МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2021
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2021
Proceedings of the Fiftieth Spring Conference
of the Union of Bulgarian Mathematicians
2021

ON SOME SPECIAL MATRIX DECOMPOSITIONS OVER FIELDS AND FINITE COMMUTATIVE RINGS*

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In order to find a suitable expression of an arbitrary square matrix over an arbitrary field, we prove that every square matrix over an infinite field is always representable as a sum of a diagonalizable matrix and a square-zero nilpotent matrix. In addition, each 2×2 matrix over any field admits such a representation. We also show that, for all natural numbers $n \geq 3$, every $n \times n$ matrix over a finite field having no less than $n + 1$ elements also admits such a decomposition. As a consequence of these decompositions, we show that every matrix over a finite field can be expressed as the sum of a potent matrix and a square-zero matrix. Moreover, we prove that every matrix over a finite commutative ring is always representable as a sum of a potent matrix and a square-zero nilpotent matrix, provided the Jacobson radical of the ring has zero-square.

Our main theorems substantially improve on recent results due to Abyzov et al. in Mat. Zametki (2017), Šter in Lin. Algebra & Appl. (2018), Breaz in Lin. Algebra & Appl. (2018) and Shitov in Indag. Math. (2019).

1. Introduction and known results. We start the frontier of this paper by recalling that an element x of an arbitrary ring R is said to be *nilpotent* if there is an integer $i > 0$ such that $x^i = 0$ whereas an element y from R is said to be *potent*, or more exactly *m-potent*, if there is a natural number $m \geq 2$ with $y^m = y$. In particular, all the idempotents are 2-potent elements.

Our current work is devoted to the further study, firstly somewhat initiated in [5], of decomposing square matrices as a sum of a potent and a nilpotent. Concretely, a brief historical retrospection of the most important results in this direction is as follows:

It was proven in [5] that each matrix from the ring $\mathbb{M}_n(\mathbb{F}_2)$ of $n \times n$ matrices over the field $\mathbb{F}_2$ of two elements is a sum of an idempotent matrix and a nilpotent matrix – even something more, if the matrix ring $\mathbb{M}_n(F)$ over an arbitrary field F possesses this property, then $F \cong \mathbb{F}_2$. This result was substantially strengthened by Šter in [15] who proved that $\mathbb{M}_n(\mathbb{F}_2)$ is actually a sum of an idempotent matrix and a nilpotent matrix of

*The first author was partially supported by the Bulgarian National Science Fund under Grant KP-06 No. 32/1 of December 07, 2019.

The second two authors were partially supported by MTM2017-84194-P (AEI/FEDER, UE), and by the Junta de Andalucía FQM264.

2020 Mathematics Subject Classification: 15A24, 15B33, 16U99.

Key words: Companion matrix, Jordan normal form, rational form, irreducible polynomial, field, finite commutative ring, nilpotent matrix, potent matrix.

index at most 4. Lately, this result was significantly improved by Shitov in [14] for certain matrix sizes n. Moreover, an important work was done by de Seguins Pazzis in [7], where a valuable discussion on the decomposition of a matrix as a sum of an idempotent and a square-zero matrix is provided.

On the other vein, Abyzov and Mukhametgaliev showed in [1] that, for all naturals $n \geq 1$, any element of the ring $\mathbb{M}_n(F)$ is presented as a sum of a nilpotent and a q-potent element, provided that F is a field of cardinality q – specifically, in [1, Theorem 2] it was showed that some square matrix over finite fields are expressible as a sum of a potent and a nilpotent but the order of the existing nilpotent is, in general, greater than 2. Also, a recent paper [4] by Breaz deals with the more exact presentation of matrices over fields of odd cardinality q as a sum of a q-potent matrix and a nilpotent matrix of order 3. Besides, an ingenious example of a 3×3 matrix over the field $\mathbb{F}_3$ of three elements that cannot be presented as the sum of a 3-potent and a nilpotent matrix of order 2 was constructed in [4, Example 6] (in other terms, the latter matrix is also called *square-zero* or, equivalently, *zero-square*).

We also concretize that some related results can be found by the interested reader in [6] and [13] along with the given references therewith, respectively.

So, analyzing carefully all of the results established above, we come to mind that further non-trivial generalizations are pretty possible by minimizing the order of the existing nilpotent to not exceeding 2.

2. Main results. We will distribute our two chief theorems into two independent subsections:

2.1. Matrix decompositions over fields. Our first central decomposing theorem is the following one:

Theorem 2.1. *Given any field K, all matrices in $\mathbb{M}_2(K)$ admit a decomposition into $D+Q$, where D is a diagonalizable matrix and Q is a matrix such that $Q^2=0$.*

Let $n \geq 3$ and let K be a field with $|K| \geq n+1$. Then every matrix $A \in \mathbb{M}_n(K)$ admits a decomposition into $D+Q$, where D is a diagonalizable matrix and Q is a matrix such that $Q^2=0$. In particular, square matrices over infinite fields always admit such decomposition.

Since the diagonalizable matrices over a finite field of q elements are always q-potent, we immediately obtain the following claim.

Corollary 2.2. *Let $\mathbb{F}_q$ be the finite field of q elements, $q > 2$. Then every matrix in $\mathbb{M}_n(\mathbb{F}_q)$ with $n \leq q-1$ admits a decomposition into $D+Q$, where D is a q-potent matrix and Q is a nilpotent matrix such that $Q^2=0$.*

The key instruments in proving the statements stated above are the following ones:

Lemma 2.3. *Let K be a field, let $n \geq 3$ and let $A \in \mathbb{M}_n(K)$ be the companion matrix of a polynomial $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$. Then*

- *If $c_{n-1} = 0$ and $|K| \geq n$ then A admits a decomposition into $D+Q$ where D is diagonalizable with no multiple eigenvalues and $Q^2=0$ with $\operatorname{rank}(Q) \leq 1$.*

- *If $c_{n-1} \neq 0$ and $|K| \geq n+1$ then A admits a decomposition into $D+Q$, where D is diagonalizable with no multiple eigenvalues and $Q^2=0$ with $\operatorname{rank}(Q) \leq 1$.*

In the proof we use the following machinery: Let $A = C(p(x))$, where

$$C(p(x)) = \begin{pmatrix} 0 & 0 & 0 & 0 & -c_0 \\ 1 & 0 & 0 & 0 & -c_1 \\ 0 & 1 & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & -c_{n-1} \end{pmatrix}$$

for $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$.

Take n different elements $a_1, \ldots, a_n$ in the field such $\sum_{i=1}^{n} a_i = -c_{n-1}$ (notice that the cardinality of K was chosen to assure the existence of these pairwise different elements) and consider the polynomial $q(x) = (x-a_1)(x-a_2)\cdots(x-a_n) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$. Then

$$C(p(x)) = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & -b_0 \\ 1 & 0 & 0 & 0 & -b_1 \\ 0 & 1 & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & -b_{n-1} \end{pmatrix}}_{C(q(x))} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & -c_0 + b_0 \\ 0 & 0 & 0 & 0 & -c_1 + b_1 \\ 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 0 & -c_{n-1} + b_{n-1} \end{pmatrix}}_{Q} \qquad (*)$$

where $C(q(x))$ is diagonalizable because it corresponds to a polynomial with n different roots, while $Q^2 = 0$ because $-c_{n-1} + b_{n-1} = 0$.

Example 2.4. In the proof of Lemma 2.3, the formula labeled by $(*)$ gives an explicit decomposition $C(p(x)) = C(q(x)) + Q$, where $C(q(x))$ is diagonalizable and $Q^2 = 0$ for the companion matrix A of any polynomial of the form $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$. Here we present another decomposition. The requirements for the size of the field are the same: when $c_{n-1} = 0$ we need that $|K| \geq n$, and when $c_{n-1} \neq 0$ we need that $|K| \geq n+1$.

Given any polynomial $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \in K[x]$ and its companion matrix A, take n different elements $a_1, \ldots, a_n \in K$ such that $a_1 + \cdots + a_n = -c_{n-1}$ (those elements exist because we are assuming that $|K| \geq n$ if $c_{n-1} = 0$ or that $|K| \geq n+1$ if $c_{n-1} \neq 0$). Let us consider the following matrix

$$B = \underbrace{\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 1 & a_2 & 0 & 0 & 0 \\ 0 & 1 & a_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & a_n \end{pmatrix}}_{\hat{D}} + \underbrace{\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & 0 & x_{n-1} \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right)}_{\hat{Q}}.$$

We claim that the elements $x_1, \ldots, x_{n-1}$ can be chosen in K so that the characteristic polynomial of B coincides with $p(x)$. Indeed, if $q(x) = x^n + d_{n-1}x^{n-1} + \cdots + d_0$ denotes the characteristic polynomial of B, $d_{n-1} = c_{n-1}$ because the traces of A and B coincide. Moreover, by the Faddeev–LeVerrier algorithm [11, 6.7], d_{n-2} depends on $a_1, \ldots, a_n$

and on x_{n-1}, and x_{n-1} can be taken such that $d_{n-2} = c_{n-2}$. Again, by the Faddeev–LeVerrier algorithm, d_{n-3} depends on the $a_1, \dots, a_n$ and on x_{n-1} and x_{n-2}, and x_{n-2} can be taken such that $d_{n-3} = c_{n-3}$. We can repeat this process until we get the precise $x_1, \dots, x_{n-1} \in K$ that makes true $q(x) = p(x)$.

Finally, A and B are two non-derogative matrices with the same characteristic polynomial, so there exists an invertible P such that

$$A = P^{-1}\hat{D}P + P^{-1}\hat{Q}P = D + Q.$$

It is worthwhile noticing that the decomposition of companion matrices obtained above has the following properties:

- The matrix D is diagonalizable with no multiple eigenvalues.
- $Q^2 = 0$ and $\operatorname{rank}(Q) \le 1$.
- If K is a field with q elements, then $D^q = D$.

2.2. Matrix decompositions over finite commutative rings. The following statement somewhat generalizes [8, Corollary 3.2], where it was shown that every matrix over a finite field is a sum of potent matrix and a zero-square matrix by using a different approach. The result is stated and proved in details in [9], and it is entirely based on the primary rational canonical form of a matrix (see, e.g., [12, VII.Corollary 4.7(ii)]), which states that every matrix $A \in \mathbb{M}_n(\mathbb{F})$ where $\mathbb{F}$ is a field is similar to a direct sum of companion matrices of prime power polynomials $p_1^{m_{11}}, \dots, p_s^{m_{sk_s}} \in \mathbb{F}[x]$ where each p_i is prime (irreducible) in $\mathbb{F}[x]$. The matrix A is uniquely determined except for the order of the companion matrices of the $p_i^{m_{ij}}$ along its main diagonal. The polynomials $p_1^{m_{11}}, \dots, p_s^{m_{sk_s}}$ are called *the elementary divisors* of the matrix A.

Proposition 2.5. *Let $\mathbb{F}$ be a finite field. For any matrix $A \in \mathbb{M}_n(\mathbb{F})$ there exists $k \in \mathbb{N}$ such that $A = P + N$, where $N^2 = 0$, $P^k = P$, $E = P^{k-1}$ is an idempotent with $PE = EP = P$ and $EN = NE = N$.*

With this at hand, we arrive now to our second central theorem on decomposing any matrix over special finite commutative rings into a potent matrix and a zero-square matrix.

Theorem 2.6. *Let R be a finite commutative ring such that its Jacobson radical has zero-square. Then every matrix A in $\mathbb{M}_n(R)$ can be expressed as $P + N$, where P is a potent matrix and N is a nilpotent matrix with $N^2 = 0$.*

As an immediate consequence for a concrete finite commutative ring, one can extract the following one:

Corollary 2.7. *Suppose p is a prime number. Then every matrix in the ring $\mathbb{M}_n(\mathbb{Z}_{p^2})$ is expressible as a sum of a potent matrix and a square-zero nilpotent matrix.*

Note that this statement somewhat strengthens the assertion which can be deduced by combining [1, Lemma 1] and [1, Theorem 4], namely that, for any $m \in \mathbb{N}$, each matrix in the ring $\mathbb{M}_n(\mathbb{Z}_{p^m})$ is presentable as the sum of a p-potent matrix and a nilpotent matrix. However, the index of nilpotence is not under control.

The following constructions illustrate that the condition of having a zero-square Jacobson radical is essential in this last theorem and cannot be dropped off.

Example 2.8. There are matrices over $\mathbb{Z}_{2^3}$ that do not admit a decomposition into potent + zero-square. For example, the matrix

$$A = 2\,\mathrm{Id} \in \mathbb{M}_n(\mathbb{Z}_{2^3})$$

does not admit such a decomposition. Otherwise, since $A^2 \neq 0$ there would exist a non-zero potent matrix P and a zero-square matrix N such that $A = P + N$. Then $P^4 = ((A - N)^2)^2 = (4\,\mathrm{Id} - 4N)^2 = 0$, which is not possible if P is potent and non-zero, thus establishing our claim. However, since $A^3 = 0$ one finds that A is presentable as a sum of a potent matrix (namely, the zero one) and a nilpotent matrix of order precisely 3.

On the other hand, Theorem 2.6 remains no longer true for finite commutative rings of characteristic p^2 for some arbitrary but fixed prime p. In fact, it suffices to find a finite commutative ring R of characteristic p^2 having an element a with $a^3 = 0$ and $a^2 \neq 0$. For example, consider the ring $R = \mathbb{Z}_4[x]/I$, where I is the ideal generated by the polynomial $(x^2+x+1)^3$. The characteristic of R is then exactly 4. Choose $a = (x^2+x+1)+I \in R$, and let us consider similarly to above the matrix $A = a\,\mathrm{Id} \in \mathbb{M}_n(R)$ for some $n \in \mathbb{N}$. This matrix A has the properties $A^2 \neq 0$ and $A^3 = 0$, whence with the help of the same argument as above it surely cannot be decomposed into the sum of a potent and a zero-square nilpotent. Nevertheless, as mentioned above, A is decomposable as a sum of a potent matrix (namely, the zero one) and a nilpotent matrix of order exactly 3.

This concludes our arguments.

In order to generalize Theorem 2.6 to commutative rings of the form $\mathbb{Z}_{p^r}$ for some natural number $r \geq 2$, we first are going to show that potent elements lift modulo a nilpotent ideal. Our proof mainly follows the ideas of the classical lifting of idempotents (see, for instance, [2, Proposition 27.1]).

Proposition 2.9. *Let R be a finite ring and let I be a nilpotent ideal of R of index n. Let us suppose $A \in R$ is such that $\overline{A} \in R/I$ is a potent element of R/I. Then there exists $B \in R$ such that $\overline{A} = \overline{B}$ and B is potent in R.*

We are now ready to proceed by proving with the following assertion.

Corollary 2.10. *Let n, r be two natural numbers. Then every matrix in $\mathbb{M}_n(\mathbb{Z}_{p^r})$ can be expressed as $P + N$, where P is a potent matrix and N is a matrix such that $N^2 \in \mathbb{M}_n(p^2\mathbb{Z}_{p^r})$. In particular, each matrix in $\mathbb{M}_n(\mathbb{Z}_{p^2})$ is expressible as the sum of a potent matrix and a zero-square matrix.*

We finish off our work in this section with the following conjecture, which is motivated by the first part of Example 2.8.

Conjecture. Suppose $m, n \geq 2$ are natural numbers and p is a prime. Then every matrix in $\mathbb{M}_n(\mathbb{Z}_{p^m})$ is a sum of a potent and of a nilpotent of order at most m.

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ВЪРХУ НЯКОИ СПЕЦИАЛНИ МАТРИЧНИ РАЗЛАГАНИЯ НАД ПОЛЕТА И КРАЙНИ КОМУТАТИВНИ ПРЪСТЕНИ*

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Доказано е, че всяка квадратна матрица над безкрайно поле е винаги представима като сума на диагонализируема матрица и нилпотентна матрица от ред 2. В допълнение, всяка такава матрица над крайно поле може да се представи като сума на потентна матрица и нилпотентна матрица с индекс на нилпотентност точно 2 – този резултат може да се разшири до квадратни матрици над крайни комутативни пръстени с радикал на Джейкобсон, чиято втора степен е нула.

Тези теореми обобщават някои класически резултати, като тези на А. Абизов и др. в *Математические Заметки* (2017), Я. Щер в *Линейна алгебра и приложения* (2018), С. Брез в *Линейна алгебра и приложения* (2018) и Я. Шитов в *Indagationes Mathematicae* (2019).

*As per the authors' request the following text is in its original form, without any edits. (Ed.)