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ON SYMMETRIC POSITIVE SOLUTIONS OF p-LAPLACIAN DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS*

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In this paper we study the existence of symmetric positive solutions for p-Laplacian differential equation. Using the mountain-pass theorem and a lemma on symmetry we prove the existence of positive even solutions of the problem considered.

1. Introduction. In this paper we study the existence of symmetric and positive solutions of the Dirichlet problem for the second-order p-Laplacian equation

$$(1) \qquad (\varphi_p (u' (x)))' - a(x)\varphi_q (u (x)) + b(x)\varphi_r (u (x)) = 0, \quad x \in [-L, L],$$

where $\varphi_p (t) = |t|^{p-2} t,\ t \in \mathbf{R}$ and $L > 0$.

We assume (and denote the assumptions by **H**) that:

(**H**) the functions $a(x)$ and $b(x)$ are continuously differentiable, strictly positive and even functions on $[-L, L]$, $xa'(x) > 0$, $xb'(x) < 0$ for $x \neq 0$ and $2 \leq p < q < r$.

A partial case of the equation (1), where $p = 2, q = 3, r = 4$ appears in a biomathematics model suggested by Austin [2] in a model of an aneurysm in the circle of Willis. Grossinho and Sanchez [12] consider the periodic solutions of the equation in this case using a variational method. Periodic and homoclinic solutions are studied in [8]. Similar problems are considered in [4, 5], using a variational method. In [10] Tersian considers the following p-Laplacian differential equation

$$(2) \qquad \left(|u'|^{p-2}u'\right)' - a(x)u|u|^{p-2} + \lambda b(x)u|u|^{q-2} = 0, \qquad x \in \mathbb{R}$$

where $2 \leq p < q$ and $\lambda > 0$, $a(x)$ and $b(x)$ are continuously differentiable, strictly positive functions, $xa'(x) > 0$ and $xb'(x) < 0$ for $x \neq 0$. Higher-order equations are studied in [7] using the generalized Clark's theorem. It is applied to fourth-order p-Laplacian equations in [9].

Denote by X the Sobolev space:

$$(3) \qquad X = W_0^{1,p} (-L, L) = \{u \in \mathbf{L^p} (-L, L) : u' \in \mathbf{L^p} (-L, L) , u (-L) = u (L) = 0\} ,$$

where $\mathbf{L^p} (-L, L)$ is the usual Lebesgue space. The space X is a separable Banach space with the norm

$$\|u\|_X = \left(\int_{-L}^{L} \left(|u' (x)|^p + |u (x)|^p\right) \ dx\right)^{\frac{1}{p}},$$

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which is equivalent to the norm

$$(4) \qquad \|u\| = \left(\int_{-L}^{L} |u'(x)|^p \ dx \right)$$

by Poincare's inequality [3, p. 218], $\|u\|_X \leq C \|u'\|_{\mathbf{L}^p}$, where
$\|u\|_{\mathbf{L^p}}^p = \int_{-L}^{L} |u|^p \ dx.$

We use a variational formulation of the problem considering the functional $J : X \to \mathbf{R}$ defined as

$$J(u) = \frac{1}{p} \int_{-L}^{L} |u'(x)|^p \ dx + \frac{1}{q} \int_{-L}^{L} a(x) \left(u^+(x)\right)^q dx - \frac{1}{r} \int_{-L}^{L} b(x) \left(u^+(x)\right)^r dx$$

where $u^+ = \max(0, u)$. We look for the critical points of the functional J in order to find the solution of the problem

$$(5) \qquad \begin{cases} (\varphi_p(u'(x)))' - a(x)\varphi_q(u(x)) + b(x)\varphi_r(u(x)) = 0, x \in (-L, L) \\ u(-L) = u(L) = 0 \end{cases}$$

Using the well-known classical mountain-pass theorem, we conclude that the functional $J(x)$ has a nontrivial critical point u_l, which is a solution of the problem (5). By *a solution of problem* (5) we mean a function $u \in \mathbf{C}([-L, L])$, such that $\varphi_p(u') \in \mathbf{AC}([-L, L])$ and $u(x)$ satisfies Eq.(1) for $x \in [-L, L]$ and boundary condition $u(-L) = u(L) = 0$. Here $\mathbf{AC}([-L, L])$ denotes the space of absolutely continuous functions on $[-L, L]$ [1, 3].

Since $w = \varphi_p(u') \in \mathbf{AC}([-L, L]) = W^{1,1}(-L, L)$

$$\int_{-L}^{L} wv' \ dx = -\int_{-L}^{L} w'v \ dx,$$

for every $v \in X$. Since $u' = \varphi_{p'}(w) = \varphi_p^{-1}(w) \in \mathbf{C}([-L, L])$, where $\frac{1}{p} + \frac{1}{p'} = 1$, it follows that

$$\int_{-L}^{L} \left(\left(|u'|^{p-2} u' \right)' - a(x) \left(u^+\right)^{q-1} + b(x) \left(u^+\right)^{r-1} \right) v \ dx = 0$$

for every $v \in \mathbf{C}_0^\infty([-L, L])$. Then it follows that u is a solution of (5). To obtain the symmetry property, we extend the Lemma of Korman and Ouyang [6] to the p-Laplacian equations.

Our main result is:

Theorem 1. *Suppose that* $2 \leq p < q < r$ *and the* ($\mathbf{H}$) *assumptions hold. Then the problem* (5) *has a positive even solution* u_l *in the interval* $[-L, L]$ *for which* $\max\{u_l(x) : x \in [-L, L]\} = u_l(0)$ *and* $u_l'(x) < 0$ *for* $x > 0$.

Further, in Section 2, we present the variational formulation of the problem, and we formulate the symmetric lemma and mountain-pass theorem. We prove a lemma for the (PS) condition for the functional J. In Section 3 we prove Theorem 1.

2. Preliminary results. Let X be the Sobolove space as defined in (3) equipped with the norm

$$\|u\| = \left(\int_{-L}^{L} |u'(x)|^p \ dx \right)^{\frac{1}{p}}.$$

Note that the embedding $X \in \mathbf{C}([-L,L])$ is compact, $u \in \mathbf{L}^q(-L,L)$ for $q \geq p$ and

$$\int_{-L}^{L} |u(x)|^q \ dx \leq \|u\|_{\mathbf{L}^\infty}^{q-p} \|u\|_{\mathbf{L}^p}^p$$

[3, Chapter 8].

We consider the functional $J : X \to \mathbb{R}$

$$J(u) = \frac{1}{p}\int_{-L}^{L} |u'(x)|^p \ dx + \frac{1}{q}\int_{-L}^{L} a(x)|u(x)|^q \ dx - \frac{1}{r}\int_{-L}^{L} b(x)|u(x)|^r \ dx,$$

and we are looking for critical points of J, which are solutions of (5).

Let $f(x,u) = -a(x)\varphi_q(u) + b(x)\varphi_r(u)$. Then the problem (5) can be rewritten as

$$\begin{cases} (\varphi_p(u'(x)))' + f(x,u(x)) = 0, & x \in (-L,L) \\ u(-L) = u(L) = 0 \end{cases} \tag{6}$$

where $f \in C^1([-L,L] \times \mathbb{R}^+)$ and satisfies (**H1**) conditions:

$$(\mathbf{H1}) \quad \begin{cases} f(-x,u) = f(x,u), & x \in (-L,L) \\ xf_x(x,u) < 0, & x \in (-L,L) \setminus \{0\}, u > 0. \end{cases}$$

We will apply the following symmetric lemma due to Korman and Ouyang [6]:

Lemma 1 ([6])**.** *Assume that $f \in C^1((-L,L) \times \mathbb{R}^+)$ satisfies* (**H1**)*. Then any positive solution* (6) *is an even function, such that $u'(x) < 0$ for $x \in (0,L]$.*

We consider the modified problem

$$\begin{cases} (\varphi_p(u'(x)))' + f(x,u^+(x)) = 0, & x \in (-L,L) \\ u(-L) = u(L) = 0 \end{cases} \tag{7}$$

where $u^+ = \max(0,u)$ and we use variational statement of the problem (7). The solutions of the problem (7) are positive solutions of the problem (5) and (6).
In [10] Tersian considers Eq. (2) and proves that if $f(x,u) \in C^1([-L,L] \times [0,\infty))$ and satisfies **(H1)**, then any positive solution $u_L(x)$ of (6) is an even function. Moreover $u_L'(x) < 0$ for $x > 0$ and $u_L(0) = \max\{u_L(x), x \in (-L,L)\}$.

We define the functional $J(u)$ for (7) as follows:

$$J(u) = \frac{1}{p}\int_{-L}^{L} |u'(x)|^p \ dx + \frac{1}{q}\int_{-L}^{L} a(x)\left(u^+(x)\right)^q \ dx - \frac{1}{r}\int_{-L}^{L} b(x)\left(u^+(x)\right)^r \ dx,$$

Recall

Theorem 2 (Mountain-pass theorem [11])**.** *Let E be a Banach space with a norm $\|.\|$, $I \in C^1(E,\mathbb{R})$, $I(0) = 0$ and I satisfy the (PS) condition. Suppose that there exist $r > 0$, $\alpha > 0$ and $e \in E$ such that $\|e\| > r$ and*

(i) *$I(u) \geq \alpha$ if $\|u\| = r$;*

(ii) *$I(e) < 0$. Let $c = \inf_{\gamma \in} \left\{ \max_{0 \leq t \leq 1} I(\gamma(t)) \right\} \geq \alpha$, where*

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Then c is a critical value of I, i.e. there exists u_0 such that $I(u_0) = c$ and $I'(u_0) = 0$.

The use of the critical points theory needs the well known Palais-Smale (PS) conditions which plays a central role:

For $J \in C^1(X, \mathbb{R})$ we say that it satisfies the (PS) condition if any sequence $\{u_n\} \in X$ for which $J(u_n)$ is bounded and $J'(u_n)$ converges to 0 as $n \to \infty$ possesses a convergent subsequence [11].

Now we prove:

Lemma 2. *Let $2 \leq p < q < r$, $a(x)$ and $b(x)$ are continuous positive functions on $[-L, L]$ and (5) is satisfied. Then, the functional $J : X \to \mathbb{R}$ satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS)-sequence in X, i.e. $\{J(u_n)\}$ is a bounded sequence and $J'(u_n) \to 0$ in $X^\star$. We have

$$\frac{1}{r}\langle J'(u_n), u_n\rangle = \frac{1}{r}\int_{-L}^{L}\left(|u_n'(x)|^p + a(x)\left(u_n^+(x)\right)^q - b(x)\left(u_n^+(x)\right)^r\right)\,dx$$

and

$$\begin{aligned} J(u_n) &- \frac{1}{r}\langle J'(u_n), u_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{r}\right)\int_{-L}^{L}|u_n'(x)|^p\,dx + \left(\frac{1}{q} - \frac{1}{r}\right)\int_{-L}^{L}\left(u_n^+(x)\right)^q\,dx \\ &\geq \left(\frac{1}{p} - \frac{1}{r}\right)\|u_n\|^p, \end{aligned}$$

which implies that

$$|J(u_n)| + \frac{1}{r}\|J'(u_n)\|_\star\|u_n\| \geq \left(\frac{1}{p} - \frac{1}{r}\right)\|u_n\|^p. \tag{8}$$

Then, the sequence $\{u_n\}$ is bounded in X. Indeed, if we suppose that there is a subsequence $\{u_{n_k}\}$ still denoted by $\{u_n\}$ such that $\|u_n\| \to \infty$, by (7) we obtain

$$\frac{|J(u_n)|}{\|u_n\|^p} + \frac{1}{r}\frac{\|J'(u_n)\|_\star}{\|u_n\|^{p-1}} \geq \frac{1}{p} - \frac{1}{r} > 0,$$

which implies a contradiction as $n \to \infty$, because $|J(u_n)|$ is bounded and $\|J'(u_n)\|_\star \to 0$. Hence $\{u_n\}$ is a bounded sequence in X. Let $u_n \rightharpoonup u$ weakly in X. By compact embedding $X \subset \mathbf{C}([-L, L])$, it follows that:

$$\begin{aligned} &\lim_{n\to\infty}\int_{-L}^{L}a(x)\left(u_n^+(x)\right)^q\,dx = \int_{-L}^{L}a(x)\left(u^+(x)\right)^q\,dx, \\ &\lim_{n\to\infty}\int_{-L}^{L}b(x)\left(u_n^+(x)\right)^r\,dx = \int_{-L}^{L}b(x)\left(u^+(x)\right)^r\,dx. \end{aligned} \tag{9}$$

We have $\langle J'(u_n), u_n\rangle \to 0$ as $n \to \infty$ by

$$|\langle J'(u_n), u_n\rangle| \leq \|J'(u_n)\|_\star\|u_n\|.$$

Then

$$\begin{aligned} 0 &= \lim_{n\to\infty}\langle J'(u_n) - J'(u), u_n - u\rangle \\ &= \lim_{n\to\infty}\int_0^L\left(\varphi_p(u_n') - \varphi_p(u')\right)(u_n' - u')\,dx \end{aligned} \tag{10}$$

By the inequality

$$(\varphi_p(x) - \varphi_p(y))(x - y) \geq \frac{2}{p(2^{p-1} - 1)}|y - x|^p$$

for $x, y \in \mathbb{R}$; for $p \geq 2$ we have

$$\left(\varphi_p\left(u_n'\right) - \varphi_p\left(u'\right)\right)\left(u_n' - u'\right) \geq \frac{2}{p\left(2^{p-2}-1\right)}\left|u_n' - u'\right|^p$$

which implies by (9) that $u_n \to u$ strongly in X. $\square$

3. Proof of the Main result. To prove Theorem 1 we show that the geometric assumptions of Theorem 2 are satisfied and apply Lemma 1.

Since $X = W_0^{1,p}(-L, L) \subset \mathbf{C}([-L, L]) \subset \mathbf{L}^q(-L, L)$ and $X = W_0^{1,p}(-L, L) \subset \mathbf{C}([-L, L]) \subset \mathbf{L}^r(-L, L)$ for $2 \leq p < q < r$ there are constants c_1 and c_2 such that for $u \in X$

$$\|u\|_{\mathbf{L}^q} = \left(\int_{-L}^{L} |u(x)|^q \, dx\right)^{\frac{1}{q}} \leq c_1 \|u\|, \tag{11}$$

$$\|u\|_{\mathbf{L}^r} = \left(\int_{-L}^{L} |u(x)|^r \, dx\right)^{\frac{1}{r}} \leq c_2 \|u\|.$$

Proof of Theorem 1. We prove that conditions (i) and (ii) are satisfied.

(i) Since $a(x)$ and $b(x)$ are positive, even and continuous functions we have $A = \max\limits_{x\in[-L,L]} a(x)$, $a = \min\limits_{x\in[-L,L]} a(x)$, $b = \min\limits_{x\in[-L,L]} b(x)$ and $B = \max\limits_{x\in[-L,L]} b(x)$ then

$$0 < a \leq a(x) \leq A, \; 0 < b \leq b(x) \leq B \tag{12}$$

By (11) and (12) we obtain:

$$J(u) = \frac{1}{p}\|u\|^p + \frac{1}{q}\int_{-L}^{L} a(x)\left(u^+\right)^q \, dx - \frac{1}{r}\int_{-L}^{L} b(x)\left(u^+\right)^r \, dx \geq$$

$$\geq \frac{1}{p}\|u\|^p - \frac{Bc_2}{r}\|u\|^r = \|u\|^p\left(\frac{1}{p} - \frac{Bc_2}{r}\|u\|^{r-p}\right).$$

Since $r > p$, for $\|u\| = \rho < \left(\frac{r}{c_2 pB}\right)^{\frac{1}{r-p}}$ sufficiently small there exist $\alpha > 0$ such that $J(u) \geq \alpha > 0$.

(ii) Let $u_0(x) \in X$ be such that $u_0(x) > 0$ if $x \in (-L, L)$ and also $u_0(-L) = u_0(L) = 0$.

Consider the function

$$\tilde{u}_0(x) = \begin{cases} tu_0(x) & \text{if } x \in [-1, 1], \\ 0 & \text{if } x \in [-L, L] \setminus [-1, 1]. \end{cases}$$

Then by (11) and (12) it follows that:

$$
\begin{aligned}
J\left(\tilde{u}_0\right) = t^p \int_{-L}^{L} \frac{|u_0'|^p}{p}\, dx + \int_{-L}^{L} \left(t^q a\left(x\right) \frac{|u_0|^q}{q} - t^r b\left(x\right) \frac{|u_0|^r}{r} \right) dx \leq \\
\leq \frac{t^p}{p} \left\| u_0 \right\|^p + \int_{-L}^{L} \left(\frac{t^q}{q} A \left| u_0 \right|^q - \frac{t^r}{r} b \left| u_0 \right|^r \right) dx < \\
< \frac{t^p}{p} \left\| u_0 \right\|^p + \int_{-L}^{L} \left(\frac{t^q}{q} A \left| u_0 \right|^q - \frac{t^r}{r} b \left| u_0 \right|^r \right) dx = \\
= \frac{t^p}{p} \left\| u_0 \right\|^p + \frac{At^q}{q} \int_{-L}^{L} \left| u_0 \right|^q \, dx - \frac{bt^r}{r} \int_{-L}^{L} \left| u_0 \right|^r \, dx = \\
= \frac{t^p}{p} \left\| u_0 \right\|^p + \frac{At^q}{q} \left\| u_0 \right\|_{L^q}^q - \frac{bt^r}{r} \left\| u_0 \right\|_{L^r}^r \leq \\
\leq \frac{t^p}{p} \left\| u_0 \right\|^p + c_1 \frac{t^q}{q} A \left\| u_0 \right\|^q - c_2 \frac{t^r}{r} b \left\| u \right\|^r = \\
= \left\| u_0 \right\|^p t^p \left(\frac{1}{p} + Ac_1 \frac{t^{q-p}}{q} \left\| u_0 \right\|^{q-p} - bc_2 \frac{t^{r-p}}{r} \left\| u_0 \right\|^{r-q} \right) < 0
\end{aligned}
$$

for $t > 0$ small enough.

By Lemma 2 and Theorem 2, there exists a solution $u_l \in X$ such that $J\left(u_l\right) = c$ and $J'\left(u_l\right) = 0$. Moreover, if u_l is a positive solution of (5), by Lemma 1 we obtain that u_l is an even solution, $u_l\left(0\right) = \max\left\{u_l\left(x\right), x \in \left(-L, L\right)\right\}$ and $u_l'\left(x\right) < 0$ for $x \in (0, L]$. □

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ВЪРХУ ЧЕТНИТЕ ПОЛОЖИТЕЛНИ РЕШЕНИЯ НА ЗАДАЧА ЗА p-ЛАПЛАСОВО ДИФЕРЕНЦИАЛНО УРАВНЕНИЕ

Гергана Цветкова

В статията се изследва съществуването на положителни четни решения на задача на Дирихле за едномерни p-Лапласови уравнения. Приложени са теоремата за хребета и лема за симетрия.