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# ESTIMATION OF THE CURE RATE OF NON-PERFORMING LOANS USING MARKOV CHAINS*

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A Markov-chain model is developed for the purpose of estimation of the cure rate of non-performing loans. The technique is performed collectively, on portfolios and it can be applicable in the process of calculation of credit impairment. It is efficient in terms of data manipulation costs which makes it accessible even to smaller financial institutions.

**Introduction.** In calculating the credit impairment the IFRS 9 standard permits, under certain conditions, the usage of cure rate in order to reduce the amount of bank provisions. The logic behind this allowance is, that if an impaired amount will eventually return to regular status, the bank need not calculate provisions on it. Several methodological manuals are available to the banking community which (cf. e.g. [2]), often without stipulation on the assumptions on the model, provide recipes for calculation of a cure rate. Estimates made in this way turn out to be often overly conservative and, sometimes, dissatisfactory, because the basic assumptions of the model could not be verified. The presented technique allows to calculate with any desired accuracy and any desired frequency. (E.g., monthly, quarterly, etc.) The model uses data only from the past 12 months in order to provide a *most recent measurement* of the cure rate. This is, sometimes, required by regulators for financial quantities measured collectively.

This very fact is the reason why in *low-default* portfolios, which are often those found in small banks, the results of this type of method do not satisfy even very basic assumptions about the cure rate in general. For this reason we apply a smoothing method from Survival analysis. This method is the topic of Section 3.

The usage of Markov-chain models, in general, is a technique accessible to the banking management and is part of their routine in accessing credit risk and expected credit losses, several studies document and contribute to this practice, including [3], [4] and [6]. It appears that Monte Carlo techniques similar to those demonstrated in e.g., [7] are applicable in the study of cure rate and it is my belief that future interest in this subject would move in this direction.

The model was developed particularly for the needs of Bulgarian small banks, which face significan challenges in meeting the regulatory requirements in transition to the IFRS 9, often notwithstanding their limited experience and computational power[1].

In addition to calculation of cure rate this method can be useful in the management of portfolios of loans. This application is left to the future.

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**Key words:** Cure rate estimation, Markov chains, survival analysis, IFRS 9 provisioning.

[1] Since the beginning of 2018 Bulgarian banks are bound to report expected credit losses and to

**1. Cure rate.** Cure rate is meant to measure the propensity of loans to return to regular status after they have been found delinquent. In a portfolio, collectively, the cure rate estimates what proportion of non-performing loans will be, in the end, repaid. Given the possibility of a loan which is once "cured" to relapse or to move back and forth between categories it is not enough to simply measure the proportion over a certain horizon of time.

For the purposes of this study we assume that a loan is considered non-performing after it is found more than 90 days late. We make the following assumptions:

1. The loan is finally cured after it becomes less than one month past due.
2. Loans which are N or more months past due are considered lost and are written off.
3. We should distinguish performing loans which have been granted forbearance. According to [1] these would be loans to parties experiencing financial difficulties in meeting their obligations and the bank has agreed to offer them special contractual terms. If such loan preserves its regular status for a year we consider it cured, otherwise we consider it lost.
4. States are assigned to all loans in the portfolio, based on $m$, the whole number of months past-due at time $t = 0$. The state where $m = 0$ is an absorbing state, as well as the one with $m \geq N$. The forborne loans are assigned in a separate state. Hence, the number of states is $N + 2$.
5. We assume the time periodicity of observation to the loan tape is annual. We measure the probabilities of transition between states by observing the migration between states within a year prior to time $t = 0$.
6. We assume that the migration in the previous years is irrelevant to the further development of the portfolio. Moreover, we assume that transition rates do not vary in time.

Assumptions 1–3 are a question of bank policy and, although they satisfy the requirement of IFRS 9, an alternative configuration may be set. Assumption 5 is inessential, although it should be noted that small banks often have shallow, low-default portfolios and high frequency observation leads to volatile cure rates.

**2. Markov process.** For a typical loan of the considered portfolio we have thus constructed a finite Markov chain of random variables $\{X_t : t = 0, 1, \dots\}$, which take as value the state of the loan at year $t$. It has $N + 2$ states $\{S_i : i = 0, \dots, N + 1\}$, describing the state the loan is. With an appropriate ordering we can assume that $S_0$ denotes the state where $m = 0$, $S_1$ denotes $m \geq N$, $S_2$ is the *forborne* state and, for any $i >= 3$, the state $S_i$ is characterized by $M = i - 2$. Hence, $S_5$ is the first non-performing state, corresponding to $M = 3$.

- Assumption 6 from Section 1 implies time homogeneity.
- Assumptions 1 and 2 imply that $S_0$ and $S_1$ are absorbing states and, hence, they form, each by itself two recurrent communication classes.
- If any part of the set of states $\mathcal{T} = \{S_i : i = 2, \dots, N + 2\}$ forms a recurrent class this would imply that the loan contract for this particular portfolio can be

calculate provisions based on IFRS 9, which is compatible with the *expected-loss approach* of Basel III. (Cf. e.g. [5].) They are, in general, computing cure rates for their retail loan portfolios and for other portfolios of standardized products.

optimized. We are giving an example to illustrate this in Section 4. For this reason we assume that $\mathcal{T}$ is the set of transitive states.

- Assumption 3 implies that $S_2$ is a transitive. In fact,

$$P[X_n = S_0|X_{n-1} = S_2] = p, \quad P[X_n = S_1|X_{n-1} = S_2] = q,$$

$$P[X_n = S_i|X_{n-1} = S_2] = 0, \text{ for } i \geq 1,$$

where $p$ and $q$, satisfying $p + q = 1$ are the probabilities to survive and fail, respectively.

We write the transition matrix $A = (p(i,j) = P[X_1 = S_j|X_0 = S_i])$, therefore as follows:

$$A = \left( \begin{array}{cc|ccc} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \hline \multicolumn{2}{c|}{T} & \multicolumn{3}{c}{S} \end{array} \right).$$

Furthermore, for the limit matrix $A_\infty = \lim\limits_{n\to\infty} A^n$ we have:

$$A_\infty = \left( \begin{array}{cc|ccc} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \hline \multicolumn{2}{c|}{T_\infty} & \multicolumn{3}{c}{O} \end{array} \right),$$

where $O$ is the zero matrix. For the matrix $T_\infty$ we have

$$T_\infty = \begin{pmatrix} p & q \\ p_1 & q_1 \\ \vdots & \vdots \\ p_{N-1} & q_{N-1} \end{pmatrix},$$

where $p_i$ and $q_i$ satisfy $p_i + q_i = 1$ and are the probabilities of a loan showing $i$ months of payment delay to be cured or lost, respectively. Hence, in search for the cure rate, our goal is to study the vector $(p_0, \dots, p_N)$.

**Proposition 1.** *In the notation defined above, the probability to cure for a loan which is $i$ months past due at time $t = 0$ can be found on the $i^{th}$ row of the first column of the matrix*

$$T_\infty = (I - S)^{-1} T.$$

**Proof.** Since $S$ is a substochastic matrix, representing the transition rates of transitive states, we know that $S^n \to O$ as $n \to \infty$. For this reason the matrix $I - S$, with $I$ — the identity matrix of size $N - 1$ is, indeed, invertible.

Denote by $t_{ij}$ the probability of a loan with initial state $X_0 = S_i$ to reach eventually the state $S_j$, $j = 0, 1$. (In the notation above, these are the entries of the matrix $T$, $t_{i0} = p_i$, and $t_{i1} = q_i$.) We have

$$\begin{aligned} rClt_{ij} &= P[X_n = S_j \text{ for some } n|X_0 = S_i] \\ &= P[X_1 = S_j|X_0 = S_i] \\ &\quad + \sum_{k=0}^{N} P[X_n = S_j \text{ for some } n|X_1 = S_k] P[X_1 = S_k|X_0 = S_i] \end{aligned}$$

$$= \quad p(i,j) + \sum_{k=2}^{N} t_{kj} p(i,k).$$

That is,

$$T_\infty = T + T_\infty S$$

Hence

$$T_\infty = (I - S)^{-1} T. \qquad \square$$

**3. Survival model.** Let $S(x)$ be probability of a loan to be cured if the initial state is at least $x$ months past due. For *non-preforming* states, $i \geq 3$,

$$S(x) = P(X_n = S_0 \text{ for some } n | X_0 = S_i, i \geq x + 2).$$

This function needs to satisfy the following conditions:

1. $S(0) = 1$;
2. $S(x) = 0$ for $x \geq N$;
3. $S(x)$ is non-increasing.

In addition, one would expect that chances of failure would increase as a function of $x$, the months past-due. This is due, in part, to two reasons. First, the longer delay signifies a more dire economic status. And second, portfolio manager would make more effort to increase the opportunities of survival of these loans which are less in delay, since they have better chance. This gives us an extra condition:

4. The logarithmic derivative $\dfrac{1}{S(x)} \dfrac{dS}{dx}(x)$ is decreasing.

Generally the outcome of calculating the Markov chain need not satisfy these conditions. In order to smoothen the results we apply tools from the Survival Analysis. (Cf. e.g. [8].) A common choice of survival function is a best-fitting Weibull curve:

$$S(x) = e^{-\left(\frac{x}{\lambda}\right)^k},$$

corresponding to a Weibull distribution with CDF $F(x) = 1 - S(X)$. Condition 4 simply means that the hazard rate is an increasing function which would imply that the shape parameter $k$ of the curve satisfies $k > 1$. The parameters $k$ and $\lambda$ must be chosen by fitting the CDF of this distribution to the points

$$(0,1), (1, p_1), (2, p_2), \ldots, (N-1, p_{N-1}), (N, 0).^2$$

After this, the cure rate of the portfolio is equal to $S(3)$.

**4. A numerical example.** We now consider the portfolio of select credit cards from a small bank at the end of March 2007. The total size of the portfolio is €328.9 Thousand, consisting of 1185 loans. The bank management assumes that a loan not serviced for 8 or more months is lost, $N = 8$. The transition matrix $A$, according to the notation above is:

[2] The point $(\delta, p)$ may be added to the sequence with an appropriately chosen small value of the parameter $\delta$.

$$A = \left(\begin{array}{cc|cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0.37 & 0.63 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.39 & 0.11 & 0.1 & 0.157 & 0.008 & 0.015 & 0.11 & 0.06 & 0.02 & 0.03 \\
0.37 & 0.12 & 0.02 & 0.003 & 0.012 & 0.045 & 0.09 & 0.04 & 0 & 0.3 \\
0.05 & 0.32 & 0.09 & 0.004 & 0.107 & 0.113 & 0.141 & 0.102 & 0.073 & 0 \\
0 & 0.45 & 0 & 0 & 0 & 0.19 & 0.119 & 0.149 & 0.012 & 0.08 \\
0 & 0.4 & 0 & 0 & 0 & 0.08 & 0.01 & 0.31 & 0 & 0.2 \\
0 & 0.21 & 0 & 0 & 0 & 0.05 & 0.009 & 0.111 & 0.41 & 0.21 \\
0 & 0.47 & 0.004 & 0 & 0 & 0 & 0 & 0.037 & 0.27 & 0.219
\end{array}\right)$$

Next, we compute the matrices $(I-S)^{-1}$:

$$(I-S)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.127 & 1.187 & 0.018 & 0.08 & 0.166 & 0.179 & 0.121 & 0.148 \\
0.033 & 0.004 & 1.024 & 0.109 & 0.127 & 0.172 & 0.254 & 0.519 \\
0.114 & 0.006 & 0.132 & 1.221 & 0.216 & 0.288 & 0.254 & 0.215 \\
0.029 & 0.002 & 0.032 & 0.299 & 1.192 & 0.348 & 0.187 & 0.274 \\
0.017 & 0.001 & 0.018 & 0.164 & 0.048 & 1.549 & 0.237 & 0.472 \\
0.018 & 0.001 & 0.018 & 0.162 & 0.053 & 0.396 & 2.016 & 0.656 \\
0.012 & 0 & 0.007 & 0.064 & 0.021 & 0.21 & 0.708 & 1.529
\end{pmatrix}$$

and $T_\infty$:

$$T_\infty = \begin{pmatrix}
0.37 & 0.63 \\
0.52 & 0.48 \\
0.398 & 0.602 \\
0.155 & 0.845 \\
0.038 & 0.962 \\
0.021 & 0.979 \\
0.021 & 0.979 \\
0.01 & 0.99
\end{pmatrix}.$$

Next we fit a Weibull curve on ten points:

$$(0,1), (0.5, 0.37), (1, 0.52), (2, 0.398), (3, 0.155),$$

$$(4, 0.038), (5, 0.021), (6, 0.021), (7, 0.01), (8, 0),$$

using a linear regression. The results are shown in Table 1.

Table 1. The fitting of the Weibull curve for a Credit Card portfolio. In parenthesis are shown the $t$-statistic values from the linear regression. The $^*$ denotes significance to the 99% level

| $\lambda$ | $k$ | $R^2$ |
|---|---|---|
| $1.51^*$ | $1.14^*$ | 0.96 |
| (2.91) | (15.03) | |

The shape coefficient $k = 1.14$ is different from 0 at the 99% level of significance. The one-sided hypothesis for $k \leq 1$ is rejected at the 95% level[3]. The expected cure rate is computed as $S(3) = 11.26\%$.

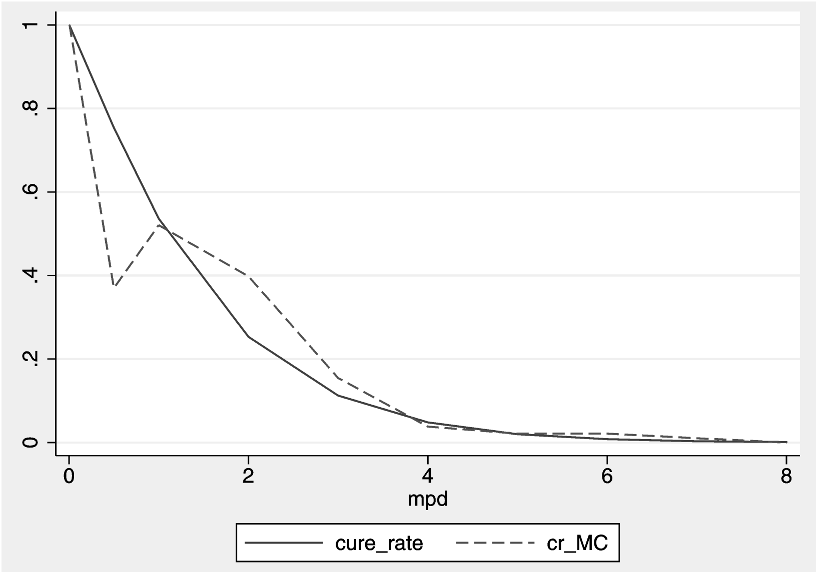


Fig. 1. The effect of Weibull curve fitting. The curve labeled *cr_MC* represents the result of the Markov chain model; the final result is labeled *cure_rate*. Both are drawn as functions of months-past-due, *mpd*

Figure 1 shows the cure rate computed from the survival function compared with the raw Markov chain results.

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## ОЦЕНЯВАНЕ НА КОЕФИЦИЕНТА НА ОЗДРАВЯВАНЕ НА БАНКОВИ ЗАЕМИ В НЕИЗПЪЛНЕНИЕ ПОСРЕДСТВОМ ВЕРИГИ НА МАРКОВ

**Вилислав Бучакчиев**

Представеният статистически модел посредством вериги на Марков се прилага за оценка на коефициента на оздравяване на банкови кредити в неизпълнение. Методът изисква минимални разходи за манипулация и съхранение на данни и е достъпен за разработка дори и за малка финансова институция. Посочени са няколко идеи за анализ на оптималността на портфейла.