МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2021
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2021
Proceedings of the Fiftieth Spring Conference of the Union of Bulgarian Mathematicians 2021

LINEAR STABILITY OF PERIODIC WAVES OF THE COUPLED NONLINEAR WAVE EQUATIONS*

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We study the linear stability of periodic traveling wave solutions for the coupled nonlinear wave equations. It is shown that for some values of the parameters of the system the solutions of cnoidal type are spectrally unstable with respect to co-periodic perturbations.

1. Introduction. Traveling waves play a central role in the study of nonlinear differential equations. The two fundamental problems of study are existence and stability of these waves. There are mainly the following kinds of stability: linear (spectral) stability and nonlinear stability. It is well known that in the periodic case the spectrum of linearized equation depends on the choice of function space. In the space of periodic functions the spectrum consists of isolated eigenvalues, while in the space of bounded functions the spectrum is continuous.

In the present paper we study the following nonlinear wave system

$$\left|\begin{array}{l} u_{tt} - u_{xx} + u - (u^2+v^2)u = 0 \\ v_{tt} - v_{xx} + v - (u^2+v^2)v = 0. \end{array}\right. \tag{1}$$

This system arises as a model of the interaction of two fields [8]. The system (1) can also be considered as coupled equations of the nonlinear wave equation

$$u_{tt} - u_{xx} + u - u^3 = 0. \tag{2}$$

Recently, the linear stability of traveling wave solutions to second order in time nonlinear differential equations has been studied extensively [1, 4, 9] . In [9] the question of the stability analysis for second order in time PDEs is reduced to the study of stability of quadratic pencils in the form $\lambda^2 + 2c\lambda\partial_x + \mathcal{H} = 0$, where $\mathcal{H}$ is a self-adjoint operator. If $\mathcal{H}$ has a simple negative eigenvalue and a simple eigenvalue at zero, the authors in [9] derived the index of stability theory for traveling waves on the whole line and the abstract results were applied to Boussinesq equation, Klein-Gordon equation and beam equation. In [4] the stability of periodic waves for Boussinesq equation is considered. In [7], using the theory developed in [3] the orbital stability of periodic wave of snoidal type for (2) is considered.

In this paper we are interested in the stability of periodic traveling wave solutions of (1) with respect to perturbations that are periodic and of the same period as the corresponding wave solutions. We adapt the abstract results developed in [6] to the periodic

*2020 Mathematics Subject Classification: Primary 35B35, Secondary 35L05.
Key words: periodic traveling waves, linear stability, nonlinear wave equation.

case. First we need to obtain the required spectral information about the operator of linearization. Then we investigate the index of stability defined in [6].

The paper is organized as follows. In Section 2, we set up the linearized problem and give the general abstract result that we use. In Section 3, we demonstrate how it works.

2. Linear stability overview. For the system (1), we consider periodic waves of the form $(\varphi(x-ct),0)$, $|c|<1$. For φ, we have the following differential equation

$$-w\varphi'' + \varphi - \varphi^3 = 0, \qquad w = 1-c^2. \tag{3}$$

We make the substitution $u(t,x) = \varphi(x-ct)+p(t,x-ct)$, $v(t,x) = q(t,x-ct)$ in (1), where the functions p and q are periodic with respect to x with the same period as the function φ. Ignoring all quadratic and higher order terms yields the following equation

$$\vec{U}_{tt} - J\vec{U}_t + H\vec{U} = 0, \tag{4}$$

where

$$\vec{U} = (p,q), \quad J = \begin{pmatrix} 2c\partial_x & 0 \\ 0 & 2c\partial_x \end{pmatrix},$$

and

$$H = \begin{pmatrix} -w\partial_{xx} + 1 - 3\varphi^2 & 0 \\ 0 & -w\partial_{xx} + 1 - \varphi^2 \end{pmatrix}.$$

If we consider the eigenvalue problem associated with (4), that is $\vec{U} = e^{\lambda t}\vec{V}$, we arrive at

$$\lambda^2\vec{V} - \lambda J\vec{V} + H\vec{V} = 0. \tag{5}$$

Definition 1. *We say that the traveling wave solution ϕ is linearly unstable, if there exist a T-periodic function $\psi \in D(H)$ and $\lambda : \Re\lambda > 0$, such that*

$$\lambda^2\psi - 2c\lambda\psi_x + H\psi = 0. \tag{6}$$

Otherwise, we say that ϕ is stable.

We can write an equivalent to (5) Hamiltonian eigenvalue problem, namely,

$$\mathcal{J}\mathcal{H}\vec{\boldsymbol{V}} = \lambda\vec{\boldsymbol{V}}, \quad \vec{\boldsymbol{V}} = \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} \in X \times X, \quad X = H^2_{per}[-T,T] \tag{7}$$

where

$$\mathcal{J} = \begin{pmatrix} 0_2 & I_2 \\ -I_2 & J \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} H & 0_2 \\ 0_2 & I_2 \end{pmatrix}.$$

We use the instability index count theory in [6]. We present a corollary, which is sufficient for our purposes. For an eigenvalue problem in the form (7), we assume that $\mathcal{H} = \mathcal{H}^*$ has $\dim(\mathrm{Ker}(\mathcal{H}) < \infty$, and also a finite number of negative eigenvalues, $n(\mathcal{H})$, a quantity sometimes referred to as *Morse index of the operator* ch. We consider the eigenvalue problem

$$\mathcal{J}\mathcal{H}\vec{\mathbf{U}} = \lambda\vec{\mathbf{U}}. \tag{8}$$

Let k_r be the number of positive eigenvalues of the spectral problem (8) (i.e. the number of real instabilities or real modes), k_c be the number of quadruplets of eigenvalues with non-zero real and imaginary parts, and k_i^- be the number of pairs of purely imaginary eigenvalues with negative Krein-signature. For a simple pair of imaginary eigenvalues $\pm i\mu, \mu \neq 0$, and the corresponding eigenvector $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, the Krein index is $\mathrm{sgn}(\langle \mathcal{H}\vec{z}, \vec{z}\rangle)$.

Also of importance in this theory is a finite dimensional matrix $\mathcal{D}$, which is obtained from the adjoint eigenvectors for (8). More specifically, consider the generalized kernel of $\mathcal{JH}$

$$gKer(\mathcal{JH}) = \mathrm{span}[(\mathrm{Ker}(\mathcal{JH}))^l, l = 1, 2, \ldots].$$

Assume that $\dim(g\,\mathrm{Ker}(\mathcal{JH})) < \infty$. Select a basis in $gKer(\mathcal{JH}) \ominus \mathrm{Ker}(\mathcal{JH}) = \mathrm{span}[\eta_j, j = 1, \ldots, N]$. Then $\mathcal{D} \in M_{N\times N}$ is defined via

$$\mathcal{D} := \{\mathcal{D}_{ij}\}_{i,j=1}^N : \mathcal{D}_{ij} = \langle \mathcal{H}\eta_i, \eta_j\rangle.$$

Then, following [6], we have the following formula, relating the number of "instabilities" or Hamiltonian index of the eigenvalue problem (8) and the Morse indices of the operators $\mathcal{H}$ and $\mathcal{D}$

$$k_{Ham} := k_r + 2k_c + 2k_i^- = n(\mathcal{H}) - n(\mathcal{D}). \tag{9}$$

Remark. If the right side of (9) is odd number, then $k_r > 0$ and hence we have instability.

Now we will give some information about spectrum of Hill operators with Lame potential. It is well-known that the first five eigenvalues of $\Lambda_1 = -\partial_y^2 + 6k^2 sn^2(y,k)$, with periodic boundary conditions on $[0, 4K(k)]$ are simple. These eigenvalues and the corresponding eigenfunctions are:

$$\begin{array}{ll}
\nu_0 = 2 + 2k^2 - 2\sqrt{1-k^2+k^4}, & \phi_0(y) = 1 - (1 + k^2 - \sqrt{1-k^2+k^4})sn^2(y,k), \\
\nu_1 = 1 + k^2, & \phi_1(y) = cn(y,k)dn(y,k) = sn'(y,k), \\
\nu_2 = 1 + 4k^2, & \phi_2(y) = sn(y,k)dn(y,k) = -cn'(y,k), \\
\nu_3 = 4 + k^2, & \phi_3(y) = sn(y,k)cn(y,k) = -k^{-2}dn'(y,k), \\
\nu_4 = 2 + 2k^2 + 2\sqrt{1-k^2+k^4}, & \phi_4(y) = 1 - (1 + k^2 + \sqrt{1-k^2+k^4})sn^2(y,k).
\end{array}$$

For the operator $\Lambda_2 = -\partial_y^2 + 2\kappa^2 sn^2(y,\kappa)$, first three eigenvalues are simple. These eigenvalues and corresponding eigenfunctions are:

$$\begin{cases}
\epsilon_0 = k^2, & \theta_0(y) = dn(y,k), \\
\epsilon_1 = 1, & \theta_1(y) = cn(y,k), \\
\epsilon_2 = 1 + k^2, & \theta_2(y) = sn(y,k).
\end{cases}$$

3. Stability of cnoidal waves. Integrating once equation (3), we get

$$\varphi'^2 = \frac{1}{2w}(-\varphi^4 + 2\varphi^2 + a), \tag{10}$$

where a is a constant of integration. For $a > 0$ equation (10) has a solution in the form

$$\varphi(x) = \varphi_0 cn(\alpha x, \kappa), \qquad |\varphi_0| > \sqrt{2} \tag{11}$$

where

$$\kappa^2 = \frac{\varphi_0^2}{2\varphi_0^2 - 2}, \ \ \alpha^2 = \frac{2\varphi_0^2 - 2}{2w} = \frac{1}{w(2\kappa^2 - 1)}. \tag{12}$$

Since the fundamental period of cn is $4K(\kappa)$, then the fundamental period of $\varphi(x)$ is $2T = \frac{4K(\kappa)}{\alpha}$. Here and below $E(\kappa)$ and $K(\kappa)$ are complete elliptic integrals of first kind.

For the operator $H_1 = -w\partial_x^2 + 1 - 3\varphi^2$ using that $sn^2(y) + cn^2(y) = 1$ and (12), we get

$$\begin{aligned} H_1 &= -w\partial_x^2 + 1 - 3\varphi_0^2 cn^2(\alpha x, \kappa) \\ &= w\alpha^2\left[-\partial_y^2 + 6\kappa^2 sn^2(y,\kappa) - (1+4\kappa^2)\right] = w\alpha^2[\Lambda_1 - (1+4\kappa^2)], \end{aligned}$$

where $y = \alpha x$.

It follows that the first three eigenvalues of the operator L_1, equipped with periodic boundary condition on $[0, 4K(k)]$ are simple and zero is the third eigenvalue.

For the operator $H_2 = -w\partial_x^2 + 1 - \varphi^2$, we have

$$\begin{aligned} H_2 &= -w\partial_x^2 + 1 - \varphi_0^2 cn^2(\alpha x, \kappa) \\ &= w\alpha^2\left[-\partial_y^2 + 2\kappa^2 sn^2(y,\kappa) - 1\right] = w\alpha^2[\Lambda_2 - 1]. \end{aligned}$$

Hence $n(\mathcal{H}) = 3$, kernel of $\mathcal{H}$ is two dimensional and spanned by $(\varphi', 0, 0, 0)$ and $(0, \varphi, 0, 0)$.

We have [see [5]], $g\,\mathrm{Ker}(\mathcal{J}\mathcal{H}) \ominus \mathrm{Ker}(\mathcal{H})$ is spanned by

$$\vec{\eta_1} = \begin{pmatrix} 2cH_1^{-1}\varphi'' \\ \varphi \\ \varphi' \\ 0 \end{pmatrix}, \quad \vec{\eta_2} = \begin{pmatrix} \varphi' \\ 2cH_2^{-1}\varphi' \\ 0 \\ \varphi \end{pmatrix}. \tag{13}$$

and

$$\begin{cases} \mathcal{D}_{11} = ||\varphi'||^2 + 4c^2\langle H_1^{-1}\varphi'', \varphi''\rangle \\ \mathcal{D}_{12} = \mathcal{D}_{21} = 0 \\ \mathcal{D}_{22} = ||\varphi||^2 + 4c^2\langle H_2^{-1}\varphi', \varphi'\rangle. \end{cases} \tag{14}$$

Since $\varphi' \perp \{\theta_0(\alpha x), \theta_1(\alpha x)\}$, then $\langle H_2^{-1}\varphi', \varphi'\rangle > 0$. Now we will estimate $\langle H_1^{-1}\varphi'', \varphi''\rangle$ and $\langle \varphi', \varphi'\rangle$. We have the following representation

$$\langle H_1^{-1}\varphi'', \varphi''\rangle = \frac{1}{w^2}\langle H_1^{-1}\varphi, \varphi\rangle + \frac{1}{2w^2}\langle \varphi, \varphi\rangle - \frac{1}{2w}\langle \varphi', \varphi'\rangle.$$

First, we will compute $\langle H_1^{-1}\varphi, \varphi\rangle$. We have $H_1\varphi' = 0$. The function

$$\psi(x) = \varphi'(x)\int^x \frac{1}{\varphi'^2(s)}ds, \quad \begin{vmatrix} \varphi' & \psi \\ \varphi'' & \psi' \end{vmatrix} = 1$$

is also solution of $H_1\psi = 0$. Formally, since φ' has zeros using the identity

$$\frac{1}{sn^2(y,\kappa)} = -\frac{1}{dn(y,\kappa)}\frac{\partial}{\partial_y}\frac{cn(x,\kappa)}{sn(y,\kappa)}$$

and integrating by parts, we get

$$\psi(x) = \frac{1}{\alpha^2\varphi_0}\left[cn(\alpha x) - \alpha\kappa^2 sn(\alpha x,\kappa)dn(\alpha x,\kappa)\int_0^x \frac{1+cn^2(\alpha s,\kappa)}{dn^2(\alpha s,\kappa)}ds\right].$$

After integrating by parts, we get

$$\langle H_1^{-1}\varphi,\varphi\rangle = -\frac{1}{w}\langle\varphi^3,\psi\rangle + \frac{\varphi^2(T)+\varphi(0)^2}{2w}\langle\varphi,\psi\rangle + C_\varphi\langle\varphi,\psi\rangle. \tag{15}$$

Similarly as in [2], integrating by parts yields

$$\langle\psi'',\varphi\rangle = 2\psi'(T)\varphi(T) + \langle\psi,\varphi''\rangle.$$

Using that $H\varphi = -2\varphi^3$, we get

$$\langle\psi,\varphi^3\rangle = -w\psi'(T)\varphi(T).$$

We have

$$C_\varphi = -\frac{\varphi''(T)}{2w\psi'(T)}\langle\varphi,\psi\rangle + \frac{\varphi^2(T)-\varphi^2(0)}{2w}.$$

Hence

$$\langle H_1^{-1}\varphi,\varphi\rangle = \psi'(T)\varphi(T) + \frac{\varphi^2(T)}{w}\langle\varphi,\psi\rangle - \frac{\varphi''(T)}{2w\psi'(T)}\langle\varphi,\psi\rangle^2$$

and

$$\langle H_1^{-1}\varphi,\varphi\rangle = -\frac{2}{\alpha}\frac{E^2(\kappa)-2(1-\kappa^2)E(\kappa)K(\kappa)+(1-\kappa^2)K^2(\kappa)}{(2\kappa^2-1)E(\kappa)+(1-\kappa^2)K(\kappa)}.$$

By direct estimates, we have

$$\begin{cases} \|\varphi\|^2 = \dfrac{\varphi_0^2}{\alpha}\dfrac{4[E(\kappa)-(1-\kappa^2)K(\kappa)]}{\kappa^2} \\[2ex] \|\varphi'\|^2 = 4\alpha\varphi_0^2\dfrac{(2\kappa^2-1)E(\kappa)+(1-\kappa^2)K(\kappa)}{3\kappa^2}. \end{cases}$$

Finally, we get

$$\begin{aligned} ||\varphi'||^2 + 4c^2\langle H_1^{-1}\varphi'',\varphi''\rangle &= \frac{8}{\alpha w^2}\left[\frac{(2\kappa^2-1)E(\kappa)+(1-\kappa^2)K(\kappa)}{3(2\kappa^2-1)^2}\right. \\ &\left. - \frac{\kappa^2(1-\kappa^2)K^2(\kappa)}{(2\kappa^2-1)^2[(2\kappa^2-1)E(\kappa)+(1-\kappa^2)K(\kappa)]}c^2\right] \end{aligned}$$

If the above expression is positive, then the right side of (9) is odd number. With this we proved the following theorem

Theorem 1. *Periodic traveling wave solutions of cnoidal type are spectrally unstable for all values of c, satisfying the following inequality*

$$c^2 < \frac{[(2\kappa^2-1)E(\kappa)+(1-\kappa^2)K(\kappa)]^2}{3\kappa^2(1-\kappa^2)K^2(\kappa)}.$$

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ЛИНЕЙНА УСТОЙЧИВОСТ НА ПЕРИОДИЧНИ ВЪЛНИ НА НЕЛИНЕЙНА СИСТЕМА ОТ ВЪЛНОВИ УРАВНЕНИЯ

Турхан Сюлейманов

В тази статия се разглежда линейната устойчивост на периодични вълни за система от нелинейни вълнови уравнения. Показано е, че за някои стойности на параметрите вълните от „кноидален“* тип са линейно неустойчиви при копериодични пертурбации.

2020 Mathematics Subject Classification: Основен: 35B35, Вторичен: 35L05.

Ключови думи: периодични вълни, линейна устойчивост, нелинейно вълново уравнение.

*Този тип описва повърхнинни вълни, чиято дължина е значително по-голяма от дълбочината на океана. За да добиете нагледна представа за поведението им, може да използвате демонстрационния модел Cnoidal Waves from Korteweg-de Vries Equation https://demonstrations.wolfram.com/CnoidalWavesFromKortewegDeVriesEquation/ (бел. ред.)