p}{p}\|u\|^p - \sum_{k=1}^n \int_{J_k} F(x, t\lambda_k v_k(x))dx \\ &\le \frac{t^p}{p}\|u\|^p - ct^q \sum_{k=1}^n |\lambda_k|^q \int_{J_k} |v_k(x)|^q dx \\ &= \frac{t^p}{p}\|u\|^p - ct^q\|u\|_q^q \le \frac{t^p}{p}\|u\|^p - c_1 t^q d_{2n}^q \|u\|^q. \end{aligned}$$

By the last inequality and $1 < q < p$ it follows that $I(v) < 0$ for $v \in S_n^t := \{tu : u \in S_n\}$ with t sufficiently small. Finally, the functional I satisfies all conditions of Theorem 2.4 and the assertion follows. □

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СЪЩЕСТВУВАНЕ НА БЕЗБРОЙНО МНОГО РЕШЕНИЯ НА ЗАДАЧИ ЗА p-ЛАПЛАСОВИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ ЧРЕЗ ВАРИАЦИОНЕН МЕТОД

Степан Терзиян

В статията се изучава съществуването на безкрайно много решения на задача на Дирихле за едномери p-Лапласови уравнения. Дадени са исторически бележки за теорията на критични точки, вариационен метод и теорема на Кларк.