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HAMILTON'S INSTABILITY INDEX AND APPLICATIONS TO PERIODIC WAVES*

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In this survey we present some results on the stability of periodic waves. The main objective is to study their stability with respect to co-periodic perturbations. In our considerations, we rely on an instability index count theories, which in turn critically depend on a detailed spectral analysis of a self-adjoint both scalar and matrix Hill operators, and in the computations of the quantities involved on the stability index.

1. Introduction. One of the important aspects in the theory of nonlinear dispersive evolution equations concerns the traveling wave solutions. Traveling wave solutions are of special importance because of the distinguished role they play in the solution of the initial value problem for evolution equations. This is perhaps best known for completely integrable equations such as the Korteweg-de Vries (KdV) equation and the one-dimensional cubic Schrodinger equation, where inverse scattering theories specify their special role. The two fundamental problems of study are existence and stability of these waves.

The problem of the stability of solitary waves is well-studied (cf. [1, 4, 5, 9, 10, 11, 18]). The situation regarding the study of periodic travelling waves is very different ([2, 3, 12]). It is well known that the spectrum of linearized equation depends on the choice of function space. In the space of periodic functions the spectrum consists of isolated eigenvalues, while in the space of bounded functions the spectrum is continuous.

Recently, the linear stability of traveling wave solutions for the second order in time nonlinear differential equations has been studied extensively [6, 13, 16, 17] . In [16] the question of the stability analysis for the second order in time PDEs is reduced to the study of stability of quadratic pencils in the form $\lambda^2 + 2c\lambda\partial_x + \mathcal{H} = 0$, where $\mathcal{H}$ is a self-adjoint operator. If $\mathcal{H}$ has a simple negative eigenvalue and a simple eigenvalue at zero, the authors in [16] derived the index of stability and the abstract results were applied to Boussinesq equation, Klein-Gordon equation and the beam equation. The authors of [6] derived index counting theory for more general class of quadratic pencils.

Here we present some applications of these theories in the periodic settings.

2. Index counting theory for quadratic pencils. In this section we are interested in the stability of spatially periodic waves for certain models, which involve the second-order derivative in time.

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2.1. Stability of periodic wave of the Boussinesq equation. Consider the following Boussinesq equation

$$u_{tt} + u_{xxxx} - u_{xx} + (u^3)xx = 0. \tag{1}$$

We look for traveling wave solutions for (1) in the form $u(t,x) = \varphi(x+ct)$. For the function φ, we have

$$\varphi'^2 = b + w\varphi^2 - \frac{1}{2}\varphi^4,$$

where $w = 1-c^2$ and b is a constant of integration. Hence, the periodic solutions are given by the periodic trajectories $H(\varphi,\varphi') = b$ of the Hamiltonian $H(x,y) = y^2 + \frac{1}{4}x^4 - \frac{1}{2}wx^2$. There are then two possibilities.

– (Outer case.) For any $b > 0$ the orbit defined by $H(\varphi,\varphi') = b$ is periodic and oscillates around the eight-shaped loop $H(\varphi,\varphi') = 0$ through the saddle at the origin.

– (Left and right cases.) For any $b \in (-\frac{1}{2}w^2, 0)$ there exist two periodic orbits defined by $H(\varphi,\varphi') = b$ (the left and right ones). These are located inside the eight-shaped loop and oscillate around the centers at $(\pm\sqrt{w}, 0)$, respectively.

We will consider only the right one. In these case, denote by $\varphi_0 > \varphi_1 > 0$ the positive roots of the quadratic equation $\frac{1}{2}\rho^4 - w\rho^2 - b = 0$. Then, up to a translation, we obtain the respective explicit formulae

$$\varphi(x) = \varphi_0 dn(\alpha x) \tag{2}$$

and

$$\kappa^2 = \frac{\varphi_0^2 - \varphi_1^2}{\varphi_1^2} = \frac{2\varphi_0^2 - 2w}{\varphi_1^2}, \quad \alpha = \frac{\varphi_0}{\sqrt{2}}.$$

We now set up the linear stability/instability problem for (1). Set the ansatz $u = \varphi(x+ct) + v(t, x+ct)$ and ignore all terms $O(v^2)$. We get $v_{tt} + 2cvtx + Mv = 0$, where

$$Mv = \partial_x^4 v - (1-c^2)\partial_x^2 v + (3\varphi^2 v)_{xx}.$$

Note that this operator M is not self-adjoint. However, if we introduce the variable $z : z_x = v$, we get the following linearized equation in terms of z:

$$z_{ttx} + 2cz_{txx} + M[z_{tx}] = 0.$$

The question of equations in the form

$$z_{tt} + 2cz_{tx} + \mathcal{H}z = 0$$

or what is equivalent (at least in this case) to the solvability of

$$\lambda^2\psi + 2w\lambda\psi + \mathcal{H}\psi = 0,$$

where $\mathcal{H}$ is self-adjoint and is in the form

$$\mathcal{H} = -\partial_x \mathcal{L} \partial_x \quad \mathcal{L} = -\partial_x^2 + (1-c^2) - 3\varphi^2.$$

Next, we give precise statements of the results in [17] for quadratic pencil in the form

$$\lambda^2\psi + 2c\lambda\psi' + H\psi = 0. \tag{3}$$

Definition 1. *We say that the travelling wave φ is spectrally/linearly unstable if there exists a T-periodic mean value zero function $\psi \in D(\mathcal{H})$ and $\lambda : \Re\lambda > 0$ satisfying*

$$\lambda^2\psi + 2c\lambda\psi' + H\psi = 0.$$

Assume that

$$\overline{\mathcal{H}u} = \mathcal{H}\overline{u}, \mathcal{H}^* = \mathcal{H}, \tag{4}$$

$$(\mathcal{H}+\tau)^{-1/2}\partial_x(\mathcal{H}+\tau)^{-1/2}, \ \ (\mathcal{H}+\tau)^{-1} \in B(L^2). \tag{5}$$

In addition to (4) and (5), we assume following for the spectrum of $\mathcal{H}$

$$\begin{cases} \mathcal{H}\phi = -\delta^2\phi, \mathcal{H}\mid_{\{\phi\}^\perp} \geq 0 \\ \mathrm{Ker}[\mathcal{H}] = \mathrm{span}[\vec{\psi}_0]. \end{cases} \tag{6}$$

The following theorem is proved in [16]

Theorem 1. *Let $\mathcal{H}$ be a self-adjoint operator on a Hilbert space H. Assume that it satisfies the assumptions* (4), (5), *and* (6).

Then, if $\langle \mathcal{H}^{-1}[\psi_0'], \psi_0'\rangle \geq 0$, one has a solution of (4), *that is instability in the sense of Definition* 1. *Otherwise, supposing that $\langle \mathcal{H}^{-1}[\psi_0'], [\psi_0']\rangle < 0$*

- *the problem* (4) *has solution, if w satisfies the inequality*

$$|w| < \frac{1}{2\sqrt{-\langle \mathcal{H}^{-1}[\psi_0'], [\psi_0']\rangle}} =: w^*(\mathcal{H}); \tag{7}$$

- *the problem* (4) *does not have solutions (i.e. stability), if w satisfies the reverse inequality*

$$|w| > w^*(\mathcal{H}). \tag{8}$$

In the case of Boussinesq equation, all conditions of Theorem 1 are satisfied and we have the following result on the stability of periodic wave of dnoidal type.

Theorem 2. *The two-parameter family of dnoidal solutions, described in* (2), *is then linearly stable, if and only if*

$$|c| \geq \sqrt{\frac{M(\kappa)}{4+M(\kappa)}},$$

where

$$M(\kappa) = \frac{[4E(\kappa) - \pi^2/K(\kappa)][(2-\kappa^2)E(\kappa) - 2(1-\kappa^2)K(\kappa)]}{(2-\kappa^2)[E^2(\kappa) - (1-\kappa^2)K^2(\kappa)]},$$

in which $E(\kappa)$ and $K(\kappa)$ are elliptic integrals of the first and second kind in a Legendre form.

Remark. Using the results of Theorem 2, one can reconstruct the results on linear stability of the whole-line waves [16]:

$$\varphi(x) = \left[2(1-c^2)\right]^{\frac{1}{2}} \mathrm{sech}[(1-c^2)x].$$

Recall that the results of [24] predict linear stability if and only if $|c| \geq \frac{\sqrt{2}}{2}$. The periodic waves described in (2) in the limit $\kappa \to 1-$ correspond to the whole-line waves. Note that, since $\lim\limits_{\kappa\to 1-} E(\kappa) = 1$, $\lim\limits_{\kappa\to 1-} K(\kappa) = \infty$ and $\lim\limits_{\kappa\to 1-} (1-\kappa^2)K^2(\kappa) = 0$, we can easily conclude that $\lim\limits_{\kappa\to 1-} M(\kappa) = 4$, whence

$$\lim_{\kappa\to 1-} \sqrt{\frac{M(\kappa)}{4+M(\kappa)}} = \frac{\sqrt{2}}{2}.$$

2.2. Index counting theory for quadratic pencil. We now present the theory developed in [6] for quadratic pencil in the form

$$\lambda^2\psi + \lambda J\psi + \mathcal{H}\psi = 0. \tag{9}$$

We start with some notations. For a self-adjoint operator $\mathcal{H}$, define the number of the negative eigenvalues

$$n(\mathcal{H}) = \#\{\lambda \in (-\infty, 0) \cap \sigma(\mathcal{H}).$$

Next, for a subspace M denote $P_M : M \to M$ to be the orthogonal projection onto the subspace M and $P_M^\perp := Id - P_M$. For an operator T, denote $T_M := P_M T P_M$. In particular, if M is finite dimensional, with orthogonal basis $\{\chi_j\}_{j=1}^n$, P_M is given in a matrix form by $\{\langle T\chi_i, \chi_j\rangle\}_{i,j=1}^n$.

Theorem 3 ([6])**.** *Let $\mathcal{H} = \mathcal{H}^*$, so that $n(\mathcal{H}) < \infty$ and $\dim(\mathrm{Ker}(\mathcal{H})) < \infty$. Assume that*

1.

$$(P^\perp_{\mathrm{Ker}(\mathcal{H})}\mathcal{H}P^\perp_{\mathrm{Ker}(\mathcal{H})})^{-1}, (P^\perp_{\mathrm{Ker}(\mathcal{H})}\mathcal{H}P^\perp_{\mathrm{Ker}(\mathcal{H})})^{-1}P^\perp_{\mathrm{Ker}(\mathcal{H})}JP^\perp_{\mathrm{Ker}(\mathcal{H})}, \tag{10}$$

are both compact operators on L^2.

2. *There exists a positive operator S, so that $X_S = \{u : \langle Su, Su\rangle < \infty\}$ is dense in L^2.*
3. *$S^{-1}JS^{-1}, S^{-1}$ are compact, whereas $S(\mathcal{H} + \lambda)^{-1}S$ is compact for $\lambda >> 1$.*
4. *$J : \mathrm{Ker}(\mathcal{H}) \to \mathrm{Ker}(\mathcal{H})^\perp$*
5. *$(Id - J\mathcal{H}^{-1}J)|_{\mathrm{Ker}(\mathcal{H})}$ is invertible*[1]

Then,

$$k_r + k_c + k_- = n(H) - n((Id - J\mathcal{H}^{-1}J)|_{\mathrm{Ker}(\mathcal{H})}). \tag{11}$$

where k_r is the number of positive solutions λ of (9)*, k_- is the number of solutions λ of* (9) *with positive real part, whereas k_c is the total Krein index for the quadratic pencil.*

Remark. The non-negative integers k_c, k_- are even. It follows as a corollary from (11) that if $n(\mathcal{H}) = 1n(\mathcal{H}) = 1$, then $k_c = k_- = 0$ and

$$k_r = 1 - n((Id - J\mathcal{H}^{-1}J)|_{\mathrm{Ker}(\mathcal{H})}). \tag{12}$$

Now consider the application of Theorem 3 to the second order in time differential equation. Consider Klein-Gordon equation

$$u_{tt} - u_{xx} + u - |u|^2u = 0. \tag{13}$$

We seek a periodic wave in the form $u(t,x) = e^{iwt}e^{iq(x-ct)}\varphi(x-ct)$. Up to translation, the solution φ is given by

$$\varphi(x) = \varphi_0 dn(\alpha x).$$

[1]Note that this is well-defined since $J : \mathrm{Ker}(\mathcal{H}) \to \mathrm{Ker}(\mathcal{H})^\perp$ and hence $J\mathcal{H}^{-1}J$ is well-defined.

We take the perturbation in the form $u(t,x) = e^{iwt}e^{iq(x-ct)}(\varphi(x-ct) + v(t,x-ct)$. Plugging in (13) and ignoring quadratic and higher order terms, and splitting the real and imaginary parts, we can rewrite the linearized problem as

$$\mathbf{v}_{tt} + \mathcal{J}\mathbf{v}_t + \mathcal{H}\mathbf{v} = 0,$$

where

$$\mathcal{H}\begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} -c\partial_x & w+qc \\ -(w+qc) & c\partial_x \end{pmatrix}$$

and

$$L_+ = -\mu^2\partial_x^2 + \nu - 3\varphi^2, \quad L_- = -\mu^2\partial_x^2 + \nu - \varphi^2.$$

If we consider the eigenvalue problem, that is $v = e^{\lambda t}\psi$, we arrive at

$$\lambda^2\psi + \lambda\mathcal{J}\psi + \mathcal{H}\psi = 0.$$

Next, we list the spectral properties of the operator $\mathcal{H}$. Since it is a diagonal operator, with entries $L_\pm$, we have $\sigma(\mathcal{H}) = \sigma(L_+) \cup \sigma(L_-)$. We also have $\dim\ker L_+ = \dim\ker L_- = 1$, $n(L_+) = 1$, and $n(L_-) = 0$. Hence, 0 is an eigenvalue of multiplicity two of $\mathcal{H}$ and $n(\mathcal{H}) = 1$. The normalized elements of the kernel of $\mathcal{H}$ are

$$\psi_0 = \begin{pmatrix} 0 \\ \varphi||\varphi||^{-1} \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} \varphi'||\varphi'||^{-1} \\ 0 \end{pmatrix}.$$

We need to consider the matrix representation of $(Id - \mathcal{J}\mathcal{H}^{-1}\mathcal{J})|_{\ker(\mathcal{H})}$. It is the symmetric matrix

$$U = \begin{pmatrix} 1 + \langle \mathcal{H}^{-1}\mathcal{J}\psi_0, \mathcal{J}\psi_0\rangle & \langle \mathcal{H}^{-1}\mathcal{J}\psi_0, \mathcal{J}\psi_1\rangle \\ \langle \mathcal{H}^{-1}\mathcal{J}\psi_1, \mathcal{J}\psi_0\rangle & 1 + \langle \mathcal{H}^{-1}\mathcal{J}\psi_1, \mathcal{J}\psi_1\rangle \end{pmatrix}.$$

Note that all elements of the matrix U are computable in terms of complete elliptic integrals. Since $n(\mathcal{H}) = 1$ we have stability if $n(U) = 1$, and instability if $n(U) = 0$.

3. Index counting theory for Hamiltonian system. We start this section with some results about the instability index count theories developed in [15]. These allow us to count the number of unstable eigenvalues for eigenvalue problems of the form (14) (see below) based on the information about the the spectrum of various self-adjoint operators, both scalar and matrix, and some specific estimates. For eigenvalue problem in the form

$$\mathcal{J}\mathcal{H}V = \lambda V \tag{14}$$

we assume that $\mathcal{H} = \mathcal{H}^*$ has $\dim(\mathrm{Ker}(\mathcal{H}) < \infty$, and also a finite number of negative eigenvalues, $n(\mathcal{H})$, a quantity, sometimes referred to as *Morse index of the operator $\mathcal{H}$*. In addition, $\mathcal{J}^* = -\mathcal{J}$. Let k_r be the number of positive eigenvalues of the spectral problem (14) (i.e. the number of real instabilities or real modes), k_c be the number of quadruplets of eigenvalues with non-zero real and imaginary parts, and k_i^-, the number of pairs of purely imaginary eigenvalues with negative Krein-signature. For a simple pair of imaginary eigenvalues $\pm i\mu, \mu \neq 0$, and the corresponding eigenvector $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, the Krein index is $\mathrm{sgn}(\langle \mathrm{ch}\, \vec{z}, \vec{z}\rangle)$.

A finite dimensional matrix $\mathcal{D}$, which is obtained from the adjoint eigenvectors for (14) is also of importance in this theory. More specifically, consider the generalized kernel of $\mathcal{J}\mathcal{H}$

$$\mathrm{gKer}(\mathcal{J}\mathcal{H}) = \mathrm{span}[(\mathrm{Ker}(\mathcal{J}\mathcal{H}))^l, l = 1, 2, \ldots].$$

Assume that $\dim(\mathrm{gKer}(\mathcal{J}\mathcal{H})) < \infty$ (note that under minimal Fredholm assumptions on $\mathcal{J}, \mathcal{H}$, this is indeed the case). Select an orthonormal basis in $\mathrm{gKer}(\mathcal{J}\mathcal{H}) \ominus \mathrm{Ker}(\mathcal{H}) = \mathrm{span}[\eta_j, j = 1, \ldots, N]$. Then $\mathcal{D} \in M_{N\times N}$ is defined via

$$\mathcal{D} := \{\mathcal{D}_{ij}\}_{i,j=1}^{N} : \mathcal{D}_{ij} = \langle \mathcal{H}\eta_i, \eta_j \rangle.$$

Then, following [15], we have the following formula, relating the number of "instabilities" or Hamiltonian index of the eigenvalue problem (14) and the Morse indices of the operators $\mathcal{H}$ and $\mathcal{D}$:

$$k_{Ham} := k_r + 2k_c + 2k_i^- = n(\mathcal{H}) - n(\mathcal{D}). \tag{15}$$

Remark. As a corollary, if $n(\mathcal{H}) = 1$, it follows from (15) that $k_c = k_i^- = 0$ and

$$k_r = 1 - n(\mathcal{D}). \tag{16}$$

4. Stability of periodic waves of the short-pulse equation. We consider the short pulse models in a symmetric spatial interval, subject to periodic boundary conditions

$$(u_t + (f(u))_x)_x = u. \tag{17}$$

Our main interest is the stability of explicit and classical traveling waves for the short pulse equation (17) of the form $u(t,x) = \varphi(x - ct)$. The profile equation for φ is

$$((\varphi^{p-1} - c)\varphi_\xi)_\xi = \varphi, \quad -L \le \xi \le L. \tag{18}$$

Clearly, (18) is *not a very nice* object to deal with. We perform a change of variables

$$\xi = \Xi(\eta) := \eta - \frac{\Psi(\eta)}{c}, \quad \varphi(\xi) = \Phi(\eta) = \Psi'(\eta). \tag{19}$$

which leads to the profile equation

$$-c^2\Phi'' - c\Phi + \Phi^p = 0. \tag{20}$$

One has to keep in mind however, that the solutions of (20) are equivalent, as long as the transformation (19) is invertible. For $p = 2$ and $p = 3$ we construct the explicit expression for Φ,

$$\Phi = \Phi_0 + (\Phi_1 - \Phi_0)sn^2(\alpha x, \kappa), \tag{21}$$

and

$$\Phi = \Phi_2 sn(\alpha x, \kappa), \tag{22}$$

respectively. Our next task is to derive the linearized problem for such solutions φ, assuming that they exist and the transformation (19) is invertible in the appropriate interval. To this end, we take the ansatz $u(t,x) = \varphi(x - ct) + v(t, x - ct)$ in (17) and ignore all quadratic terms. We obtain the following linearized equation

$$(v_t + ((\varphi^{p-1} - c)v)_\xi)_\xi = v \quad -L < \xi < L. \tag{23}$$

Next, we turn (23) into an eigenvalue problem, by letting $v(t,\xi) = e^{\lambda t}w(\xi), w \in H^2[-L, L]$. This results in

$$(\lambda w + ((\varphi^{p-1} - c)w)_\xi)_\xi = w \quad -L < \xi < L. \tag{24}$$

Again we perform a change of variables (19) and we get the eigenvalue problem

$$-c^2 Z_{\eta\eta} - cZ + \Phi^{p-1}Z = -\lambda c Z_\eta, Z \in L^2(-M, M). \tag{25}$$

Definition 2. *Assume that the periodic wave φ is a solution of (21), with some $c \neq 0$. Assume also that there exists an one-to-one mapping $\Xi : (-M, M) \to (-L, L), M \in$*

$(0,\infty]$ *satisfying* (19). *We say that the wave is spectrally unstable, if there exists* μ : $\Re\mu > 0$ *and a function* $Z \in H^2[-M,M] \cap C^2(-M,M)$*, so that*

$$\mathcal{L}[Z] := -c^2 Z_{\eta\eta} - cZ + \Phi Z = \mu Z'. \tag{26}$$

Similarly, we say that the solution φ *of* (22) *is unstable, if there is* μ : $\Re\mu > 0$ *and* $Z \in H^2[-M,M] \cap C^2(-M,M)$*, so that*

$$\mathcal{L}[Z] := -c^2 Z_{\eta\eta} - cZ + \Phi^2 Z = \mu Z'. \tag{27}$$

Our main results are the following two theorems:

Theorem 4. *The waves described in* (21) *are spectrally stable for all wave speeds* $c > 0$.

Theorem 5. *The waves described in* (22) *are spectrally stable for all wave speeds* $c > 0$.

The real instabilities for the eigenvalue problem (27) are ruled out provided

$$\langle \mathcal{L}^{-1}[\Phi'], [\Phi'] \rangle < 0. \tag{28}$$

This is achieved by constructing Green function for operator $\mathcal{L}$.

Now we will discuss an interesting limiting case. To that end, take $\kappa \to 1-$ in the solution Φ (22) and then apply the transformation (19). We obtain the so-called *parabolic peakons* in the form

$$\lim_{\kappa\to 1-} \Phi_\kappa(\eta(\xi)) = \varphi(\xi) = \frac{\xi^2}{6} - \frac{c}{2}, c = \frac{L^2}{9}, \quad -L < \xi < L. \tag{29}$$

Taking derivative in η in (19), we obtain

$$\frac{d\Xi}{d\eta} = 1 - \frac{\Psi'(\eta)}{c} = 1 - \frac{\varphi(\xi)}{c} = \frac{3}{2L^2}(L^2 - \xi^2). \tag{30}$$

Treating the latter equality as a separable ODE, we have $\dfrac{d\xi}{L^2 - \xi^2} = \dfrac{3}{2L^2} d\eta$. By integrating the ODE, we obtain the formula

$$\xi = L \frac{Ce^{\frac{e^{3\eta}}{L}} - 1}{Ce^{\frac{e^{3\eta}}{L}} + 1}.$$

This gives us an arbitrary solution of (30). We set $C = 1$ as a particular solution, which gives us

$$\xi = \Xi(\eta) = L \frac{e^{\frac{e^{3\eta}}{L}} - 1}{e^{\frac{e^{3\eta}}{L}} + 1} = Ltanh(3\eta/(2L)). \tag{31}$$

Clearly, the function $\Xi : (-\infty,\infty) \to (-L,L)$ is one-to-one and it has an inverse function $\eta = \eta(\xi) : (-L,L) \to (-\infty,\infty)$. Thus, in this case the interval $(-M,M)$ (which we used to get from inverting the transformation $\xi = \Xi(\eta)$) is actually degenerate, in that $M = \infty$. Using the latter in (29), we obtain a formula for Φ. More precisely,

$$\Phi(\eta) = \varphi(\Xi(\eta)) = \frac{L^2}{6}\tanh^2(3\eta/(2L)) - \frac{L^2}{18} = \frac{L^2}{9}\left(1 - \frac{3}{2}\operatorname{sech}^2(3\eta/(2L))\right). \tag{32}$$

Note that $-c + \Phi = -\dfrac{L^2}{6}\operatorname{sech}^2(3\eta/(2L))$. We will show that for some $\mu > 0$, the resulting eigenvalue problem

$$(33) \qquad (-\frac{L^2}{81}\partial_{\eta\eta} - \frac{1}{6}\operatorname{sech}^2(3\eta/(2L)))Z = \pm\mu Z_\eta, \quad -\infty < \eta < \infty,$$

has a solution Z.

We make a few more transformation for (33). If we take $f : Z(\eta) = f(3\eta/(2L))$, it leads us into the equivalent eigenvalue problem

$$(34) \qquad -f''(x) - 6\operatorname{sech}^2(x)f(x) = \pm\mu_1 f'(x),$$

where $\mu_1 = \dfrac{54\mu}{L}$.

Theorem 6. *The parabolic peakons φ, given by* (29), *are spectrally unstable for all $L > 0$ in the sense that there exists $z \in C^2(-L, L)$ (in fact $z \in C^\infty(-L, L)$) so that* (27) *is satisfied in a classical sense for $\xi \in (-L, L)$.*

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ХАМИЛТЪН ИНДЕКС НА НЕСТАБИЛНОСТ И ПРИЛОЖЕНИЯ КЪМ СТАБИЛНОСТ НА ПЕРИОДИЧНИ ВЪЛНИ

Севджан Хаккъев

В този обзор представяме някои резултати, свързани с устойчивоста на периодични вълни. За целта разглеждаме копериодични пертурбации. В основата на тези разглеждания се използват индексни теории, които се основават на детайлен спектрален анализ, както на скаларни, така и матрични Хил оператори, и на пресмятането на определени величини, свързани с тези индекси.