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A DIOPHANTINE TRANSPORT PROBLEM FROM 2016 AND ITS POSSIBLE SOLUTION FROM 1903*

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Motivated by a recent Diophantine transport problem about how to transport profitably a group of persons or objects, we survey classical facts about solving systems of linear Diophantine equations and inequalities in nonnegative integers. We emphasize on the method of Elliott from 1903 and its further development by MacMahon in his "Ω-Calculus" or Partition Analysis. As an illustration we obtain the solution of the considered transport problem in terms of a formal power series in several variables which is an expansion of a rational function of a special form.

1. Introduction. The idea for this paper came from the very interesting recent papers by Robles-Pérez and Rosales [37] and [38]. Starting with a specific transport problem about how to transport profitably a group of persons or objects the authors of [37] and [38] have generalized it to the following system of linear Diophantine inequalities.

Let $\mathbb{N}$ be the set of nonnegative integers, let $(a_1,\ldots,a_k)$ and $(b_1,\ldots,b_k)$ belong to $\mathbb{N}^k$, and let $a,b\in\mathbb{N}$. How to find the set T of all solutions $y\in\mathbb{N}$ of the system

$$\left| \begin{array}{l} y \geq a_1x_1+\cdots+a_kx_k+a \\ y \leq b_1x_1+\cdots+b_kx_k-b? \end{array} \right.$$

The approach in [37] and [38] is to prove that $T\cup\{0\}$ is a submonoid of the additive monoid $(\mathbb{N},+)$ and to develop an algorithm for computing the minimal system of its generators. This is in the spirit of results in [39], the main of which states that there exists a one-to-one correspondence between the set of numerical semigroups (i.e. submonoids S of $(\mathbb{N},+)$ such that $\mathbb{N}\setminus S$ is a finite set) with a fixed minimal nonzero element and the set of nonnegative integer solutions of a system of linear Diophantine inequalities.

Linear Diophantine equations and inequalities and their solutions in nonnegative integers are classical objects which appear in many branches of mathematics, computer science and their applications. Many methods have been developed for solving such systems. The purpose of our paper is to survey results, many of them with proofs, starting from Euler, Gordan and Hilbert. Special attention is paid to the method of Elliott [19]

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from 1903 and its further development by MacMahon [32] in his "Ω-Calculus" or Partition Analysis. Later, this method was studied in detail by Stanley [42] who added new geometric ideas. More recently, Domenjoud and Tomás [17] gave new life to the method deriving an algorithm for solving systems of linear Diophantine equations, inequalities and disequalities in nonnegative integers. As an illustration of the methods of Elliott and MacMahon in the present paper we solve one of the Diophantine transport problems which motivated our project.

The ideas of Elliott and MacMahon have many other applications. Andrews, alone or jointly with Paule, Riese, and Strehl published a series of twelve papers (I – [4], ..., XII – [5]) on MacMahon's Partition Analysis, with numerous applications to different problems, illustrating the power of the methods. The "Ω-Calculus" was further improved by developing better algorithms and effective computer realizations by Andrews, Paule, and Riese [6, 7], and Xin [47]. Other applications were given by Berele [10, 11] (to algebras with polynomial identities), Bedratyuk and Xin [9] (to classical invariant theory), and the authors in a series of papers, also jointly with Benanti, Genov, and Koev (to algebras with polynomial identities, and classical and noncommutative invariant theory), see [12] and the references there.

Let
$$a_{ij}, a_i \in \mathbb{Z}, \quad i = 1, \dots, m, j = 1, \dots, k,$$
be arbitrary integers. Consider the system of Diophantine equations and inequalities

$$(1)\qquad \left| \begin{array}{rcrcrl} a_{11}x_1 & + \cdots + & a_{1k}x_k & + & a_1 & = 0 \\ & \cdots & & & & \\ a_{l1}x_1 & + \cdots + & a_{lk}x_k & + & a_l & = 0 \\ a_{l+1,1}x_1 & + \cdots + & a_{l+1,k}x_k & + & a_{l+1} & \geq 0 \\ & \cdots & & & & \\ a_{m1}x_1 & + \cdots + & a_{mk}x_k & + & a_m & \geq 0. \end{array} \right.$$

Let the set of solutions of the system (1) in nonnegative integers be
$$S = \{s = (s_1, \dots, s_k) \in \mathbb{N}^k \mid s \text{ is a solution of the system}\}.$$
The leitmotif of the paper is to apply a slight modification of the method as presented in the original paper by Elliott [19], to show how to calculate the function
$$(2)\qquad \chi_S(t_1, \dots, t_k) = \sum_{s \in S} t_1^{s_1} \cdots t_k^{s_k},$$
which describes the solutions of the system, and to derive the parametric form of the solutions. We give concrete calculations for the example in [37]. Using similar methods one can handle also the example in [38].

2. The ideas of Euler, Gordan, and Hilbert seen from nowadays. We start with some classical facts on systems of linear Diophantine equations. For historical details and a survey of the methods for solving such systems we refer to the sections on historical and further notes on linear Diophantine equations and on integer linear programming in the book of Schrijver [41] and the section of brief historical notes in the Ph.D. Thesis of Tomás [44].

The following fact was known already by Euler in 1748, see [20, 21, 22].

Lemma 2.1. *Let all coefficients a_{ij} and b_j of the system*

$$\left| \begin{array}{l} a_{11}x_1 + \cdots + a_{1k}x_k = b_1 \\ \cdots \\ a_{m1}x_1 + \cdots + a_{mk}x_k = b_m, \end{array} \right. \tag{3}$$

be nonnegative integers such that for each $j = 1, \dots, k$ at least one of the coefficients a_{ij}, $i = 1, \dots, m$, is different from 0. Then the number of solutions in nonnegative integers of the system is equal to the coefficient of $t_1^{b_1} \cdots t_m^{b_m}$ of the expansion as a power series of the product

$$\prod_{j=1}^{k} \frac{1}{1 - t_1^{a_{1j}} \cdots t_m^{a_{mj}}}. \tag{4}$$

Proof. Using the formula

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots ,$$

we obtain immediately that

$$\prod_{j=1}^{k} \frac{1}{1 - t_1^{a_{1j}} \cdots t_m^{a_{mj}}} = \prod_{j=1}^{k} \sum_{s_j \geq 0} t_1^{a_{1j}s_j} \cdots t_m^{a_{mj}s_j}$$

$$= \sum_{s_j \geq 0} t_1^{\sum_{j=1}^{k} a_{1j}s_j} \cdots t_m^{\sum_{j=1}^{k} a_{mj}s_j} = \sum_{b_i \geq 0} c_b t_1^{b_1} \cdots t_m^{b_m},$$

where the coefficient c_b is equal to the number of k-tuples $(s_1, \dots, s_k) \in \mathbb{N}^k$ such that

$$a_{i1}s_1 + \cdots + a_{ik}s_k = b_i, \quad i = 1, \dots, m,$$

i.e. to the number of solutions of the system (3). $\square$

Remark 2.2. If we replace in Lemma 2.1 the product (4) by

$$\prod_{j=1}^{k} \frac{1}{1 - z_j t_1^{a_{1j}} \cdots t_m^{a_{mj}}},$$

then the coefficient of $t_1^{b_1} \cdots t_k^{b_k}$ will be a polynomial

$$\chi_S(z_1, \dots, z_k) = \sum_{s \in S} z_1^{s_1} \cdots z_k^{s_k}$$

in $z_1, \dots, z_k$, where $S \subset \mathbb{N}^k$ is the set of the solutions $s = (s_1, \dots, s_k)$ of the system (3).

Example 2.3. Given the system

$$\left| \begin{array}{l} x_1 + x_2 + x_3 = 10 \\ x_1 + 2x_2 + 3x_3 = 15 \end{array} \right.$$

we expand the product

$$\frac{1}{(1 - z_1t_1t_2)(1 - z_2t_1t_2^2)(1 - z_3t_1t_2^3)}$$

as a power series and find that the coefficient of $t_1^{10}t_2^{15}$ is equal to $z_1^7z_2z_3^2 + z_1^6z_2^3z_3 + z_1^5z_2^5$. Hence the system has three solutions

$$(s_1, s_2, s_3) = (7, 1, 2), (6, 3, 1), (5, 5, 0).$$

Now we shall consider systems of homogeneous linear Diophantine equations

$$(5)\qquad \left| \begin{array}{llll} a_{11}x_1 & +\cdots+ & a_{1k}x_k & =0 \\ & \cdots & & \\ a_{m1}x_1 & +\cdots+ & a_{mk}x_k & =0, \end{array} \right.$$

where the coefficients a_{ij} are arbitrary integers. As before, we shall be interested in solutions in nonnegative integers. We introduce a partial order on $\mathbb{N}^k$:

$$(6)\qquad q' = (q'_1, \dots, q'_k) \preceq (q''_1, \dots, q''_k) = q'' \text{ if } q'_j \leq q''_j, \quad j = 1, \dots, k.$$

If $q' \prec q''$ are two solutions, then

$$q = q'' - q' = (q_1, \dots, q_k) = (q''_1 - q'_1, \dots, q''_k - q'_k) \in \mathbb{N}^k$$

is also a solution and q'' is a sum of two smaller solutions q and q'. Hence every solution of the system (5) is a sum of *minimal* (or *fundamental*) solutions. In 1873 Gordan [24] called the minimal solutions *irreducible.* He proved that every system (5) has a finite number of minimal solutions. Here we give the proof from the book by Grace and Young [26, Chapter VI, Section 97] which is very close to the original proof of Gordan.

Theorem 2.4. *The system* (5) *has a finite number of minimal solutions.*

Proof. We start with a single equation. Changing the order of the unknowns we rewrite the equation in the form

$$(7)\qquad a_1x_1 + \cdots + a_mx_m = b_1y_1 + \cdots + b_ny_n,$$

where all a_i and b_j are positive integers. The equation has mn solutions

$$x_r = b_s, y_s = a_r$$

and all other variables equal to 0. Now, let us assume that in the solution $(r, s) = (r_1, \dots, r_m, s_1, \dots, s_n)$ one of the coordinates r_i (e.g. r_1) is greater than $b_1 + \cdots + b_n$. Hence

$$b_1s_1 + \cdots + b_ns_n = a_1r_1 + \cdots + a_mr_m \geq a_1r_1 > a_1(b_1 + \cdots + b_m),$$

$$b_1(s_1 - a_1) + \cdots + b_n(s_n - a_1) > 0$$

and there exists an s_j (e.g. s_1) such that $s_j > a_1$. Then

$$(r, s) = (r_1 - b_1, r_2, \dots, r_m, s_1 - a_1, s_2, \dots, s_n) + (b_1, 0, \dots, 0, a_1, 0, \dots, 0)$$

and the solution (r, s) is not minimal. In this way the minimal solutions satisfy the condition

$$r_i \leq b_1 + \cdots + b_n, s_j \leq a_1 + \cdots + a_m.$$

Hence they are a finite number and can be found explicitly. Let $(r^{(1)}, s^{(1)}), \dots, (r^{(p)}, s^{(p)})$ be all minimal solutions of the first equation of the system (5). Then all solutions of the first equation are in the form

$$(8)\qquad q = (r, s) = (r^{(1)}, s^{(1)})z_1 + \cdots + (r^{(p)}, s^{(p)})z_p, \quad z_i \in \mathbb{N}.$$

Replacing (r, s) in the second equation of the system (5) we obtain an equation

$$(9)\qquad c_1z_1 + \cdots + c_pz_p = 0$$

with unknowns $z_1, \dots, z_p$. Then the minimal solutions $q = (q_1, \dots, q_k)$ of the first two equations of (5) are among the solutions (8) obtained from the minimal solutions of the equation (9). Again, we can find them explicitly. Continuing in the same way we can find all minimal solutions of the system (5). □

Remark 2.5. We can find the candidates for the minimal solutions of the equation (7) combining the method of the proof of Theorem 2.4 with the method of Euler from Lemma 2.1 as modified in Remark 2.2. We consider the power series

$$T(t_1,\dots,t_m,z)=\prod_{i=1}^{m}\frac{1}{1-t_iz^{a_i}}=\sum_{x_i\geq 0}t_1^{x_1}\cdots t_m^{x_m}z^{a_1x_1+\cdots+a_mx_m}=\sum_{k\geq 0}T_k(t_1,\dots,t_m)z^k,$$

$$U(u_1,\dots,u_m,z)=\prod_{j=1}^{n}\frac{1}{1-u_jz^{b_j}}=\sum_{y_j\geq 0}u_1^{y_1}\cdots u_n^{y_n}z^{b_1y_1+\cdots+b_ny_n}=\sum_{k\geq 0}U_k(u_1,\dots,u_n)z^k.$$

The polynomials $T_k(t_1,\dots,t_m)$ and $U_k(u_1,\dots,u_n)$, respectively, are sums of monomials of the form $t_1^{x_1}\cdots t_m^{x_m}$ and $u_1^{y_1}\cdots u_n^{y_n}$ and each pair of these monomials gives a solution of (7) of the form

$$a_1x_1+\cdots+a_mx_m=b_1y_1+\cdots+b_ny_n=k.$$

By the proof of Theorem 2.4 the candidates for minimal solutions satisfy the conditions $x_i\leq b_1+\cdots+b_n$ and $y_j\leq a_1+\cdots+a_m$. Hence it is sufficient to compute the polynomials $T_k(t_1,\dots,t_m)$ and $U_k(u_1,\dots,u_n)$ for $k\leq(a_1+\cdots+a_m)(b_1+\cdots+b_n)$.

Example 2.6. Let us consider the system

$$\left|\begin{array}{l} x_1+2x_2-x_3-x_4=0\\ 2x_1+3x_2-2x_3-x_4=0.\end{array}\right. \tag{10}$$

We rewrite the first equation in the form

$$x_1+2x_2=y_3+y_4.$$

By the proof of Theorem 2.4 every minimal solution $q=(r_1,r_2,s_1,s_2)$ satisfies the conditions

$$0<r_1+r_2,\quad r_1,r_2\leq 2,\quad 0<s_1+s_2,\quad s_1,s_2\leq 3.$$

There are 8 possibilities for r_1+r_2 and for each (r_1,r_2) there are $r=r_1+r_2+1$ possibilities $(r,0),(r-1,1),\dots,(0,r)$ for (s_1,s_2). Simple calculations give that there are 35 candidates for minimal solutions. We start with the cases $(r_1,r_2)=(0,1)$ and $(1,0)$ and obtain 5 solutions

$$\begin{gathered} q^{(1)}=(0,1,0,2),\quad q^{(2)}=(0,1,1,1),\quad q^{(3)}=(0,1,2,0),\\ q^{(4)}=(1,0,0,1),\quad q^{(5)}=(1,0,1,0).\end{gathered} \tag{11}$$

If $r_1>1$, then $s_1+s_2>1$ and the solution $q=(r_1,r_2,s_1,s_2)$ is not minimal because $q^{(i)}\prec q$ for some $i=4,5$. By similar argument we derive that the other solutions with $r_1+r_2>1$ are also not minimal. Thus we obtain that the minimal solutions of the first equation of (10) are those in (11) and all solutions of this equation are

$$q=\sum_{i=1}^{5}t_iq^{(i)}=(t_4+t_5,t_1+t_2+t_3,t_2+2t_3+t_5,2t_1+t_2+t_4),\quad t_i\geq 0.$$

The second equation of (10) becomes

$$\begin{aligned} 2x_1+3x_2-2x_3-x_4&=2(t_4+t_5)+3(t_1+t_2+t_3)-2(t_2+2t_3+t_5)-(2t_1+t_2+t_4)\\ &=t_1-t_3+t_4=0.\end{aligned}$$

By the proof of Theorem 2.4 again, $t_1,t_4\leq 1$, $t_3\leq 2$, and the candidates for minimal solutions of the equation $t_1 - t_3 + t_4 = 0$ are

$$(t_1, t_3, t_4) = (1, 1, 0), (0, 1, 1), (1, 2, 1).$$

Only the first two solutions are minimal. We have to add also the minimal solutions

$$(t_1, t_2, t_3, t_4, t_5) = (0, 1, 0, 0, 0), (0, 0, 0, 0, 1)$$

and obtain the candidates for minimal solutions of the original system (10)

$$q = (0, 2, 2, 2), (1, 1, 2, 1), (0, 1, 1, 1), (1, 0, 1, 0).$$

The first two solutions are not minimal and we obtain all minimal solutions of (10)

$$q^{(1)} = (0, 1, 1, 1), q^{(2)} = (1, 0, 1, 0).$$

Example 2.7. We consider the same system (10). Applying the method in Remark 2.5 to the first equation $x_1 + 2x_2 = y_3 + y_4$ of the system, we start with the power series

$$T(t_1, t_2, z) = \frac{1}{(1 - t_1 z)(1 - t_2 z^2)} = \sum_{k \geq 0} T_k(t_1, t_2) z^k,$$

$$U(u_1, u_2, z) = \frac{1}{(1 - u_1 z)(1 - u_2 z)} = \sum_{k \geq 0} U_k(u_1, u_2) z^k$$

and compute the first 6 polynomials T_k and U_k $(k = 1, \ldots, 6)$. For example

$$T_1 = t_1, U_1 = u_1 + u_2, \quad T_2 = t_1^2 + t_2, U_2 = u_1^2 + u_1 u_2 + u_2^2,$$

which gives the pair of monomials

$$(t_1, u_1), (t_1, u_2), (t_1^2, u_1^2), (t_1^2, u_1 u_2), (t_1^2, u_2^2), (t_2, u_1^2), (t_2, u_1 u_2), (t_2, u_2^2)$$

producing the solutions

$$(1, 0, 1, 0), (1, 0, 0, 1), (2, 0, 2, 0), (2, 0, 1, 1), (2, 0, 0, 2), (0, 1, 2, 0), (0, 1, 1, 1), (0, 1, 0, 2).$$

Then by direct verification we select the minimal solutions of the equation and continue with the second equation of the system (10).

By the Hilbert Basis theorem [27] every ideal of the polynomial algebra $\mathbb{Q}[x_1, \ldots, x_k]$ is finitely generated. The following property of the partial order (6) is known as the *Dickson lemma* with easy proof by induction in [16]. As it is mentioned in [16] it is a direct consequence of the Hilbert Basis theorem applied to monomial ideals in $\mathbb{Q}[x_1, \ldots, x_k]$.

Lemma 2.8. *Let J be a subset of $\mathbb{N}^k$. Then J has a finite subset*

$$\{q^{(i)} = (q_1^{(i)}, \ldots, q_k^{(i)}) \mid i = 1, \ldots, n\}$$

with the property that for any $q = (q_1, \ldots, q_k) \in J$ there exists a $q^{(i)}$ such that $q^{(i)} \preceq q$.

We should mention that this lemma was used by Gordon [25] in 1899 in his proof of the Hilbert Basis theorem. Clearly, as a corollary we immediately obtain also a nonconstructive proof of Theorem 2.4.

It is interesting to know the behavior of the minimal solutions of the system (5). It can be given in terms of recursion theory.

Definition 2.9. *Let $f : \mathbb{N} \to \mathbb{N}$ be an arbitrary function. A function $h : \mathbb{N}^k \to \mathbb{N}$ is* primitive recursive in f *if h can be obtained by a finite number of steps applying the following rules, starting with the function f, the constant function 0, the successor function $s : \mathbb{N} \to \mathbb{N}$ (defined by $s(n) = n + 1$, $n \in \mathbb{N}$) and the projection function $p_i^k : \mathbb{N}^k \to \mathbb{N}$, $i = 1, \ldots, k$ (defined by $p_i^k(n_1, \ldots, n_k) = n_i$, $(n_1, \ldots, n_k) \in \mathbb{N}^k$):*

(1) *The functions f, 0, s and p_i^k are primitive recursive in f;*

(2) Substitution: *If* $g : \mathbb{N}^k \to \mathbb{N}$ *and* $h_i : \mathbb{N}^m \to \mathbb{N}$*,* $m = 1, \dots, k$*, are primitive recursive in* f*, then the function* $g(h_1, \dots, h_k) : \mathbb{N}^m \to \mathbb{N}$ *is also primitive recursive in* f*;*

(3) Primitive recursion: *If* $g : \mathbb{N}^k \to \mathbb{N}$ *and* $h : \mathbb{N}^{k+2} \to \mathbb{N}$*, are primitive recursive in* f*, then the primitive recursion* $p : \mathbb{N}^{k+1} \to \mathbb{N}$ *of* g *and* h *defined by*

$$p(0, n_1, \dots, n_k) = g(n_1, \dots, n_k) \text{ and } p(s(m), n_1, \dots, n_k) = h(m, p(m, n_1, \dots, n_k)),$$

$m \in \mathbb{N}$*,* $(n_1, \dots, n_k) \in \mathbb{N}^k$*, is also primitive recursive in* f*.*

In the above definition, the "ordinary" primitive recursive functions are those which do not depend on the function f. Roughly speaking, from the point of view of computability theory, a primitive recursive function can be computed by a computer program such that for every loop in the program the number of iterations can be bounded from above before entering the loop.

Definition 2.10. *The function* $g : \mathbb{N}^k \to \mathbb{N}$ *is recursive if in addition to the constructions in the definition of a primitive recursive function one uses also the following.*

(4) Minimization operator μ*: If* $h(m, n_1, \dots, n_k) : \mathbb{N}^{k+1} \to \mathbb{N}$ *is partially defined (i.e. defined on a subset of* $\mathbb{N}^{k+1}$*), then the function* $\mu(h) : \mathbb{N}^k \to \mathbb{N}$ *is defined by* $\mu(h)(n_1, \dots, n_k) = m$ *if* $h(i, n_1, \dots, n_k) > 0$ *for* $i = 0, 1, \dots, m-1$ *and* $h(m, n_1, \dots, n_k) = 0$*. If* $h(i, n_1, \dots, n_k) > 0$ *for all* $i \in \mathbb{N}$ *or if* $h(i, n_1, \dots, n_k)$ *is not defined before reaching some* m *with* $h(m, n_1, \dots, n_k) = 0$*, then the search for* m *never terminates, and* $\mu(h)(n_1, \dots, n_k)$ *is not defined for the argument* $(n_1, \dots, n_k)$*.*

We shall restate the formalization of Seidenberg [40] introduced originally for the ideals of the polynomial algebra $\mathbb{Q}[x_1, \dots, x_k]$.

Problem 2.11. *Given a function* $f : \mathbb{N} \to \mathbb{N}$*, what is the maximal* $p_f(i) \in \mathbb{N}$ *with the property: There exists a set*

$$I_f = \{q^{(i)} = (q_1^{(i)}, \dots, q_k^{(i)}) \mid i = 1, \dots, p_f(i)\} \subset \mathbb{N}^k$$

such that $q_1^{(i)} + \dots + q_k^{(i)} \leq f(i)$ *and the elements of* I_f *are not comparable with respect to the partial order* $\prec$*.*

Seidenberg showed that there exists a bound $p_f^{(k)}$ depending on f and k only, which is recursive in f for a fixed k. This result was improved by Moreno-Socías [34].

Theorem 2.12. *In the notation of Problem* 2.11 *for every* k *there is a primitive recursive function* $p_f^{(k)} : \mathbb{N} \to \mathbb{N}$ *in* f*, but there is no bound* p_f *which is primitive recursive in* f *in general.*

For each $d \in \mathbb{N} \setminus \{0\}$ Moreno-Socías [33] constructed an example for the primitive recursive function $f_d(n) = d + n$, $n \in \mathbb{N}$, with the property that the bound p_{f_d} is expressed in terms of the Ackermann function $a(k, n) : \mathbb{N}^2 \to \mathbb{N}$ [2] defined by

$$a(0, n) = n + 1, \quad a(k + 1, 0) = a(k, 1), \quad a(k + 1, n + 1) = a(k, a(k + 1, n)).$$

It is known that $a(k, n)$ is recursive and grows faster than any primitive recursive function.

Theorem 2.13. *In the notation of Problem* 2.11 *let* $d \in \mathbb{N} \setminus \{0\}$ *and let* $f_d(n) = d + n$*,* $n \in \mathbb{N}$*. Then there exists a set*

$$I_{f_d} = \{q^{(i)} = (q_1^{(i)}, \dots, q_k^{(i)}) \mid i = d, d + 1, \dots, p\} \subset \mathbb{N}^k$$

of noncomparable k*-tuples such that* $q_1^{(i)} + \dots + q_k^{(i)} = i$ *and* p *is equal to* $a(k, d - 1) - 1$*, where* $a(k, n)$ *is the Ackermann function.*

See also the paper by Aschenbrenner and Pong [8] for another approach to the complexity of the problems discussed above.

3. From the homogeneous case to the solution of general linear Diophantine constraints. Let us consider the general system of linear Diophantine equations and inequalities (1). There is a standard way to bring the solution of (1) to the solution of a homogeneous system of linear Diophantine equations. We introduce new unknowns $y, y_{l+1}, \dots, y_m$ and replace the system (1) by the system

$$(12)\qquad \left| \begin{array}{lllllll} a_{11}x_1 & + \cdots + & a_{1k}x_k & + & a_1 y & & = 0 \\ \cdots & & & & & & \\ a_{l1}x_1 & + \cdots + & a_{lk}x_k & + & a_l y & & = 0 \\ a_{l+1,1}x_1 & + \cdots + & a_{l+1,k}x_k & + & a_{l+1}y & - \; y_{l+1} & = 0 \\ \cdots & & & & & & \\ a_{m1}x_1 & + \cdots + & a_{mk}x_k & + & a_m y & - \; y_m & = 0. \end{array} \right.$$

The following easy theorem shows how to reduce the solution of a general system to a homogeneous one.

Theorem 3.1. *Let*

$$(13)\qquad \{q^{(i)} = (r_1^{(i)}, \dots, r_k^{(i)}, s^{(i)}, s_{l+1}^{(i)}, \dots, s_m^{(i)}) \mid i = 1, \dots, n\} \subset \mathbb{N}^{k+1+m-l}$$

be the set of minimal solutions of the homogeneous system of equations (12) *and let* $s^{(i)} = 1$ *for* $i = 1, \dots, c$, $s^{(i)} = 0$ *for* $i = c+1, \dots, d$ *and* $s^{(i)} > 1$ *for* $i = d+1, \dots, n$. *Then the set of all solutions of the system* (1) *are of the form*

$$q = q^{(i)} + t_{c+1}q^{(i)} + \cdots + t_d q^{(d)}, \quad i = 1, \dots, c, \quad t_{c+1}, \dots, t_d \in \mathbb{N}.$$

Proof. If $q = (r_1, \dots, r_k, s, s_{l+1}, \dots, s_m)$ is a solution of (12), then $s_{l+1}, \dots, s_m \geq 0$ and the solutions $(r_1, \dots, r_k)$ of (1) are obtained from solutions q with $s = 1$. Every q is a linear combination of the minimal solutions from (13). If q has the form $q = \sum_{i=1}^{n} t_i q^{(i)}$, we obtain that $1 = s = t_1 + \cdots + t_c + t_{d+1}s^{(d+1)} + \cdots + t_n s^{(n)}$. Hence $t_1 + \cdots + t_c = 1$ and $t_{d+1} = \cdots = t_n = 0$ because $t_{d+1}, \dots, t_n > 1$. $\square$

Remark 3.2. Let some of the inequalities in the system (1), e.g.

$$a_{m1}x_1 + \cdots + a_{mk}x_k + a_m > 0$$

be strict. Then we replace it in the system (12) by

$$a_{m1}x_1 + \cdots + a_{mk}x_k + (a_m + 1)y - y_m = 0.$$

Example 3.3. Let us modify the system (10) from Example 2.6 into the system

$$(14)\qquad \left| \begin{array}{l} x_1 + 2x_2 - x_3 - 1 = 0 \\ 2x_1 + 3x_2 - 2x_3 - 1 \geq 0. \end{array} \right.$$

Following the proof of Theorem 3.1 we have to consider the system

$$(15)\qquad \left| \begin{array}{l} x_1 + 2x_2 - x_3 - y = 0 \\ 2x_1 + 3x_2 - 2x_3 - y - y_2 = 0. \end{array} \right.$$

The minimal solutions $q = (r_1, r_2, r_3, s)$ of the first equation of (15) are the same as the minimal solutions (11) of the first equation of (10). Since $s = 2 > 1$ in the solution $q^{(1)} = (0, 1, 0, 2)$, $s = 1$ in $q^{(2)}$ and $q^{(4)}$, and $s = 0$ in $q^{(3)}$ and $q^{(5)}$, we obtain that all solutions are

$$q' = q^{(2)} + t_3 q^{(3)} + t_5 q^{(5)} = (t_5, 1 + t_3, 1 + 2t_3 + t_5, 1),$$

$$q'' = q^{(4)} + t_3 q^{(3)} + t_5 q^{(5)} = (1 + t_5, t_3, 2t_3 + t_5, 1),$$

$t_3, t_5 \in \mathbb{N}$. Substituting q' in the second equation of (15) we obtain

$$2t_5 + 3(1 + t_3) - 2(1 + 2t_3 + t_5) - 1 - y_2 = -t_3 - y_2 = 0.$$

Hence $t_3 = y_2 = 0$ and we obtain the solutions $(r_1, r_2, r_3) = (1 + t_5, 0, t_5)$ of (14). Similarly, starting with q'', the second equation of (15) gives

$$2(1 + t_5) + 3t_3 - 2(2t_3 + t_5) - 1 - y_2 = 1 - t_3 - y_2 = 0.$$

We obtain two solutions $(t_3, y_2) = (0, 1)$ and $(t_3, y_2) = (1, 0)$, which give the solutions $(r_1, r_2, r_3) = (1 + t_5, 0, t_5)$ and $(r_1, r_2, r_3) = (1 + t_5, 1, 2 + t_5)$ of (14).

There are many methods based on different ideas for solving systems of linear Diophantine equations and inequalities. See for example [1, 3, 14, 15, 17, 23, 31, 35, 36, 45, 46] and the bibliography there. See also [28, 29] for relations with mathematical logic and the theory of formal grammars.

4. The method of Elliott. In this section we shall explain in detail the method of Elliott from [19]. Originally, it was developed for systems of homogeneous linear Diophantine equations. But for our applications we shall restate the method for systems of linear Diophantine inequalities. We fix arbitrary integers

$$a_{ij}, a_i \in \mathbb{Z}, \quad i = 1, \dots, m, j = 1, \dots, k,$$

a system of Diophantine inequalities

$$(16) \qquad \left| \begin{array}{c} a_{11}x_1 + \cdots + a_{1k}x_k + a_1 \geq 0 \\ \cdots \\ a_{m1}x_1 + \cdots + a_{mk}x_k + a_m \geq 0, \end{array} \right.$$

and consider the set of solutions of the system (16) in nonnegative integers

$$S = \{s = (p_1, \dots, p_k) \in \mathbb{N}^k \mid s \text{ is a solution of the system}\}.$$

4.1. The first idea of Elliott. A usual way to describe a set $A \subset \mathbb{R}^k$ is in terms of its *characteristic* (or *indicator*) *function* $\mathrm{ch}_{f_A} : \mathbb{R}^k \to \{0, 1\}$ defined by

$$\mathrm{ch}_{f_A}(p_1, \dots, p_k) = \begin{cases} 1, (p_1, \dots, p_k) \in A \\ 0, (p_1, \dots, p_k) \notin A. \end{cases}$$

Definition 4.1. *Let $P \subset \mathbb{N}^k$. By analogy with the characteristic function of P we call the formal power series*

$$\chi_P(t_1, \dots, t_k) = \sum_{p \in P} t_1^{p_1} \cdots t_k^{p_k}, \quad p = (p_1, \dots, p_k),$$

the characteristic series *of P.*

When $P \subset \mathbb{N}^k$ is the set of solutions of a system of linear Diophantine equations Elliott suggests to call $\chi_P(t_1, \dots, t_k)$ the *generating function* of the set of solutions.

4.2. The second idea of Elliott. To find the characteristic series of a set of solutions $S \subset \mathbb{N}^k$ of a system of homogeneous linear Diophantine equations Elliott involves Laurent series.

Definition 4.2. *Let $P \subset \mathbb{N}^k$, let*

$$f(t_1, \dots, t_k) = \sum_{p \in P} \alpha_p t_1^{p_1} \cdots t_k^{p_k} \in \mathbb{C}[[t_1, \dots, t_k]], \quad p = (p_1, \dots, p_k), \alpha_p \in \mathbb{C},$$

be a formal power series, and let S be a subset of P. We call the formal power series

$$\chi_S(f; t_1, \dots, t_k) = \sum_{p \in S} \alpha_p t_1^{p_1} \cdots t_k^{p_k},$$

the characteristic series of f with respect to the set S.

The next lemma is one of the key moments in the approach of Elliott. Its proof is obvious.

Lemma 4.3. *Let $P \subset \mathbb{N}^k$, let*

$$f(t_1, \dots, t_k) = \sum_{p \in P} \alpha_p t_1^{p_1} \cdots t_k^{p_k} \in \mathbb{C}[[t_1, \dots, t_k]], \quad p = (p_1, \dots, p_k), \alpha_p \in \mathbb{C},$$

be a formal power series, and let $S \subset P$ be the set of solutions in P of the Diophantine equation

$$a_1 x_1 + \cdots + a_k x_k = 0, \quad a_i \in \mathbb{Z}.$$

If the Laurent series

$$\xi_S(t_1, \dots, t_k, z) = f(t_1 z^{a_1}, \dots, t_k z^{a_k}) = \sum_{n=-\infty}^{\infty} \sum_{p \in S} \alpha_p t_1^{p_1} \cdots t_k^{p_k} z^n,$$

$p = (p_1, \dots, p_k)$, $n = a_1 p_1 + \cdots + a_k p_k$, has the form

$$\xi_S(t_1, \dots, t_k, z) = \sum_{n=-\infty}^{\infty} f_n(t_1, \dots, t_k) z^n, \quad f_n(t_1, \dots, t_k) \in \mathbb{C}[[t_1, \dots, t_k]],$$

then

$$\chi_S(f; t_1, \dots, t_k) = \sum_{p \in S} \alpha_p t_1^{p_1} \cdots t_k^{p_k} = f_0(t_1, \dots, t_k).$$

The next lemma is a slight generalization of the original approach of Elliott. Its proof is also obvious.

Lemma 4.4. *Let $P \subset \mathbb{N}^k$, let*

$$f(t_1, \dots, t_k) = \sum_{p \in P} \alpha_p t_1^{p_1} \cdots t_k^{p_k} \in \mathbb{C}[[t_1, \dots, t_k]], \quad p = (p_1, \dots, p_k), \alpha_p \in \mathbb{C},$$

be a formal power series, and let S be the solutions in P of the Diophantine inequality

$$a_1 x_1 + \cdots + a_k x_k + a \geq 0, \quad a_i, a \in \mathbb{Z}.$$

If

$$\xi_S(t_1, \dots, t_k, z) = z^a f(t_1 z^{a_1}, \dots, t_k z^{a_k})$$

$$= \sum_{n=-\infty}^{\infty} \sum_{p \in S} \alpha_p t_1^{p_1} \cdots t_k^{p_k} z^n = \sum_{n=-\infty}^{\infty} f_n(t_1, \dots, t_k) z^n,$$

$f_n(t_1, \dots, t_k) \in \mathbb{C}[[t_1, \dots, t_k]]$, $n = a_1 p_1 + \cdots + a_k p_k + a$, then

$$\chi_S(f; t_1, \dots, t_k) = \sum_{(p_1, \dots, p_k) \in S} \alpha_p t_1^{p_1} \cdots t_k^{p_k} = \sum_{n=0}^{\infty} f_n(t_1, \dots, t_k).$$

Now the problem is how to find $\chi_S(f; t_1, \dots, t_k)$ if $f(t_1, \dots, t_k)$ is an explicitly given power series which converges to a rational function and we know the set S. Even in simple cases the answer may be not trivial.

Example 4.5. Combining the results from [18, p. 409] with ideas from [12], for the formal power series

$$f(t_1,t_2)=\frac{t_1-t_2}{1-(t_1+t_2)}=(t_1-t_2)\sum_{n=0}^{\infty}(t_1+t_2)^n,\quad S=\{(p_1,p_2)\in\mathbb{N}^2\mid p_1\geq p_2\},$$

we obtain

$$\chi_S(f;t_1,t_2)=\frac{1-\sqrt{1-4t_1t_2}}{2t_2-(1-\sqrt{1-4t_1t_2})}.$$

4.3. The third idea of Elliott. The following definition is given by Berele [10].

Definition 4.6. *A* nice rational function *is a rational function with denominator which is a product of monomials of the form* $(1-t_1^{\alpha_1}\cdots t_k^{\alpha_k})$.

Nice rational functions appear in many places of mathematics. For example, the Hilbert series of any finitely generated multigraded commutative algebra is of this form. The following theorem was proved by Elliott [19]. The proof also gives an algorithm how to find the characteristic series $\chi_S(t_1,\ldots,t_k)$ of the set S of solutions.

Theorem 4.7. *Let*

$$(17)\qquad \left|\begin{array}{c} a_{11}x_1+\cdots+a_{1k}x_k=0\\ \cdots\\ a_{m1}x_1+\cdots+a_{mk}x_k=0, \end{array}\right.$$

where $a_{ij}\in\mathbb{Z}$, $i=1,\ldots,m$, $j=1,\ldots,k$, *be a system of homogenous linear Diophantine equations. Then the characteristic series* $\chi_S(t_1,\ldots,t_k)$ *of the set* S *of the solutions of* (17) *in* $\mathbb{N}^k$ *is a nice rational function.*

Proof. We start with the characteristic series of the set $\mathbb{N}^k$ of all points with nonnegative integers coordinates

$$\chi_{\mathbb{N}^k}(t_1,\ldots,t_k)=\prod_{i=1}^{k}\frac{1}{1-t_i}=\sum_{p_i\geq 0}t_1^{p_1}\cdots t_k^{p_k}.$$

Let the first equation of the system be of the form

$$(18)\qquad a_1x_1+\cdots+a_dx_d-c_{e+1}x_{e+1}-\cdots-c_kx_k=0,\quad a_i>0,c_j>0,d\leq e.$$

Then the Laurent series

$$\xi_S(t_1,\ldots,t_k,z)=\chi_{\mathbb{N}^k}(t_1z^{a_1},\ldots,t_dz^{a_d},t_{d+1},\ldots,t_e,t_{e+1}z^{-c_{e+1}},\ldots,t_kz^{-c_k})$$

from Lemma 4.3 has the form

$$\xi_S(t_1,\ldots,t_k,z)=\prod_{i=1}^{d}\prod_{j=d+1}^{e}\prod_{m=e+1}^{k}\frac{1}{(1-t_iz^{a_i})(1-t_j)(1-t_mz^{-c_m})}.$$

More general, we shall assume that $\xi_S(t_1,\ldots,t_k,z)$ is of the form

$$(19)\qquad \xi_S(t_1,\ldots,t_k,z)=\prod_{i=1}^{d}\prod_{j=d+1}^{e}\prod_{m=e+1}^{k}\frac{1}{(1-A_iz^{a_i})(1-B_j)(1-C_mz^{-c_m})},$$

where A_i,B_j,C_m are monomials in $t_1,\ldots,t_k$.

Case 1. If the equation (18) does not contain negative c_m, i.e. if $e=k$, then obviously

$$\xi_S(t_1,\dots,t_k,z)=\prod_{i=1}^{d}\prod_{j=d+1}^{k}\frac{1}{(1-A_iz^{a_i})(1-B_j)}$$

$$=\prod_{j=d+1}^{k}\frac{1}{1-B_j}\left(1+\sum_{n\geq 1}D_nz^n\right),\quad D_n\in\mathbb{Q}[[t_1,\dots,t_k]],$$

and

$$\chi_S(t_1,\dots,t_k)=f_0(t_1,\dots,t_k)=\prod_{j=d+1}^{k}\frac{1}{1-B_j}.$$

Case 2. Similar arguments work when the equation (18) does not contain positive a_i, i.e. when $d=0$. Again

$$\chi_S(t_1,\dots,t_k)=f_0(t_1,\dots,t_k)=\prod_{j=1}^{e}\frac{1}{1-B_j}.$$

Case 3. Now, let (18) contain both positive a_i and negative c_m. We shall use the Elliott tricky equality [19, equation (4)]

$$\frac{1}{(1-Az^a)(1-Cz^{-c})}=\frac{1}{1-ACz^{a-c}}\left(\frac{1}{1-Az^p}+\frac{1}{1-Cz^{-c}}-1\right),\tag{20}$$

where $a,c\in\mathbb{N}$ and A,C are again monomials in $t_1,\dots,t_k$. Applying (20) to a pair of a_i and c_m, we shall replace the product (19) by a sum of three similar products with numerators ± 1. Continuing with the application of (20) to each of the three expressions in several steps we shall obtain a sum of products with denominators containing factors which do not depend on z and factors with only positive or only negative degrees of z. Then in order to obtain the expression of $f_0(t_1,\dots,t_k)$ we handle each summand as in Case 1 and Case 2 of the proof. In this way we compute the characteristic series of the set of the solutions of the first equation of (17). Applying the same algorithm on $f_0(t_1,\dots,t_k)$ we obtain the characteristic series of the solutions of the first two equations of (17) and continue the process until we find the characteristic series $\chi_S(t_1,\dots,t_k)$ of the set S of the solutions of the whole system (17). Below we shall explain why the process stops in a finite number of steps. □

The product (19) has $d+(k-e)$ factors depending on z. The original arguments of Elliott are the following. If $a=c$, then, applying (20), in any of the three summands the number of factors depending on z is smaller. If $a>c$, then after applying (20), one of the summands (the third one) has fewer number of factors depending on z. For the first of the other two factors we replace the factor $(1-Az^a)(1-Cz^{-c})$ with negative degree of z in $1-Cz^{-c}$ by the factor $(1-ACz^{a-c})(1-Az^a)$. Since $a-c$ is between $-c$ and a, this expression is simpler than the original. For the other factor $(1-ACz^{a-c})(1-Cz^{-c})$ again $a-c$ is between $-c$ and a and the expression is simpler than the original, too. Similar arguments can be applied for the case $a<c$.

We shall formalize the arguments of Elliott following the Master Thesis [30] of the third named author of this paper. Given two sequences of nonnegative integers

$$\alpha=(a_1,\dots,a_d)\text{ and }\gamma=(c_{e+1},\dots,c_k),$$

we denote by $\theta = [\alpha, \gamma]$ the corresponding pair of partitions

$$[\alpha] = (a_{i_1}, \dots, a_{i_d}), \quad a_{i_1} \geq \dots \geq a_{i_d}, \quad [\gamma] = (c_{m_1}, \dots, c_{m_{k-e}}), \quad c_{m_1} \geq \dots \geq c_{m_{k-e}}.$$

Then we define the linear order

$$\theta = [\alpha, \gamma] \prec [\alpha', \gamma'] = \theta', \text{ if } [\alpha] < [\alpha'] \text{ or } [\alpha] = [\alpha'], [\gamma] < [\gamma'],$$

where the order $<$ in $[\alpha] < [\alpha']$ and $[\gamma] < [\gamma']$ is with respect to the usual lexicographic order. Obviously the order $\prec$ satisfies the descending chain condition.

Proposition 4.8. *The algorithm of Elliott in the proof of Theorem* 4.7 *stops after a finite number of steps.*

Proof. Applying (20) to the product $\xi = \xi_S(t_1, \dots, t_k, z)$ from (19) we shall follow the behavior of the two sequences $\alpha = (a_1, \dots, a_d)$ and $\gamma = (c_{e+1}, \dots, c_k)$ of the degrees of z and the corresponding pair of partitions $\theta = [\alpha, \gamma]$. Since the set of all finite integer sequences is well ordered with respect to $\prec$, it is sufficient to show that the statement holds for the pair θ if it holds for all pairs φ which are smaller with respect to $\prec$, and then to apply inductive arguments. Without loss of generality we may assume that $a_1 \geq \dots \geq a_d$ and $c_{e+1} \geq \dots \geq c_k$. By virtues of (20) we replace the product (19) corresponding to $\theta = [\alpha, \gamma]$ with three products

$$\xi' = \frac{1}{1 - A_1 C_1 z^{a_1 - c_{e+1}}} \prod_{i=1}^{d} \prod_{j=d+1}^{e} \prod_{m=e+2}^{k} \frac{1}{(1 - A_i z^{a_i})(1 - B_j)(1 - C_m z^{-c_m})},$$

$$\xi'' = \frac{1}{1 - A_1 C_1 z^{a_1 - c_{e+1}}} \prod_{i=2}^{d} \prod_{j=d+1}^{e} \prod_{m=e+1}^{k} \frac{1}{(1 - A_i z^{a_i})(1 - B_j)(1 - C_m z^{-c_m})},$$

$$\xi''' = \frac{1}{1 - A_1 C_1 z^{a_1 - c_{e+1}}} \prod_{i=2}^{d} \prod_{j=d+1}^{e} \prod_{m=e+2}^{k} \frac{1}{(1 - A_i z^{a_i})(1 - B_j)(1 - C_m z^{-c_m})},$$

corresponding to the pairs $\theta' = [\alpha', \gamma']$, $\theta'' = [\alpha'', \gamma'']$, $\theta''' = [\alpha''', \gamma''']$, respectively.

Case 1. Let $a_1 = c_1$. Then

$$\theta' = [(a_1 - c_{e+1} = 0, a_1, a_2, \dots, a_d), (c_{e+2}, \dots, c_k)],$$

$$\theta'' = [(a_2, \dots, a_d), (c_{e+1}, c_{e+2}, \dots, c_k)],$$

$$\theta''' = [(a_2, \dots, a_d), (c_{e+2}, \dots, c_k)].$$

Since $[c_{e+2}, \dots, c_k] < [c_{e+1}, c_{e+2}, \dots, c_k]$ and $[a_2, \dots, \dots, a_d] < [a_1, a_2, \dots, \dots, a_d]$ we obtain that $\theta', \theta'', \theta''' \prec \theta$ and we can apply inductive arguments.

Case 2. Let $a_1 > c_{e+1}$. Then $\theta', \theta'', \theta'''$ are, respectively,

$$\theta' = [\alpha', \gamma'] = [(a_1 - c_{e+1}, a_1, a_2, \dots, a_d), (c_{e+2}, \dots, c_k)],$$

$$\theta'' = [\alpha'', \gamma''] = [(a_1 - c_{e+1}, a_2, \dots, a_d), (c_{e+1}, c_{e+2}, \dots, c_k)],$$

$$\theta''' = [\alpha''', \gamma'''] = [(a_1 - c_{e+1}, a_2, \dots, a_d), (c_{e+2}, \dots, c_k)].$$

Since $[\alpha''] = [\alpha'''] < [\alpha]$ we have that $\theta'', \theta''' \prec \theta$ and we can apply inductive arguments for them. But we have that $[\alpha'] > [\alpha]$ and $\theta' \succ \theta$. Applying (20) to θ' we obtain three pairs of partitions:

$$(\theta')' = [(\alpha')', (\gamma')'] = [(a_1 - c_{e+1}, a_1 - c_{e+2}, a_1, a_2, \dots, a_d), (c_{e+3}, \dots, c_k)],$$

$$(\theta')'' = [(\alpha')'', (\gamma')''] = [(a_1 - c_{e+1}, a_1 - c_{e+2}, a_2, \dots, a_d), (c_{e+2}, c_{e+3}, \dots, c_k)],$$

$$(\theta')''' = [(\alpha')''', (\gamma')'''] = [(a_1 - c_{e+1}, a_1 - c_{e+2}, a_2, \dots, a_d), (c_{e+3}, \dots, c_k)].$$

Let us assume that

$$[\alpha] = (\underbrace{a_1, \dots, a_1}_{r \text{ times}}, a_{r+1}, \dots, a_d), \quad a_1 > a_{r+1} \geq \dots \geq a_d.$$

Then

$$[(\alpha')''] = [(\alpha')'''] = [\underbrace{a_1, \dots, a_1}_{r-1 \text{ times}}, a_1 - c_{e+1}, a_1 - c_{e+2}, a_{r+1}, \dots, a_d] < [\alpha]$$

and again $(\theta')'', (\theta')''' \prec \theta$. Hence we have a problem with $(\theta')'$ only.

The application of (20) to $(\theta')'$ gives three pairs $((\theta')')', ((\theta')')'', ((\theta')')'''$. The latter two, $((\theta')')''$ and $((\theta')')'''$, are smaller than θ and

$$((\theta')')' = [(a_1, \dots, a_d, a_1 - c_{e+1}, a_1 - c_{e+2}, a_1 - c_{e+3}), (c_{e+4}, \dots, c_k)] \succ \theta.$$

Continuing to apply (20), we obtain in each step pairs of partitions which are smaller than θ, and the pairs

$$[(a_1, \dots, a_d, a_1 - c_{e+1}, \dots, a_1 - c_{e+j}), (c_{e+j+1}, \dots, c_k)] \succ \theta.$$

Finally, we shall reach the pair

$$[(a_1, \dots, a_d, a_1 - c_{e+1}, \dots, a_1 - c_k), (0)]$$

corresponding to the product

$$\prod_{i=1}^{d} \frac{1}{1 - A_i z^{a_i}} \prod_{m=e+1}^{k} \frac{1}{1 - A_1 C_m z^{a_1 - c_m}} \prod_{j=d+1}^{e} \frac{1}{1 - B_j},$$

which we can handle as in Case 1 of Theorem 4.7.

Case 3. Let $a_1 < c_{e+1}$. Applying (20) to θ gives pairs of partitions

$$\begin{aligned}
\theta' &= [\alpha', \gamma'] = [(a_1, a_2, \dots, a_d), (c_{e+1} - a_1, c_{e+2}, \dots, c_k)], \\
\theta'' &= [\alpha'', \gamma''] = [(a_2, \dots, a_d), (c_{e+1} - a_1, c_{e+1}, c_{e+2}, \dots, c_k)], \\
\theta''' &= [\alpha''', \gamma'''] = [(a_2, \dots, a_d), (c_{e+1} - a_1, c_{e+2}, \dots, c_k)]
\end{aligned}$$

with $\theta', \theta''' \prec \theta$. As in Case 2, we replace θ'' by a sequence

$$[(a_{j+1}, \dots, a_d), (c_{e+1} - a_1, \dots, c_{e+1} - a_j, c_{e+1}, c_{e+2}, \dots, c_k)], \quad j = 2, \dots, d,$$

until we obtain $[(0), (c_{e+1} - a_1, \dots, c_{e+1} - a_d, c_{e+1}, c_{e+2}, \dots, c_k)]$ and then handle the corresponding product as in Case 2 of Theorem 4.7. $\square$

The following theorem is a modification of Theorem 4.7 for systems of Diophantine inequalities.

Theorem 4.9. *Let $a_{ij}, a_i \in \mathbb{Z}$, $i = 1, \dots, m$, $j = 1, \dots, k$, be arbitrary integers. Then the characteristic series $\chi_S(t_1, \dots, t_k)$ of the set S of the solutions in nonnegative integers of the system of Diophantine inequalities*

$$\left| \begin{array}{c} a_{11}x_1 + \dots + a_{1k}x_k + a_1 \geq 0 \\ \dots \\ a_{m1}x_1 + \dots + a_{mk}x_k + a_m \geq 0 \end{array} \right.$$

is a nice rational function.

Proof. We repeat the arguments from the proof of Theorem 4.7 using Lemma 4.4

instead of Lemma 4.3. Consider the linear Diophantine inequality

$$a_1x_1 + \cdots + a_kx_k + a \geq 0, \quad a_i, a \in \mathbb{Z},$$

and the nice rational function

$$f(t_1, \ldots, t_k) = t_1^{r_1} \cdots t_k^{r_k} \prod \frac{1}{1 - t_1^{q_{i1}} \cdots t_k^{q_{ik}}}$$

with a set S of solutions in nonnegative integers. By Lemma 4.4 we have to compute the component

$$\chi_S(t_1, \ldots, t_k) = \sum_{n=0}^{\infty} f_n(t_1, \ldots, t_k)$$

of the Laurent series

$$\xi_S(t_1, \ldots, t_k, z) = z^a f(t_1z^{a_1}, \ldots, t_kz^{a_k}) = \sum_{n=-\infty}^{\infty} f_n(t_1, \ldots, t_k)z^n.$$

Applying the algorithm of Elliott to the part $\prod \frac{1}{1 - D_iz^{d_i}}$ of

$$z^a f(t_1z^{a_1}, \ldots, t_kz^{a_k}) = z^b B \prod \frac{1}{1 - D_iz^{d_i}},$$

where B and D_i are monomials in $t_1, \ldots, t_k$, we present $z^a f(t_1z^{a_1}, \ldots, t_kz^{a_k})$ as a sum of products of the form

$$\xi^+ = z^d E \prod \frac{1}{1 - G_i} \prod_{h_j>0} \frac{1}{1 - H_jz^{h_j}} \text{ and } \xi^- = z^d E \prod \frac{1}{1 - G_i} \prod_{h_j<0} \frac{1}{1 - H_jz^{h_j}},$$

where again E, G_i, H_j are monomials in $t_1, \ldots, t_k$. In order to complete the proof we have to determine the contribution of each summand ξ^+ and ξ^- to $\chi_S(t_1, \ldots, t_k)$.

Case 1. The exponent d of z in ξ^+ satisfies $d \geq 0$. Then the whole ξ^+ contributes to $\chi_S(t_1, \ldots, t_k)$.

Case 2. The exponent d of z in ξ^- satisfies $d \leq 0$. If $d < 0$, then ξ^- does not contribute to $\chi_S(t_1, \ldots, t_k)$ because its expansion as a Laurent series contains only negative degrees of z. If $d = 0$, then the contribution of ξ^- is $E \prod \frac{1}{1 - G_i}$.

Case 3. The exponent d of z in ξ^+ satisfies $d < 0$. We expand $\prod_{h_j>0} \frac{1}{1 - H_jz^{h_j}}$ as

$$\prod_{h_j>0} \frac{1}{1 - H_jz^{h_j}} = 1 + K_1z + K_2z^2 + \cdots + K_{d-1}z^{d-1} + z^d L(z),$$

where $L(z) \in \mathbb{C}[[t_1, \ldots, t_k, z]]$. Then

$$\xi^+ = E \prod \frac{1}{1 - G_i}(z^{-d} + K_1z^{-d+1} + K_2z^{-d+2} + \cdots + K_{d-1}z^{-1}) + E \prod \frac{1}{1 - G_i} L(z)$$

and the contribution of ξ^+ to $\chi_S(t_1, \ldots, t_k)$ is $E \prod \frac{1}{1 - G_i} L(z)$.

Case 4. The exponent d of z in ξ^- satisfies $d > 0$. As in Case 3 we have

$$\prod_{h_j<0} \frac{1}{1 - H_jz^{h_j}} = 1 + K_1z^{-1} + K_2z^{-2} + \cdots + K_dz^{-d} + z^{-(d+1)} L(z),$$

where $L(z) \in \mathbb{C}[[t_1, \dots, t_k, z^{-1}]]$. Then

$$\xi^- = E \prod \frac{1}{1-G_i}(z^d + K_1 z^{d-1} + K_2 z^{d-2} + \dots + K_{d-1}z + K_d) + z^{-1} E \prod \frac{1}{1-G_i} L(z)$$

and the contribution of ξ^- is

$$E \prod \frac{1}{1-G_i}(z^d + K_1 z^{d-1} + K_2 z^{d-2} + \dots + K_{d-1}z + K_d). \qquad \square$$

There are several algorithms for solving linear systems of Diophantine equations and inequalities using the method of Elliott, see e.g. Domenjoud, Tomás [17], Pasechnik [35] and Xin [47].

Applying the result of Elliott we start with a nice rational function and obtain the result also in the form of a nice rational function. See Stanley [42] for further discussions and applications of the approach of Elliott.

The next theorem of Blakley [13] gives another point of view of the problem.

Theorem 4.10. *Let*

$$f(t_1, \dots, t_k) = \prod_{i=1}^{n} \frac{1}{1 - t_1^{a_{i1}} \cdots t_k^{a_{ik}}} = \sum_{b_j \geq 0} \beta(b_1, \dots, b_k) t_1^{b_1} \cdots t_k^{b_k}, \quad \beta(b_1, \dots, b_k) \in \mathbb{N}.$$

Then there is a finite decomposition of $\mathbb{N}^k$ such that the coefficients $\beta(b_1, \dots, b_k)$ are polynomials of degree $n-k$ in $b_1, \dots, b_k$ on each piece.

In the notation of Theorem 4.10 Sturmfels [43] proposed a method to find such a decomposition and the polynomials which express the coefficients $\beta(b_1, \dots, b_k)$.

5. The algorithm of Xin. In this section we give an idea for the algorithm of Xin [47] in a form suitable for our purposes. The algorithm is based on two easy observations.

Lemma 5.1. *Let*

$$q(t_1, \dots, t_k, z) = 1 - t_1^{a_1} \cdots t_k^{a_k} z^b, \quad a_i \in \mathbb{N}, b \in \mathbb{Z}.$$

(i) *If $b > 0$, then $q(t_1, \dots, t_k, z)$ decomposes as a product of irreducible polynomials in $\mathbb{Q}[t_1, \dots, t_k, z]$ with constant terms (as polynomials in z) equal to* 1.

(ii) *If $b < 0$, then $q(t_1, \dots, t_k, z)$ is decomposed as*

$$q(t_1, \dots, t_k, z) = z^b \prod_{i=1}^{m} u_i(t_1, \dots, t_k, z),$$

where the irreducible polynomials $u_i(t_1, \dots, t_k, z) \in \mathbb{Q}[t_1, \dots, t_k, z]$ are with leading terms (as polynomials in z) equal to z^{n_i}, $n_i \geq 1$.

Proof. (i) Let $b > 0$ and let $q(t_1, \dots, t_k, z) = 1 - t_1^{a_1} \cdots t_k^{a_k} z^b$ decompose as

$$q(t_1, \dots, t_k, z) = \prod_{i=1}^{m} u_i(t_1, \dots, t_k, z), \quad u_i(t_1, \dots, t_k, z) \in \mathbb{Z}[t_1, \dots, t_k, z].$$

Comparing the constant term 1 of $q(t_1, \dots, t_k, z)$ with respect to z with the product of the constant terms of the factors $u_i(t_1, \dots, t_k, z)$ we derive that the constant terms of $u_i(t_1, \dots, t_k, z)$ belong to $\mathbb{Q}$, i.e. we may assume that they are equal to 1.

(ii) If $b < 0$ we present $q(t_1, \dots, t_k, z)$ in the form

$$q(t_1, \dots, t_k, z) = \frac{1}{z^c} q_1(t_1, \dots, t_k, z), \quad q_1(t_1, \dots, t_k, z) = z^c - t_1^{a_1} \cdots t_k^{a_k}, \quad c = -b.$$

As in (i), comparing the leading monomials z^c of $q_1(t_1,\dots,t_k,z)$ and the product of the leading monomials $u_i(t_1,\dots,t_k,z)$ we derive the proof of (ii). $\square$

Proposition 5.2. *Let*

$$\begin{aligned} f(t_1,\dots,t_k,z) &= g(t_1,\dots,t_k,z,z^{-1})\prod_{i=1}^{m}\frac{1}{1-t_1^{a_{i1}}\cdots t_k^{a_{ik}}z^{b_i}} \\ &= \sum_{n=-\infty}^{\infty} f_n(t_1,\dots,t_k)z^n, \end{aligned} \tag{21}$$

where $a_{ij}\in\mathbb{N}$, $b_i\in\mathbb{Z}$, $g(t_1,\dots,t_k,z,z^{-1})\in\mathbb{Z}[t_1,\dots,t_k,z,z^{-1}]$ *is a polynomial in* $t_1,\dots,t_k$ *and a Laurent polynomial in* z, *and* $f_n(t_1,\dots,t_k)\in\mathbb{Q}[[t_1,\dots,t_k]]$. *Let the partial fraction decomposition of* $f(t_1,\dots,t_k,z)$ *be*

$$f(t_1,\dots,t_k,z)=p(t_1,\dots,t_k,z)+\sum_{l,d}\frac{p_{ld}(t_1,\dots,t_k,z)}{q_l^d(t_1,\dots,t_k,z)}+\sum_{j,e}\frac{r_{je}(t_1,\dots,t_k,z)}{s_j^e(t_1,\dots,t_k,z)},$$

where $p,p_{ld},q_l,r_{je},s_j\in\mathbb{Q}[t_1,\dots,t_k,z]$, q_l *and* s_j *are irreducible in* $\mathbb{Q}[t_1,\dots,t_k,z]$, $\deg_z p_{ld}<\deg_z q_l$, $\deg_z r_{je}<\deg_z s_j$, *the constant term* $q_l(t_1,\dots,t_k,0)$ *of each* q_l *be nonzero and belong to* $\mathbb{Q}$ *and the constant term* $s_j(t_1,\dots,t_k,0)$ *of each* s_j *be a polynomial of positive degree in* $\mathbb{Q}[t_1,\dots,t_k]$ *(or* $s_j(t_1,\dots,t_k,0)=z$*). Then*

$$h(t_1,\dots,t_k,z)=\sum_{n=0}^{\infty}f_n(t_1,\dots,t_k)z^n=p(t_1,\dots,t_k,z)+\sum_{l,d}\frac{p_{ld}(t_1,\dots,t_k,z)}{q_l^d(t_1,\dots,t_k,z)}$$

and $f_0(t_1,\dots,t_k)=h(t_1,\dots,t_k,0)$.

Proof. The polynomials $q^d(t_1,\dots,t_k,z)$ and $s^e(t_1,\dots,t_k,z)$ in the denominators in the expression of $f(t_1,\dots,t_k,z)$ are of the form prescribed in Lemma 5.1 (or $s(t_1,\dots,t_k,z)=z$). Hence we may assume that

$$q(t_1,\dots,t_k,z)=1+zv(t_1,\dots,t_k,z),$$

$$s(t_1,\dots,t_k,z)=z^n+w(t_1,\dots,t_k,z),$$

$v,w\in\mathbb{Q}[t_1,\dots,t_k,z]$, $\deg_z w<n$, $\deg w(t_1,\dots,t_k,0)>0$ (or $s(t_1,\dots,t_k,z)=z$). The expansion of the fractions with denominators of the form $q^d(t_1,\dots,t_k,0)$ belongs to $\mathbb{Q}[[t_1,\dots,t_k,z]]$ because the expression

$$\begin{aligned} \frac{1}{q^d(t_1,\dots,t_k,z)} &= \frac{1}{(1+zv(t_1,\dots,t_k,z))^d} \\ &= (1+zv(t_1,\dots,t_k,z)+z^2v^2(t_1,\dots,t_k,z)+\cdots)^d \end{aligned}$$

does not involve negative degrees of z. By similar arguments, when $\deg s(t_1,\dots,t_k,0)>0$, all monomials in the expansion of

$$\begin{aligned} \frac{1}{s^e(t_1,\dots,t_k,z)} &= \frac{1}{(z^n+w(t_1,\dots,t_k,z))^e}=\frac{1}{z^{nd}(1+u(t_1,\dots,t_k,z^{-1}))^e} \\ &= \frac{1}{z^{nd}}(1+u(t_1,\dots,t_k,z^{-1})+u^2(t_1,\dots,t_k,z^{-1})+\cdots)^d \end{aligned}$$

involve factors z^{-m} with negative degrees of z with $m\geq n$. Since m is larger than the degree in z of the corresponding numerator $r(t_1,\dots,t_k,z)$, the expansion of these fractions contains only negative degrees of z and does not contribute to $h(t_1,\dots,t_k,z)$.

When $s(t_1,\dots,t_k,z)=z$ the numerator $r(t_1,\dots,t_k,z)$ does not depend on z and hence these fractions do not participate in $h(t_1,\dots,t_k,z)$ again. □

Algorithm 5.3. We want to solve the homogeneous linear Diophantine system of equations and inequalities

$$
(22)\qquad \left| \begin{array}{rcl} a_{11}x_1 + \cdots + a_{1k}x_k &=& 0 \\ & \cdots & \\ a_{l1}x_1 + \cdots + a_{lk}x_k &=& 0 \\ a_{l+1,1}x_1 + \cdots + a_{l+1,k}x_k &\ge& 0 \\ & \cdots & \\ a_{m1}x_1 + \cdots + a_{mk}x_k &\ge& 0. \end{array} \right.
$$

We start with the function

$$u(t_1,\dots,t_k)=\prod_{i=1}^{k}\frac{1}{1-t_i},$$

replace the variables t_i by $t_iz^{a_{1i}}$, $i=1,\dots,k$, and expand $u(t_1z^{a_{11}},\dots,t_kz^{a_{1k}})$ in the form (21)

$$f(t_1,\dots,t_k,z)=\sum_{n=-\infty}^{\infty}f_n(t_1,\dots,t_k)z^n.$$

Applying Proposition 5.2 we obtain

$$h(t_1,\dots,t_k,z)=\sum_{n=0}^{\infty}f_n(t_1,\dots,t_k)z^n=p(t_1,\dots,t_k,z)+\sum_{l,d}\frac{p_{ld}(t_1,\dots,t_k,z)}{q_l^d(t_1,\dots,t_k,z)}.$$

All polynomials $q_l(t_1,\dots,t_k,z)$ in the denominators are divisors of some $1-t_1^{b_1}\cdots t_k^{b_k}z^c$, multiplying the numerators and denominators with suitable polynomials we present $h(t_1,\dots,t_k,z)$ as a fraction with denominator in the form $\prod(1-t_1^{b_1}\cdots t_k^{b_k}z^c)$, i.e. the result is a nice rational function. If we start with an equation $a_{11}x_1+\cdots+a_{1k}x_k=0$ from (22) we take $f_0(t_1,\dots,t_k)=h(t_1,\dots,t_k,0)$, continue the work with $f_0(t_1,\dots,t_k)$ and handle the next equation or inequality of (22). If we have an inequality $a_{11}x_1+\cdots+a_{1k}x_k\ge 0$, we make the next step with the function $h(t_1,\dots,t_k,1)$ which takes into account all $f_n(t_1,\dots,t_k)$, $n\ge 0$. Continuing in the same way, we obtain in each step a nice rational function which is the characteristic series of the solutions of the first several equations and inequalities of (22). At the final step, we obtain the characteristic series of the solutions of the whole system.

Example 5.4. We start with the system from Example 2.6

$$
\left| \begin{array}{rcl} x_1 + 2x_2 - x_3 - x_4 &=& 0 \\ 2x_1 + 3x_2 - 2x_3 - x_4 &=& 0. \end{array} \right.
$$

By Algorithm (5.3),

$$u(t_1,t_2,t_3,t_4)=\frac{1}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)},$$

$$f(t_1,t_2,t_3,t_4,z)=u(t_1z,t_2z^2,t_3z^{-1},t_4z^{-1})$$

$$=\frac{1}{(1-t_1)(1-t_2z^2)(1-t_3z^{-1})(1-t_4z^{-1})}$$

$$= \frac{t_3^2}{(t_3-t_4)(1-t_1t_3)(1-t_2t_3^2)(z-t_3)} - \frac{t_4^2}{(t_3-t_4)(1-t_1t_4)(1-t_2t_4^2)(z-t_4)}$$

$$+ \frac{(1+t_1t_3+t_1t_4+t_2t_3t_4+(t_1+t_2t_3+t_2t_4+t_1t_2t_3t_4)z)t_2}{(t_2-t_1^2)(1-t_2t_3^2)(1-t_2t_4^2)(1-t_2z^2)}$$

$$- \frac{t_1^2}{(t_2-t_1^2)(1-t_1t_3)(1-t_1t_4)(1-t_1z)},$$

$$h(t_1,t_2,t_3,z) = \frac{(1+t_1t_3+t_1t_4+t_2t_3t_4+(t_1+t_2t_3+t_2t_4+t_1t_2t_3t_4)z)t_2}{(t_2-t_1^2)(1-t_2t_3^2)(1-t_2t_4^2)(1-t_2z^2)}$$

$$- \frac{t_1^2}{(t_2-t_1^2)(1-t_1t_3)(1-t_1t_4)(1-t_1z))},$$

$$f_0(t_1,t_2,t_3,t_4) = h(t_1,t_2,t_3,0) = \frac{1+t_2t_3t_4-t_1t_2t_3^2t_4-t_1t_2t_3t_4^2}{(1-t_1t_3)(1-t_1t_4)(1-t_2t_3^2)(1-t_2t_4^2)}.$$

We continue in the same way with the second equation and present $f_0(t_1z^2, t_2z^3, t_3z^{-2}, t_4z^{-1})$ as a sum of partial fractions. Finally we obtain the characteristic series of the solutions of (10)

$$\chi_S(t_1,t_2,t_3,t_4) = \frac{1}{(1-t_1t_3)(1-t_2t_3t_4)} = \sum_{m,n\geq 0} (t_1t_3)^m(t_2t_3t_4)^n.$$

This means that all solutions of the system are

$$m(1,0,1,0)+n(0,1,1,1), \quad m,n\in\mathbb{N},$$

i.e. the minimal solutions are $(1,0,1,0)$ and $(0,1,1,1)$.

6. Our solution of the problem of Robles-Pérez and Rosales. We shall illustrate the method of Elliott and the algorithm of Xin on the example of the Diophantine transport problem given in [37] which was one of the two main motivations of the present project. As stated in [37], the example is the following.

A transport company carries cars from the factory to a dealer using small and large trucks with a capacity of three and six cars. The trucks cost for the company 1200 *and* 1500 *euros, respectively. The company receives from the dealer* 300 *euros for each transported car and offers as a bonus the transportation of an additional car without charge. The company considers that the ordered transport is profitable when it has a profit of at least* 900 *euros. How many cars must be transported at least in order to achieve that purpose?*

If y denotes the required number of cars, x_3 and x_6 are the numbers of the small and the large trucks, respectively, the problem is equivalent to the linear Diophantine system

$$\left|\begin{array}{l} 300y \geq 1200x_3+1500x_6+900 \\ y+1 \leq 3x_3+6x_6 \end{array}\right.$$

which after a simplification has the form:

$$\left|\begin{array}{l} y \geq 4x_3+5x_6+3 \\ y \leq 3x_3+6x_6-1 \end{array}\right. \quad\Longrightarrow\quad \left|\begin{array}{r} -4x_3-5x_6+y-3\geq 0 \\ 3x_3+6x_6-y-1\geq 0. \end{array}\right. \tag{23}$$

The goal of the paper [37] was to prove that the set T of the integers n for which the system (23) has a solution $(x_3,x_6,y) = (r_3,r_6,n) \in \mathbb{N}^3$ together with 0 forms a submonoid of $(\mathbb{N},+)$, and to give algorithmic procedures how to compute T. We shall extend this goal and show how to find the set S of all solutions $(r_3, r_6, n) \in \mathbb{N}^3$ of the system. In particular, we shall discuss the relations between the profit and the solutions of the system.

Remark 6.1. As in the one-dimensional case considered in [37], it is easy to see that the set S of solutions (r_3, r_6, n) of the system (23), together with $(0,0,0)$ forms a submonoid of $(\mathbb{N}^3,+)$.

Our solution 6.2. Applying Theorem 3.1 and Algorithm 5.3, the first inequality of (23) is replaced by the equation $-4x_3 - 5x_6 + y - 3t = 0$ which has to be solved for $t = 1$. We start with the function

$$u(x_3, x_6, y, t) = \frac{1}{(1-x_3)(1-x_6)(1-y)(1-t)}$$

and replace the variables x_3, x_6, y, t by $x_3z^{-4}, x_6z^{-5}, yz, tz^{-1}$, respectively. Presenting the obtained function $f(x_3, x_6, y, t, z)$ as a sum of partial fractions with respect to z

$$f(x_3, x_6, y, t, z) = u(x_3z^{-4}, x_6z^{-5}, yz, tz^{-3}) = \sum_{n=-\infty}^{\infty} f_n(x_3, x_6, y, t)z^n,$$

we obtain

$$h(x_3, x_6, y, t, z) = \sum_{n=0}^{\infty} f_n(x_3, x_6, y, t)z^n = \frac{1}{(1-x_3y^4)(1-x_6y^5)(1-y^3t)(1-yz)}.$$

By Theorem 3.1 the solutions of the first inequality in (23) are obtained from the solutions (x_3, x_6, y, t) with $t = 1$. In the expansion

$$h(x_3, x_6, y, t, z) = \sum_{d=0}^{\infty} h_d(x_3, x_6, y, z)t^d$$

these solutions correspond to the coefficient $h_1(x_3, x_6, y, z)$. Hence

$$h_1(x_3, x_6, y, z) = \frac{y^3}{(1-x_3y^4)(1-x_6y^5)(1-yz)}.$$

Instead of the second inequality of (23) we consider the equation

$$3x_3 + 6x_6 - y - v = 0.$$

We start with the function

$$p(x_3, x_6, y, v) = \frac{1}{1-v}h_1(x_3, x_6, y, 1) = \frac{y^3}{(1-x_3y^4)(1-x_6y^5)(1-y)(1-v)}$$

and applying Algorithm 5.3 we obtain

$$q(x_3, x_6, y, v, w) = p(x_3w^3, x_6w^6, yw^{-1}, vw^{-1})$$

$$= \frac{y^3}{w^3(1-x_3y^4w^{-1})(1-x_6y^5w)(1-yw^{-1})(1-vw^{-1})} = \sum_{m=-\infty}^{\infty} q_m(x_3, x_6, y, v)w^m,$$

$$r(x_3, x_6, y, v, w) = \sum_{m=0}^{\infty} q_m(x_3, x_6, y, v)w^m$$

$$= \frac{x_6^3y^{18}}{(1-x_6y^6)(1-x_3x_6y^9)(1-x_6y^5v)(1-x_6y^5w)} = \sum_{k=0}^{\infty} r_k(x_3, x_6, y, w)v^k,$$

$$r_1(x_3, x_6, y, w) = \frac{x_6^4 y^{23}}{(1 - x_6 y^5 w)(1 - x_6 y^6)(1 - x_3 x_6 y^9)},$$

$$\chi_S(x_3, x_6, y) = r_1(x_3, x_6, y, 1) = \frac{x_6^4 y^{23}}{(1 - x_6 y^5)(1 - x_6 y^6)(1 - x_3 x_6 y^9)}.$$

Hence the set S of all solutions (r_3, r_6, n) of the system (23) are

$$(r_3, r_6, n) = (0, 4, 23) + c_1(0, 1, 5) + c_2(0, 1, 6) + c_3(1, 1, 9), \quad c_1, c_2, c_3 \in \mathbb{N}.$$

Replacing x_3 and x_6 with 1 in $\chi_S(x_3, x_6, y)$ we obtain the characteristic series of the set T of the possible values of n:

$$\chi_T(y) = \chi_S(1, 1, y) = \frac{y^{23}}{(1 - y^5)(1 - y^6)(1 - y^9)}$$

which expresses the original solution of the problem in [37] in the form of its characteristic series.

Remark 6.3. Using the expression for $\chi_S(x, y, z)$ found above, it is easy to find a relation between the solutions and the corresponding profit. Since for each transported car the firm gains 300 euros and pays, respectively, 1200 euros and 1500 euros for each small and large truck, we shall consider the function

$$\omega(x_3, x_6, y, t) = \chi_S(x_3 t^{-4}, x_6 t^{-5}, yt) = \frac{x_6^4 y^{23} t^3}{(1 - x_6 y^5)(1 - x_3 x_6 y^9)(1 - x_6 y^6 t)}$$

$$= \sum_{k=3}^{\infty} \omega_k(x_3, x_6, y) t^k = \sum_{k=3}^{\infty} \frac{x_6^{k+1} y^{6k+5} t^k}{(1 - x_6 y^5)(1 - x_3 x_6 y^9)}.$$

The firm will have a profit $300k$ euros for all (r_3, r_6, n) such that $x_3^{r_3} x_6^{r_6} y^n$ participates with a nonzero coefficient $\alpha(r_3, r_6, n)$ in the expansion of $\omega_k(x_3^{r_3} x_6^{r_6} y^n)$ as a power series:

$$\omega_k(x_3, x_6, y) = \frac{x_6^{k+1} y^{6k+5}}{(1 - x_6 y^5)(1 - x_3 x_6 y^9)} = \sum_{r_3, r_6, n \geq 0} \alpha(r_3, r_6, n) x_3^{r_3} x_6^{r_6} y^n.$$

In particular, it is easy to see that the minimal number of transported cars to gain a profit $300k$ is $n = 6k + 5$ (plus one car as a bonus) and for this purpose the firm has to use $k + 1$ large trucks.

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ЕДНА ДИОФАНТОВА ТРАНСПОРТНА ЗАДАЧА ОТ 2016 ГОДИНА И НЕЙНОТО ВЪЗМОЖНО РЕШЕНИЕ ПРЕЗ 1903 ГОДИНА

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Този проект е мотивиран от неотдавнашен диофантов транспортен проблем как да транспортираме изгодно група от хора или обекти. Ние правим обзор на класически факти за решаване на системи линейни диофантови уравнения и неравенства в неотрицателни цели числа. Специално внимание отделяме на метода на Елиът от 1903 година и неговото по-нататъшно развитие от МакМахън в неговото „Омега смятане“. Като илюстрация намираме решение на разглеждания транспортен проблем на езика на формални степенни редове на няколко променливи, които са развития на рационални функции от специален вид.