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EXPLICIT SOLUTION OF A BOUNDARY VALUE PROBLEM FOR A HEAVY STRING[*]

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In this paper we find an explicit solution of a problem for hanging string using operational calculus approach, based on a non-classical convolution. First, using Sonyn's transformation T we transform this problem to BVP with constant coefficient but with a nonlocal boundary condition. Then we find the explicit solution. This explicit solution is suitable for numerical calculation too.

1. BVP for a Heavy String. In this paper we find an explicit solution of a problem for hanging chain, described by the following initial-boundary value problem:

$$\frac{\partial^2}{\partial t^2}v = \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)v, \quad 0 < x < 1, \quad 0 < t, \tag{1}$$

$$v(x,0) = v_0(x), \quad 0 \le x \le 1,$$

$$\frac{\partial v(x,0)}{\partial t} = v_1(x), \quad 0 \le x \le 1,$$

$$v(1,t) = \alpha(t), \quad 0 \le t,$$

$$|v(0,t)| < \infty, \quad 0 \le t. \tag{2}$$

This problem arise when we consider heavy string, fixed on the top and free at the bottom.

We solve this problem in the flowing way.

First by Sonyn's transformation T [5], [4] we transform this problem to BVP with constant coefficient but with a nonlocal boundary condition.

Second we solve this nonlocal boundary value problem. We find explicit - duhamel representation of the solution u using nonstandard convolution $\overset{x}{*}$ acting on the space variable. This convolution is introduced by Dimivski (see [2], p. 119). At the end we use inverse transform T^{-1} to transform the explicit solution u of nonlocl BVP. This transformed solution $v = T^{-1}u$ is the solution of BVP (1)–(2).

2. Sonyn's transformation. We use the following notation

$$B = \frac{d}{dx}\left(x\frac{d}{dx}\right) \tag{3}$$

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for Bessel operator and

(4) $$D = \frac{d^2}{dx^2}$$

for the operator of the square of differentiation.

We consider both operators B and D in $C^2(0,1)$, i.e. for $f \in C^2(0,1)$

$$Bf(x) = \frac{d}{dx}\left(x\frac{d}{dx}\right) f(x).$$

Next, we consider Sonyn's transformation

(5) $$Tf(x) = 3x\int_0^1 \frac{f\left(\frac{x^2\xi}{4}\right)}{\sqrt{1-\xi}} d\xi$$

and its inverse

(6) $$T^{-1}f = \frac{1}{6\pi}\frac{d}{dx}\int_0^x \frac{f(2\sqrt{\xi})}{\sqrt{x-\xi}} d\xi.$$

Next, we use the following representation of T^{-1}:

(7) $$T^{-1}f(x) = \frac{1}{6\pi}\left(\frac{1}{2\sqrt{x}}\int_0^1 \frac{f(2\sqrt{x\tau})}{\sqrt{1-\tau}} d\tau + \int_0^1 \frac{f'(2\sqrt{x\tau})\sqrt{\tau}}{\sqrt{1-\tau}} d\tau\right).$$

We show that the transformation T transmutes the Bessel operator B to the operator of square of the differentiation D.

Lemma 1. *We have*

(8) $$TB = DT$$

in $C^2(0,1)$.

Proof. First we prove $TB = DT$ for the functions of the form $f(x) = x^\alpha$, $\alpha \in [0,\infty)$. Let us consider the left-hand side of $TBx^\alpha = DTx^\alpha$:

$$TBx^\alpha = T\frac{d}{dx}\left(x\frac{d}{dx}\, x^\alpha\right) = T(\alpha^2 x^{\alpha-1}) = \alpha^2\, T(x^{\alpha-1}) =$$

$$= 3\alpha^2 x\int_0^1 \frac{\frac{x^{2(\alpha-1)}\xi^{\alpha-1}}{4^{\alpha-1}}}{\sqrt{1-\xi}} d\xi = \frac{3\alpha^2}{4^{\alpha-1}} x^{2\alpha-1}\int_0^1 \frac{\xi^{\alpha-1}}{\sqrt{1-\xi}} d\xi.$$

Next we consider the right-hand side of $TBx^\alpha = DTx^\alpha$:

$$DTx^\alpha = D\left(3x\int_0^1 \frac{\frac{x^{2\alpha}\xi^\alpha}{4^\alpha}}{\sqrt{1-\xi}} d\xi\right) = \frac{3}{4^\alpha} D\left(x^{2\alpha+1}\int_0^1 \frac{\xi^\alpha}{\sqrt{1-\xi}} d\xi\right)$$

$$= \frac{6\alpha(1+2\alpha)}{4^\alpha} x^{2\alpha-1}\int_0^1 \frac{\xi^\alpha}{\sqrt{1-\xi}} d\xi.$$

By

$$\int_0^1 \frac{\xi^\alpha}{\sqrt{1-\xi}} d\xi = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})}\sqrt{\pi}$$

we obtain that

$$TBx^\alpha = DTx^\alpha.$$

Using the linearity of operators D, B and T and the Weierstrass' approximation theorem we finish the proof. $\square$

Lemma 2. *The transformation* (5)

$$Tf(x) = 3x \int_0^1 \frac{f\left(\frac{x^2\xi}{4}\right)}{\sqrt{1-\xi}} d\xi$$

transforms the boundary value problem

$$\frac{d}{dx}\left(x\frac{d}{dx}\right)\varphi(x) + c\ \varphi(x) = F(x), \quad 0 < x < 1, \quad c = \text{const}, \tag{9}$$

$$\varphi(0) = a_0, \quad \varphi(1) = a_1$$

to the boundary value problem

$$\frac{d^2}{dx^2}\psi(x) + c\ \psi(x) = TF(x), \quad 0 < x < 1, \tag{10}$$

$$\psi(0) = 0, \qquad (T^{-1}\psi)(1) = a_1,$$

where

$$\psi(x) = T\varphi(x).$$

Proof. Using notation (3), we can write (9) as

$$B\varphi(x) + c\ \varphi(x) = F(x).$$

Let us apply the transformation T to equation (9). We find

$$TB\varphi(x) + c\ T\varphi(x) = TF(x).$$

By Lemma 1 we obtain

$$DT\varphi(x) + c\ T\varphi(x) = TF(x).$$

Using the notation

$$\psi(x) = T\varphi(x)$$

we find (10)

$$D\psi(x) + c\ \psi(x) = TF(x).$$

Let us consider the boundary condition $\varphi(0) = a_0$. From

$$\psi(x) = T\varphi(x) = 3x \int_0^1 \frac{\varphi\left(\frac{x^2\xi}{4}\right)}{\sqrt{1-\xi}} d\xi$$

we obtain that

$$\psi(0) = T\varphi(0) = 0.$$

Next we consider the boundary condition $\varphi(1) = a_1$. We apply the inverse transformation T^{-1} to $\psi(x) = T\varphi(x)$ and find

$$\varphi(x) = T^{-1}\psi(x).$$

Using the boundary condition $\varphi(1) = a_1$, we find

$$(T^{-1}\psi)(1) = \varphi(1) = a_1.$$

Here it is important to make the following observation. We change the local condition $\varphi(1) = a_1$ with respect to function φ by the nonlocal condition $(T^{-1}\psi)(1) = a_1$ with respect to function ψ. $\square$

Using Lemma 1 and Lemma 2 we obtain that the transformation T transforms the boundary value problem (1)–(2) to the boundary value problem:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad 0 < t,\\
&u(x,0) = f(x), \qquad 0 \le x \le 1,\\
&\frac{\partial u(x,0)}{\partial t} = g(x), \quad 0 \le x \le 1,\\
&u(0,t) = 0, \qquad 0 \le t,\\
&\Phi_\xi\{u(\xi,t)\} = \alpha(t), \quad 0 \le t,
\end{aligned} \tag{11}$$

where

$$\Phi_\xi\{u(\xi,t)\} = T^{-1}u(1,t) = \frac{1}{\pi}\left(\frac{d}{dx}\int_0^x \frac{u(2\sqrt{\xi},t)}{\sqrt{x-\xi}}d\xi\right)|_{x=1} = \tag{12}$$

$$= \frac{1}{\pi}\left(\frac{1}{2}\int_0^1 \frac{u(2\sqrt{\tau},t)}{\sqrt{1-\tau}}d\tau + \int_0^1 \frac{u_x(2\sqrt{\tau},t)\sqrt{\tau}}{\sqrt{1-\tau}}d\tau\right)$$

and

$$f(x) = Tv_0(x), \quad g(x) = Tv_1(x),$$

$$u(x,t) = Tv(x,t) = 3x\int_0^1 \frac{v\left(\frac{x^2\xi}{4},\ t\right)}{\sqrt{1-\xi}}d\xi, \quad 0 \le x \le 1, \quad 0 \le t. \tag{13}$$

From

$$|v(0,t)| < \infty, \quad 0 \le t$$

we obtain that

$$u(0,t) = (Tv(x,t))|_{x=0} = \left(3x\int_0^1 \frac{v\left(\frac{x^2\xi}{4},\ t\right)}{\sqrt{1-\xi}}d\xi\right)|_{x=0} = 0, \quad 0 \le t.$$

3. Duhamel-type representation of the solution of boundary value problem (1)–(2). Using operational calculus approach of Dimovski [1], [2], we obtain an explicit representation of the solution of BVP (12).

Theorem 1. *If $f(x) \in C^2([0,1])$, $f(0) = f'(0) = 0$ and $\Phi_\xi\{f(\xi)\} = 0$, then*

$$u = \frac{\partial^4}{\partial x^4}(\Omega(x,t) \overset{x}{*} f(x)) =$$

$$-\frac{1}{2}\Phi_\xi\left\{\int_0^\xi (\Omega_x(\xi + x - \eta, t) - \Omega_x(\xi - x - \eta, t))f''(\eta)d\eta\right\}$$

$$-\frac{1}{4\pi}\left[\int_0^1 \left(\frac{1}{\sqrt{1-\tau}}\int_0^{2\sqrt{\tau}} \Omega_x(2\sqrt{\tau}+x-\eta,t)-\Omega_x(2\sqrt{\tau}-x-\eta,t))f''(\eta)d\eta\right) d\tau\right]+$$

$$-\frac{1}{4\pi}\left[\int_0^1 \frac{1}{\sqrt{1-\tau}}\left(\int_0^{2\sqrt{\tau}} (\Omega_{xx}(2\sqrt{\tau}+x-\eta,t)-\Omega_{xx}(2\sqrt{\tau}-x-\eta,t))f''(\eta)\, d\eta+\right.\right.$$

$$\left.\left. 2\Omega_x[x]f''[2\sqrt{\tau}]\right) d\tau\right],$$

where

$$\Omega(x,t) = -\sum_{n=1}^{\infty} \frac{\sin\lambda_n t}{\lambda_n^5 J_1(2\lambda_n)} \sin\lambda_n x,$$

$n = 1, 2, \dots.$ *is a solution of* (11) *for* $g(x) = \psi(t) = F(x,t) \equiv 0$.

$J_1(x)$ is the Bessel function of order one and $2\lambda_n$ are the zeros of Bessel function $J_0(x)$ of order zero.

The proof may be accomplished by a direct check, too.

From $u(x,t) = Tv(x,t)$ where T^{-1} is given by (6) we obtain the solution $v(x,t) = T^{-1}u(x,t)$ of (1)–(2).

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ТОЧНО РЕШЕНИЕ НА ГРАНИЧНАТА ЗАДАЧА ЗА ТЕЖКА СТРУНА

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В статията е намерено точно решение на задачата описваща тежка струна, като е използвано операционно смятане основано на некласическа конволюция. Използвайки трансформацията на Сонин, трансформираме задачата в гранична задача с постоянни коефициент но с нелокално гранично условие. След това намираме явно решение. Намереното явно решение е подходящо и за числено смятане.